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Beauty Contests Under Private Information and Diverse Beliefs: How Different?

by

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Abstract: The paper contrasts theories that explain diverse belief by asymmetric private information (in short PI) with theories which postulate agents use subjective heterogenous beliefs (in short HB). We focus on problems where agents forecast *aggregates* such as profit rate of the S&P500 and our model is similar to the one used in the literature on asset pricing (e.g. Brown and Jennings (1989), Grundy and McNichols (1989), Allen, Morris and Shin (2003)).

We first argue there is no a-priori conceptual basis to assuming PI about economic aggregates. Since PI is not observed, models with PI offer no testable hypotheses, making it possible to prove anything with PI. In contrast, agents with HB reveal their forecasts hence data on market belief is used to test hypotheses of HB. We show the common knowledge assumptions of the PI theory are implausible. The theories differ on four main analytical issues. (1) The pricing theory under PI implies prices have infinite memory and at each t depend upon *unobservable* variables. In contrast, under HB prices have finite memory and depend only upon observable variables. (2) The "Beauty Contest" implications of the two are different. Under PI today's price depends upon today's market belief about tomorrow's *mean belief about "fundamental"* variables. Under HB it depends upon today's market belief about tomorrow's *market beliefs.* Tomorrow's beliefs are, in part, beliefs about future beliefs and are often mistaken. *Market forecast mistakes are key to Beauty Contests*, and are a central cause of market uncertainty called "endogenous uncertainty." (3) Contrary to PI, theories with HB have wide empirical implications which are testable with available data. (4) PI theories assume unobserved data and hence do not restrict behavior, while rationality conditions impose restrictions on any HB theory. We explain the tight restrictions on the model's parameters imposed by the theory of *Rational Beliefs.*

JEL classification: D82, D83, D84, G12, G14, E27.

Keywords: private information; Bayesian learning; updating beliefs; heterogenous beliefs; asset pricing; Rational Beliefs.

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Diversity of belief is an empirical fact. A large and growing body of work has used this diversity to explain various market phenomena, and there are two theories inspired by it. One follows the Harsanyii doctrine which views people as Bayesian decision makers who hold the same probability belief but who have asymmetric private information which they use in forecasting. Examples of papers that are applicable here includes Phelps (1970), Lucas (1972), Diamond and Verrecchia (1981), Singleton (1987), Brown and Jennings (1989), Grundy and McNichols (1989), Wang (1994), He and Wang (1995), Hellwig (2002), Judd and Bernardo (1996), (2000), Woodford (2003), Allen, Morris and Shin (2003) and others. An alternative view holds that there is nothing to justify a common prior and heterogeneity of probability models is inevitable in a complex world. Moreover, agents clearly do not have and do not use private information to forecast aggregates such as the S&P 500, GNP growth rate, exchange rates, inflation or interest rates, yet there is a vast diversity of such forecasts. A sample of papers which use this approach includes Harrison and Kreps (1978), Varian (1985), (1989), Harris and Raviv (1993), Detemple and Murthy (1994), Kurz (1994), (1997a), Kurz and Motolese (2001), Kurz Jin and Motolese (2005a), (2005b), Motolese (2001), (2003), Nielsen (1996),(2003), Wu and Guo (2003), (2004). In particular, Kurz's (1994), (1997a) theory of belief diversity stresses the impossibility of perfect learning. It holds that our environment is non-stationary with technological and institutional changes occurring faster than we can learn them. But then, how different are these two theories of belief diversity? What are the differences in their theoretical and empirical implications?

This paper explores the economic structure of asset pricing theories under private information (in short, PI) compared with the structure of heterogenous beliefs approach (in short, HB), aiming to

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highlights the different theoretical and empirical implications of the two theories. To that end we keep the formalism down to a minimum, focusing on ideas and concepts. Our discussion is confined to theories where optimizing agents forecast aggregates such as future S&P500 returns, exchange rates, interest rates, GDP growth etc. We do not address the problem of forecasting future conditions of individual firms or establishments. Our main conclusions are that models with PI are not appropriate to the problem of forecasting economic aggregates and offer contrived solutions. On the other hand, theories where agents have diverse beliefs and use diverse models constitute a natural setting for problems of this type. We argue that PI models have virtually no empirical implications and hence with private information one can prove almost anything. In contras, models with HB have clear empirical implications and testable hypotheses since market beliefs are observable.

To explore the key ideas we first outline a simple model used to study asset pricing with private information. In Section 2 we adapt the model to an environment with HB but without private information. After fully developing the equilibrium asset pricing theory under HB we compare in Section 3 the results to those obtained under private information. We explore in Section 3.5 the restrictions on beliefs proposed by the theory of Rational Beliefs (see Kurz (1994), (1997a)).

1. Asymmetric Information and Asset Pricing

The model reviewed here is an adaptation of the short lived trader model used by Brown and Jennings (1989), Grundy and McNichols (1989), Allen, Morris and Shin (2003) and others. Specifying the model will also provide us with terminology and notation used throughout the paper.

There is a unit mass of traders, indexed by the [0, 1] interval and only one homogenous aggregate asset (e.g. S&P500 index fund) with unknown intrinsic value Q. The economy is static with one period divided into three dates (no discounting): in dates 1 traders first receive a public and private signals about the asset value and then they trade. In date 2 they trade again. In date 3 (or end of date 2) uncertainty is resolved, the true liquidation value Q of the asset is revealed and traders receive this value for their holdings. The initial information of traders is that Q is distributed normally with $E(Q) = y$ and variance $\frac{1}{\alpha}$. At date 1 each trader also observes a private signal about Q, $x^i = Q + \varepsilon^i$ where ε^i are, independently normally distributed across all i with mean 0 and variance $\frac{1}{\beta}$. Since these facts are common knowledge, agents know that the true unknown value Q is “in the market” at all time since by the law of large numbers the mean of all private signals is the future value Q. All have the same CARA utility over wealth W, with constant absolute coefficient of risk aversion. They maximize expected utility $u(W^i) = -e^{-(W^i/\tau)}$ where $W^i = S^i p_1 + D_1^i(p_2 - p_1) + D_2^i(Q - p_2)$. Trader i starts with S^i units of the aggregate asset and can borrow at zero interest to finance trading in it. (D_1^i, D_2^i) are i's demands in the first and second rounds and (p_1, p_2) are market prices in the two rounds. Aggregate supplies (S_1, S_2) of ownership shares traded in each of the rounds are random, *unobserved* and normally distributed. This noise is crucial since it ensures that traders cannot deduce from prices the true value of Q. In a noisy Rational Expectations Equilibrium (in short, REE) traders maximize expected utility while markets clear after traders deduce from prices all possible information. Indeed, Brown and Jennings (1989) show equilibrium price at date 1 is

(1a) $$p_1 = \kappa_1(\lambda_1 y + \mu_1 Q - S_1)$$

and since S_1 is normally distributed p_1 is also normally distributed. (1a) shows that since Q and S_1 are both unknown, prices are not fully revealing. Since over trading dates Q is fixed, more rounds of trading generate more price data from which traders deduce added information about Q. But with additional supply shocks the inference problem becomes more complicated. That is, at date 2 the price p_2 contains more information about Q but it depends upon *two* unobserved noise shocks (S_1, S_2). Hence, as in Brown and Jennings (1989), the price function takes the form

(1b) $$p_2 = \hat{\kappa}_2(\hat{\lambda}_2 y + \hat{\mu}_2 Q - S_2 + \psi S_1).$$

Since the realized noise S_1 is not known at date 2, traders condition on the known price p_1 to infer the information about S_1. They thus use a date 2 price function which takes an equivalent form

$$p_2 = \kappa_2(\lambda_2 y + \mu_2 Q - S_2 + \xi_{21} p_1).$$

Using (1a) equivalence implies that $\kappa_2 = \hat{\kappa}_2$, $\lambda_2 = (\hat{\lambda}_2 + \lambda_1\psi)$, $\mu_2 = (\hat{\mu}_2 + \mu_1\psi)$ and $\xi_{21} = -\frac{\psi}{\kappa_1}$. Denote by (H_1^i, H_2^i) the information of i in the two rounds. The linearity of the equilibrium price map implies that the payoff is normally distributed. Brown and Jennings (1989) then show in Appendix A that there exist constants (G_1, G_2) determined by the covariance matrix of the model's random variables such that the demand functions of trader i are

(2a) $$D_2^i(p_2) = \frac{\tau}{\mathrm{Var}^i(Q|H_2^i)}[E^i(Q|H_2^i) - p_2].$$

(2b) $$D_1^i(p_1) = \frac{\tau}{G_1}[E^i(p_2|H_1^i) - p_1] + \frac{(G_2 - G_1)}{G_1}[E^i(D_2^i|H_1^i)].$$

It is typically assumed that $\mathrm{Var}^i(Q|H_2^i) = \sigma_Q^2$ independent of i. The second term in (2b) is the "hedging demand" arising from risk perception of traders at date 1 about price change at date 2. The hedging demand in a noisy REE complicates the inference problem and raises problems regarding the existence of equilibrium. As a result, most writers ignore this demand and study the *myopic-investor economy*. This concept is framed by regarding traders as long or short lived. A "short lived" trader lives one period only. He first trades in date 1, gains utility from p_2 and leaves the economy. He is replaced by a new short lived trader who knows the information of the first trader but trades in date 2 and gains utility from the revealed Q. Neither trader has a hedging demand. A "long lived" trader lives through *both* periods, trades in dates 1 and 2 hence has a hedging demand. It is then common to ignore the second term in (2b), average on i, equate to supply and conclude that

(1c) $$p_2 = \bar{E}_2(Q) - \frac{\sigma_Q^2}{\tau}(S_1 + S_2) \;, \qquad p_1 = \bar{E}_1(p_2) - \frac{G_1}{\tau} S_1 \,.$$

$\bar{E}_2(Q)$ is date 2 average market forecast of Q and $\bar{E}_1(p_2)$ is average market forecast of p_2. In this case $G_1 = \mathrm{Var}_1^i(p_2)$ and it is assumed this variance is independent of i.

(2a)-(2b) depend only upon the condition that prices are normally distributed *but not upon any private information assumption*. Hence, the difference between the two theories on which we focus in this paper result from differences between their implications to the conditional expectations in (2a)-(2b). For example, (2a) shows p_2 depend upon date 2 expectations which are updated based on the information deduced from p_2 and p_1. This is different from date 1 information which consists of public signal, private signals and inference from p_1. Allen, Morris and Shin (2003) present in their Appendix A computations of the closed form solution. To get an idea of the inference involved we review the steps they take. What does a trader learn in round 1? Given prior belief $Q \sim N(y, \frac{1}{\alpha})$ trader i observes $p_1 = \kappa_1(\lambda_1 y + \mu_1 Q - S_1)$. Since $S_1 \sim N(0, 1/\gamma_1)$ all he infers from date 1 price is that

$$\frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y) = Q - \frac{S_1}{\mu_1} \sim N(Q\,,\, 1/(\mu_1^2\gamma_1)).$$

But now, his added piece of information is the private signal $x^i = \theta + \varepsilon^i$, $\varepsilon^i \sim N(0, \frac{1}{\beta})$. Using a standard Bayesian inference from these three sources, his posterior belief becomes

(3a) $$E^i(Q|H_1^i) = \frac{\alpha y + \beta x^i + \mu_1^2\gamma_1 \frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y)}{\alpha + \beta + \mu_1^2\gamma_1} = \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x^i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1}$$

(3b) with precision $\alpha + \beta + \mu_1^2\gamma_1$.

Averaging (3a) over the population we can see that the average market forecast at date 1 is then

$$\bar{E}_1(Q|H_1) = \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta Q + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1} \equiv \frac{\alpha y + (\beta + \mu_1^2\gamma_1)Q}{\alpha + \beta + \mu_1^2\gamma_1} - \frac{\mu_1\gamma_1 S_1}{\alpha + \beta + \mu_1^2\gamma_1}.$$

In round 2 a trader observes p_2 which is a function of the same three variables and of p_1. Given p_1 and the fact that $S_2 \sim N(0, \frac{1}{\gamma_2})$, he infers from $p_2 = \kappa_2(\lambda_2 y + \mu_2 Q - S_2 + \xi_{21}p_1)$ that

$$\frac{1}{\kappa_2\mu_2}(p_2 - \kappa_2\lambda_2 y - \kappa_2\xi_{21}p_1) = Q - \frac{S_2}{\mu_2} \sim N(Q, \frac{1}{\mu_2^2\gamma_2}).$$

He now updates (3a)-(3b). Since supply shocks are i.i.d. the updated posterior is standard

$$E^i(Q|H_2^i) = \frac{\left[\frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x^i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1}\right](\alpha + \beta + \mu_1^2\gamma_1) + \frac{1}{\kappa_2\mu_2}(p_2 - \kappa_2\lambda_2 y - \kappa_2\xi_{21}p_1)(\mu_2^2\gamma_2)}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}.$$

Simplification leads to

(4b) $$E^i(Q|H_2^i) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 - \mu_2\gamma_2\lambda_2]y + \beta x^i + [\frac{\mu_1\gamma_1}{\kappa_1}p_1 + \frac{\mu_2\gamma_2}{\kappa_2}p_2 - \mu_2\gamma_2\xi_{21}p_1]}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}$$

(4c) $$\mathrm{Var}(Q|H_2^i) = \frac{1}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}.$$

To compute (1c) we average (4b) to conclude that

(5a) $$\bar{E}_2(Q) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 - \mu_2\gamma_2\lambda_2]y + \beta Q + [\frac{\mu_1\gamma_1}{\kappa_1}p_1 + \frac{\mu_2\gamma_2}{\kappa_2}p_2 - \mu_2\gamma_2\xi_{21}p_1]}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}$$

(5b) $$\bar{E}_1(p_2) = \kappa_2(\lambda_2 y + \mu_2\bar{E}_1(Q) + \xi_{21}p_1).$$

When (5a)-(5b) are inserted into (1c) we end up with two equations in the two unknown prices which can now be computed. The final step is to match coefficients of the price functions (1a)-(1b) in order to identify $(\kappa_1, \lambda_1, \mu_1, \kappa_2, \lambda_2, \mu_2, \xi_{21})$. For details of these computations see Allen, Morris and Shin

(2003) , Appendix A. It is useful to write the forecasts (4b) and (5a) in terms of unknown variables:

$$\text{(4b')} \qquad E^{i}(Q|H_2^{i}) = \frac{\alpha y + \beta x^{i} + (\mu_1^2\gamma_1 + \mu_2^2\gamma_2)Q}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2} - \frac{\mu_1\gamma_1 S_1 + \mu_2\gamma_2 S_2}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}$$

$$\text{(5a')} \qquad \bar{E}_2(Q) = \frac{\alpha y + (\beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2)Q}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2} - \frac{\mu_1\gamma_1 S_1 + \mu_2\gamma_2 S_2}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}.$$

What is the length of memory in prices? The model is static but multiple trading rounds provide opportunities to deduce more information from prices about Q, revealed after N rounds. As trading continues, the memory of *all past prices is preserved* since prices depend upon all unobserved supply shocks. In such a case the price system can never be a finite memory Markovian process. The model has, indeed, been extended to multi period trading where Q is revealed N periods later (see Brown and Jennings (1989), Grundy and McNichols (1989), He and Wang (1995) and Allen, Morris and Shin (2003)). In these models the complexity of inference depends upon the presence of a hedging demand of long lived traders[3]. However, for both long and short lived traders *the number of trading rounds is an arbitrary modeling construct*. It would thus be instructive to examine the limit behavior of the model. In a third round of trading by the short lived traders the price map becomes

$$p_3 = \kappa_3(\lambda_3 y + \mu_3 Q - S_3 + \xi_{31}p_1 + \xi_{32}p_2).$$

Hence, the independent supply shock leads to an updating rule which is again standard

$$E^{i}(Q|H_3^{i}) = \frac{E^{i}(Q|H_2^{i})(\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2) + \frac{1}{\kappa_3\mu_3}(p_3 - \kappa_3\lambda_3 y - \kappa_3\xi_{31}p_1 - \kappa_3\xi_{32}p_2)(\mu_3^2\gamma_3)}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2 + \mu_3^2\gamma_3}.$$

Simplification and averaging over the population leads to the market forecast

$$\bar{E}_2(Q) = \frac{[\alpha - \mu_1\gamma_1\lambda_1 - \mu_2\gamma_2\lambda_2 - \mu_3\gamma_3\lambda_3]y + \beta Q}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2} + \\ + \frac{\frac{\mu_1\gamma_1}{\kappa_1}p_1 + \frac{\mu_2\gamma_2}{\kappa_2}p_2 + \frac{\mu_3\gamma_3}{\kappa_3}p_3 - \mu_3\gamma_3\xi_{31}p_1 - \mu_3\gamma_3\xi_{32}p_2 - \mu_2\gamma_2\xi_{21}p_1}{\alpha + \beta + \mu_1^2\gamma_1 + \mu_2^2\gamma_2}.$$

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[3] For discussion of the "long lived" traders see He and Wang (1995) and Appendix A of Allen, Morris and Shin (2003). For a simple exposition of the hedging demand in a two period economy see Brown and Jennings (1989).

As in (4b') individual and market forecasts can be expressed in terms of the unobserved variables. They can easily be extended to N rounds of trade and take the general form

$$(6) \qquad E^i(Q|H_N^i) = \frac{\alpha y + \beta x^i + \sum_{j=1}^{N} \mu_j^2 \gamma_j Q}{\alpha + \beta + \sum_{j=1}^{N} \mu_j^2 \gamma_j} - \frac{\sum_{j=1}^{N} \mu_j \gamma_j S_j}{\alpha + \beta + \sum_{j=1}^{N} \mu_j^2 \gamma_j}$$

A standard argument shows the μ_j converge. For simplicity assume the precision of S_j is constant hence $\gamma_j = \gamma$. The independence property of the noise with (6) and the law of large numbers imply that the first term converges to Q and the second converge with probability 1 to 0. Hence, in the limit, with probability 1 all forecasts converge to the true Q and the effect of the public signal y disappears. Hence, *repeated trade leads to a full revelation of the true value Q*. Moreover, in the limit $p = Q$ and traders do not forecast prices at all. If the unit of time is, say, a month the rounds of trade are not really limited. Hence the result contradicts Allen, Morris and Shin's (2003) claim that the effect of the public signal y on the price lingers on forever. With sufficient trading the effect of y disappears.

To conclude, the study of markets with private information has advanced our understanding of risk sharing and insurance markets. Here we examine its limits. With different information agents clearly make different forecasts. But private information is a very sharp sword. Hence, when diverse forecasting is an important component of a theory, the temptation is to assume private information to model diversity. A large literature has done just that. It is so common that for some, thinking of agents with different opinions is *synonymous* to thinking of them as having different private information. For forecasting market aggregates this equivalence is wrong and the assumption of private information has no merit. We identify three areas of forecasting where the model of diverse beliefs is the correct one:

(i) Market prices such as interest rates, indices of stock prices, foreign exchange rates ;

(ii) Macroeconomic variables such as rates of GNP growth, inflation, unemployment, monetary policy actions;

(iii) Exogenous shocks like productivity shocks, aggregate factor supplies etc.

Unfortunately there are many contributions which use models with asymmetric private information to solve problems in which traders forecast variables in the above three categories. Examples include Phelps (1970) and Lucas (1972) but recent examples include Romer and Romer (2000), Hellwig (2002), Woodford (2003), Amato and Shin (2003) Bacchetta and van Wincoop (2005a) and others.

Our view is then that the *economic explanation* provided by these papers is flawed and questionable[4].

To compare with theories under diverse beliefs, we interpret the asset value Q in the Noisy REE literature to be an aggregate value such as the S&P500, an interest rate or an exchange rate. Before formulating our HB model, we observe that the simple model discussed above leads to several *natural* objections against models where traders use private information to forecast variables in the three categories listed above. These natural objections do not depend upon the formulation of any specific heterogenous belief model. For this reason we outline these first.

2. When Should the Assumption of Asymmetric Information Be Avoided?

In casting significant doubt on the validity of the PI assumption we recall that the typical problem studied with PI include market volatility, aggregate risk premia, foreign exchange dynamics, business cycles, the effects of monetary policy, etc. Apart from the fact that the assumption of private information is not plausible, we also argue that the explanations offered for these phenomena, driven by Private Information, are unconvincing. Thus, PI offers a distorted "solution" for such problems.

(i) *What is the data that constitutes "private" information*? If forecasters of GNP growth or future interest rates use PI, one must be able to specify the data to which such forecasters have an exclusive access. Forecasters of macroeconomic variables, including the Federal Reserve itself, state their data sources and universally claim they use only published data. More important, without an explicit identification of the private information used by a forecaster, a model with PI does not make sense. Indeed, all empirical implications the model has are deduced from restrictions imposed by that information. As illustrated in Section 1, a model with PI specifies an unknown parameter Q about which agents receive private signals x_t^i with $i = 1, 2, \ldots$ For this to have meaning one must know what the x_t^i are or what they could conceivably be. When agents forecast aggregate variables in the

[4] To illustrate, Kurz (1997b) explains the volatility of foreign exchange rates and the forward discount bias in foreign exchange markets by demonstrating that these are consequences of diverse beliefs of traders about future exchange rates. In rejecting the REE framework he assumes agents hold diverse Rational Beliefs which are restricted as explained in Section 3.5 below. In such a market the center of uncertainty is the uncertainty of traders about future *beliefs* of other traders. Bacchetta and van Wincoop (2005a) adopt the same idea by using a noisy REE but assume that at each date traders have random *private information about future aggregate money supply*. Hence traders are uncertain about future private information of other traders. Our argument here is that in the context of exchange rates determination such an assumption does not have empirical validity and hence leads to an implausible explanation of the forward discount bias.

three categories above, no such imaginary data exist.

(ii) *Asymmetric information imply a Secretive Economy*. Forecasters take pride in their models and are eager to make their forecasts public. As a result, there are vast data files on market forecasts of most of the variables mentioned. These include data of the Blue Chip Economic Indicators (BLU), Blue Chip Financial Forecasts (BLUF), the Survey of Professional Forecasters (SPF), forecasts by individual firms engaged in forecasting and even detailed forecast data of the staff of the Federal Reserve System. Such data are being used more and more in economic research as (e.g. Romer and Romer (2000), Swanson (2006), Kurz (2005), Kurz and Motolese (2006)). In addition to making public their forecast data, forecasters stress their opinions are different from others. In discussing public information they explain their own interpretation of such information often framed as "their thesis", the weight they place on it and their disagreement with others' use of that same information. Trade journals are used to debate forecasting techniques and in public competitions prizes are awarded to the best forecaster in specified categories. Since PI gives clear advantage to those who have it other forecasters would not compete since there is nothing to compete about. In short, forecasters view their work as model formulation and interpretation of information, not a reflection of secret information to which they are privy. Such behavior is not compatible with an equilibrium with PI.

In contrast, an equilibrium with PI is secretive. Individuals are careful not to divulge their PI since it would deprive them of the advantage they have. In such an equilibrium all private forecast data of any state variable (e.g. productivity) are treated as sources of *new information.* Agents use forecast data of other forecasters to update their posterior beliefs about that state variable. Had such PI been deduced from forecasts, the mean market forecast would change. Since in reality all forecasters happily reveal their forecasts, the economy must converge to an equilibrium with uniform information. The eagerness of agents to reveal their forecasts is thus not compatible with PI being the cause of the persistent divergence of opinions and forecasts.

(iii) *For the problems considered, asymmetric information is not sufficient*. Implicit in (ii) is the fact that in REE with PI, there is basic tension between information asymmetry and revelation\learning. If prices reveal PI the model has noise to prevent such revelation. Noise must be unobserved and the cause for the noise is often unspecified. When specified, it takes strange forms such as an *unobserved* random supply of the asset. But then, the implications of the theory do not depend only upon the private information available but, more important, on the investigator's noise. The problem does not end there. As we have seen, repeated trading overcomes the effect of noise and leads to full revelation. Since the number of rounds of trade is a model construct, the empirical implications of the model are affected by an artificial component constructed in the model. Finally, there are other channels that affect the revelation of PI. For example, private forecast data is available and is extensively used (otherwise the data would not be collected). Given the assumption of PI, much information could then be deduced from private forecasts. Hence, any implications of theories based on PI cannot depend only upon prices; they must also depend upon other channels for inference. Without credible and observable ways to measure these channels of revelation the theory lacks empirical implications. Also, there are other formulations of the private information model in real time (e.g. Judd and Bernardo (1996), (2000), Bacchetta and van Wincoop (2005a), Wang (1994)) but we do not review them here.

(iv) *If private signals are unobserved, how could common knowledge of the structure be attained?* To permit a deduction of PI from public data the *structure* of the private signals must be common knowledge. For example, they may take the form $x^i = Q + \varepsilon^i$ where ε^i are pure noise, independent across traders. But then one asks the simpler question: if these signals are not publically observed, how does the common knowledge come about? How does agent i know that his own signal takes the form $x^i = Q + \varepsilon^i$ and that x^i is an unbiased estimate of θ? How does trader i knows that the signal of k takes the form $x^k = Q + \varepsilon^k$? Are these not merely devices used by the investigator to enable a closed form solution of the Bayesian inference problem, rather than an empirically verifiable hypothesis?

(v) *Why are private signals more informative than audited public signals?* One peculiar assumption that drives the results of Morris and Shin (2002), Allen, Morris and Shin (2003), Bacchetta and van Wincoop (2005a) and others, is explained in the model of Section 1. It says that traders get a public signal y which is the mean value of the unknown Q. Knowing the prior mean of Q is clearly inferior to knowing the true Q. It is then assumed there is a continuum of agents on [0 , 1] with $x^i = Q + \varepsilon^i$ and with ε^i i.i.d. Hence, if you knew all private signals you would use the law of large numbers to aggregate them and learn the true Q. In an REE it is assumed there is some agent who aggregates the information and hence equilibrium price becomes a function of the true Q, which nobody knows. But this procedure raises two questions.

(a) Why do private signals contain more precise information than the professionally audited statements? Does it make sense to postulate that audited statements are less reliable than the sum of all the fragmentary signals that individuals obtain?

(b) Who is doing the aggregation? How does he know the i.i.d. structure needed to arrive at an aggregation? What are the incentives of this aggregating agent? If he is a neutral agent with a duty not to exploit the public, why does he not simply announce Q? Or else, he must be part of the model.

(vi) *With asymmetric information you can prove anything*. A typical model with PI is based on the fact that crucial components of the theory can never be observable. We shall never observe the private signals agents had about GNP growth or about future value of the S&P500. This lack of observability is contrasted with the case of insurance markets where driving records or health records can confirm the assumption that agents have PI which, ex ante, is not available to firms in the insurance market. But if there is no way to ever obtain data on the crucial component of the theory, the theory cannot be falsified: for any hypothesis about market behavior one can find a pattern of PI that would induce that behavior as an equilibrium behavior. The theory has no empirical restrictions and without restrictions it has no scientific content.

3. Modeling Asset Pricing Under HB with Public Information Only

We now turn to the alternative paradigm of HB instead of private information. What are the differences between these two theories and do these differences matter?

3.1 Adaptation of the Earlier Model

To adapt the model of Section 1 with PI to a market with HB and only public information, we clearly reject the common knowledge assumptions made. But then what is common knowledge among traders with diverse beliefs? Our *unequivocal* answer is *past data on observable variables*. Traders know they all observe the same data. They have diverse beliefs about the future because they have diverse interpretations of past data. Hence, a mechanical adaptation of the two- period economy in Section 1 is not suitable for an economy with HB. A meaningful model with HB must be anchored in real time with past data available at each date. To permit a comparison we thus adapt the earlier model by preserving its key assumptions. Apart from private information, the key assumptions are: (i) traders live finite life and derive utility from the terminal value of their net wealth; (ii) at date 1 agents cannot trade futures contracts for delivery of the stock at date 2; (iii) at date 1 traders must form beliefs about the price at date 2 and the true liquidation value $\hat{Q}$. This changed notation will be clarified later.

Our adaptation is then based on two principles. First, we maintain the above assumptions. Second, we require *that our model generates exactly the same demand functions as the PI model in (2a) -(2b)* so the comparison is reduced to differences between the implied probabilities used. Since under HB traders need price history to form beliefs, we assume trading is carried out by generations of traders, each of whom trades for two periods. In our setting a trader who starts trading at date t trades again at date t +1 and retires at the end of t+1, after $\hat{Q}_{t+1}$ is revealed and the value of his holdings is set. At retirement he exchanges his stock for consumption goods. Hence, at each t there are two types of overlapping traders: one group whose trading career is launched at t-1 and who retire at the end of trading at t, and a second group launched at date t, and who retires at t+1. Our economy consists of a continuum of traders of each type. As was the case in the PI model, we do not explicitly model the entire economy with consumption, investment, and production. The real economy is the background and the model is used to study the behavior of risk taking investors who use financial markets to trade risk. As in the PI model we assume their utility is defined only over gains from trading risk hence comparison of asset returns is a comparison of risk premia in an economy under PI vs risk premia under HB. With a real economy in the background we follow the PI literature and assume a constant riskless interest rate and without loss of generality let it be zero. The traded stock reflects an aggregate collection of assets kept in the background about which true audited information is revealed at the end of each date. These valuations are then used to compensate the retiring traders for risk taking.

$\hat{Q}_t$ is the value revealed at date t and the long history of $\hat{Q}_k$ for k = 1, 2, ..., t is known at date t hence traders use past data to compute the finite dimensional distributions of the observations. Clearly, all compute the same empirical moments. Using standard extension of measures they all deduce from the data a unique probability measure on infinite sequences denoted by m. It can be shown that m is stationary (see Kurz (1994)) and we call it “the stationary measure.” This is the empirical knowledge shared by all. To conform to the earlier model assume the data reveals the $\hat{Q}_t$ are conditionally normally distributed with mean μ and precision α[5]. Now define $Q_t = \hat{Q}_t - \mu$. A theory of belief diversity flows from the fact that traders do not know the true probability distribution of the Q_t's. That is, the stochastic process $\{ Q_t , t = 1,2,\ldots \}$ has an *unknown* probability Π. Traders know only the stationary probability m deduced from data. The distinction between m and Π is central to our development and is explored later when we describe the belief structure. Here we note traders' beliefs at date t are conditioned on common information H_t which consists of past values of Q_k for k = 1, 2, ..., t and prices. As in the PI model, trader i is launched at t (he is "date t" trader) with an endowment S_t^i of shares but the total supply is *a constant*, not random. Our notation is:

S_t^i - the endowment of shares with which trader i is launched at date t;

D_t^{i1} - date t demand of trader i who is launched at date t;

D_{t+1}^{i2} - date t+1 demand of trader i who is launched at date t;

S - total constant supply of shares.

Traders borrow or hold cash at the riskless rate hence they trade between the aggregate asset and cash. Under the utility function in (7) the assumption of an endowment S_t^i of shares is a convenient assumption with absolutely no effect on the results.[6] With endowment and borrowing a trader purchases his initial stock position D_t^{i1} at the cost of $D_t^{i1} p_t$. At t+1 he traders again into the position D_{t+1}^{i2}. At the end of date t+1 the audited valuation of the asset Q_{t+1} is revealed. Given Q_{t+1} the trader exchanges his stock position D_{t+1}^{i2} for real commodities and retires. The shares of retiring traders are then used for the initial endowment to the next generation of traders[7]. A trader has a preference over risky capital gains. His net terminal wealth is $W_{t+1}^i = S_t^i p_t + D_t^{i1}(p_{t+1} - p_t) + D_{t+1}^{i2}(Q_{t+1} + \mu - p_{t+1})$ and his date t+1 utility is

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[5] It would probably be more realistic to assume that the values Q_t grow and the growth rate of the values has a mean μ rather than the values themselves. This added realism is useful when we motivate the model later but is not essential for the analytic development.

[6] Without altering any of our results we could initiate trading with an endowment of a real commodity as in ordinary overlapping generation models. This is a consequence of the fact that under the utility function in (7) there are no income effects. Had we included such endowment, the definition of wealth would simply include it.

[7] Model consistency clearly requires the sum of shares surrendered by date t-1 retiring traders to equal the sum of shares allotted to new traders at date t. This assumption is inconsequential since young traders take the share allotment as exogenous and with free borrowing and without wealth effects the rule for initial shares allotment has no effect on optimal portfolios. An alternative procedure would be to treat the initial endowment as a loan in the form of shares borrowed. This would then lead to the requirement that the trader must return the loan and the amount $S_t^i(Q_{t+1} + \mu)$ would be subtracted from terminal wealth.

(7) $$u(W^i_{t+1}) = -e^{-(\frac{W^i_{t+1}}{\tau})} \quad , \quad W^i_{t+1} = S^i_t p_t + D^{i1}_t (p_{t+1} - p_t) + D^{i2}_{t+1}(Q_{t+1} + \mu - p_{t+1}).$$

(7) shows that on the demand D^{i1}_t trader i makes gains or losses of $D^{i1}_t(p_{t+1} - p_t)$ while gains on D^{i2}_{t+1} are $D^{i2}_{t+1}(Q_{t+1} + \mu - p_{t+1})$. The realized Q_t has informational value to a date t trader since it is a signal for Q_{t+1}. Apart from this, it has no impact on his wealth since Q_t is payment to retiring portfolios at t. In short, with a real economy in the background agents in our model redistribute risk in accord with their beliefs or information. This is exactly the spirit of the PI model.

Trader i who is launched at date t selects an optimal trading strategy which sequentially solves

(7a) $$J^{i2}_{t+1}(D^{i1}_t) = \underset{D^{i2}_{t+1}}{\text{Max}}\ E^i\left(-\exp[-\frac{1}{\tau}(S^i_t p_t + D^{i1}_t(p_{t+1} - p_t) + D^{i2}_{t+1}(Q_{t+1} + \mu - p_{t+1})] \mid H_{t+1}\right)$$

(7b) $$J^{i1}_t = \underset{D^{i1}_t}{\text{Max}}\, E^i\left(\underset{D^{i2}_{t+1}}{\text{Max}}\, E^i\left(-\exp[-\frac{1}{\tau}(S^i_t p_t + D^{i1}_t(p_{t+1} - p_t) + D^{i2}_{t+1}(Q_{t+1} + \mu - p_{t+1})] \mid H_{t+1}\right) \mid H_t\right).$$

(7a) solves for D^{i2}_t, given date 1 *demand function*, while (7b) solves for i's demand in date 1. The reasoning presented earlier for computing the demand functions applies here as well. They are

(8a) $$D^{i2}_{t+1}(p_{t+1}) = \frac{\tau}{\text{Var}^i(Q_{t+1} | H_{t+1})}[E^i(Q_{t+1} | H_{t+1}) + \mu - p_{t+1}].$$

(8b) $$D^{i1}_t(p_t) = \frac{\tau}{G_1}[E^i(p_{t+1} \mid H_t) - p_t] + \frac{(G_2 - G_1)}{G_1}[E^i(D^{i2}_{t+1} \mid H_t)].$$

Is our adaptation of the model reasonable? Since our trader lives for two periods (he is "long lived") we incorporate the hedging demand. But, as required, our demand functions are identically the same as in the model with PI: (8a)-(8b) and (2a)-(2b) *are exactly the same functions*. The crucial difference between the private information and the heterogenous belief models are the expectations of traders in (8a)-(8b) and (2a)-(2b) and the information they are assumed to have. We also observe that, although somewhat artificial, the assumption of a share endowment S^i_t to new traders removes all intergenerational effects of a trader's decision. Indeed, the equality of the demand functions together with the device of the share endowment attains model consistency and ensures that the infinite time horizon in our model has no independent effect. That is, the facts that the first model is of a *finite horizon* economy and the second is imbedded in *an infinite horizon* economy do not lead, on their own, to different implications of the two models.

The infinite repetition introduces the driving force of diverse beliefs which is the fact that Π, the true probability of the process $\{ Q_t , t = 1, 2, \ldots \}$, is unknown. The model is given an economic interpretation via a collection of real assets, kept in the background. These experience changes in innovation and organization so the time variability of the mean values of $\{Q_t, t = 1, 2, \ldots\}$ is driven by the forces of change. The terminal wealth of trader i, who is initiated at date t, depends upon Q_{t+1}. If he does not trade, his terminal net wealth is $W^i_{t+1} = S^i_t(Q_{t+1} + \mu)$. But then, what does the liquidation value reflect? It is clear this value is a compensation for taking risk associated with net profits of the background assets and results from the fact that *date t uncertainty is resolved only after date t trading*. Risk taking of this sort takes place in diverse sectors such as agriculture, mining, oil extraction, real estate and others. In these arrangement an investor buys an equity position which is tradable. The capital in the venture typically consists of the cumulative net output of the venture. In agriculture it may be the grain produced at the risky harvest, in oil extraction it may be oil discoveries, in mining it may be minerals discovered, in venture capital it is the realized valuation at the public offering. Thus, ownership shares allow risk sharing of the *prospects* involved and liquidation by the retiring members is permitted when the outcome of date t venture is known (i.e. size of crops, amount of oil found, outcome of a venture capital project, etc.). More generally, the market price reflects the valuation of the *risky prospect* while the liquidation value is the known benefit of the venture when it matures. When trading is resumed at date t+1 the venture continues into its next phase with new activity, new members and a new true value that will become known after trading. This, of course, is the assumption made in the PI model and since we want the demand functions of the two models to be identically the same, we must adopt this same concept as well.[8]

In the next section we model the structure of traders' beliefs, which is central to this paper. We have stressed that disagreements arise from diverse interpretation of the same empirical record. Thus, to conclude this section we make the simple assumption that the empirical frequencies of recorded past values is known by all to imply a first order Markov process described by a stationary transition

(9) $$Q_t = \lambda_Q Q_{t-1} + \rho^Q_t \quad , \quad \rho^Q_t \sim N(0, \sigma^2_Q).$$

[8] The model could be modified to the more famialr form where $\{Q_t, t = 1, 2, \ldots\}$ *are the usual risky dividends*. In that case date t trader buy assets D^{i1}_t at date t and D^{i2}_{t+1} at date t+1. He receives dividends $Q_{t+1} + \mu$ and $Q_{t+2} + \mu$ for investments made at dates t and t+1 respectively. Dividend payments are paid, as usual, at the *start* of a period and are known at the time of trading. As a result, date t trader retires at the start of date t +2 and when he liquidates his position by selling it into the market for the value of $W^i_{t+2} = D^{i1}_t(p_{t+1} + Q_{t+1} + \mu - p_t) + D^{i2}_{t+1}(p_{t+2} + Q_{t+2} + \mu - p_{t+1})$. Uncertainty about (Q_{t+1}, Q_{t+2}) is now the uncertainty about profits. Computing the implied demand functions we find that they are *slightly different* from (2a)-(2b). We elected to stay with the problem (7a)-(7b) and demand functions (8a)-(8b), which offer an entirely reasonable analytical platform with which to carry out the comparison we seek.

Since the implied stationary probability is denoted by m, we write $E^{m}[Q_t|Q_{t-1}] = \lambda_Q Q_{t-1}$.

Is the stationary model (9) with probability m the true data generating mechanism and hence is it the case that $m = \Pi$? If the environment was stationary and if all traders knew it was stationary, the Ergodic Theorem says that all would know the true data generating process. Indeed, in that case it would be common knowledge that (9) is the truth. In reality such conditions do not hold. The economy undergoes rapid changes with structural breaks associated with periods of high or low productivity. The process $\{Q_t, t = 1, 2, \ldots\}$ is then non-stationary under the true probability Π which is not known to anyone. A stationary Markov empirical record is simply an average over different regimes. In particular, the first order Markov property is a result of diverse dynamic patterns, averaged out statistically over time. The simple analogy we can give for the empirical frequencies of past values is like running a single regression over a long data set with many unobserved regimes. Such a procedure estimates the average over different structures. But, for long data sets, this is all that they could ever agree on. The fact is that traders do not believe the empirical distribution of the past is adequate to forecast the future. All surveys of forecasters show that subjective judgment contributes more than 50% to the final forecast (e.g. Batchelor and Dua (1991)). In this environment each trader forms his own beliefs about Q_t and other state variables to be explored in the next section. With such complexity how do we describe an equilibrium? For such a description do we really need to give a full, detailed, development of all the diverse theories of the traders?

3.2 Heterogeneity of belief: The Question is How!

Diverse beliefs is the result of the fact that agents do not know the exact structure of a complex economy. Since one cannot be declared irrational if one cannot hold Rational Expectations, the concept of rationality must be modified. The theory of Rational Beliefs (in short, RB due to Kurz (1994), (1997a)) defines a trader to be *rational* if his model cannot be falsified by the data and if simulated, it reproduces the empirical distribution. Under this theory rational traders may hold diverse forecasting models based on different interpretations of the data. More generally, without a compelling known "true" model, any meaningful concept of rationality of belief will embrace a wide collection of models. Such a conclusion raises a clear methodological question. In formulating an asset pricing theory should we provide a detailed description and motivate the subjective models of each trader in the model? With diversity of traders such a task is formidable. But if the objective is an understanding of the dynamics of asset prices, is such a detailed description necessary? An examination of the subject reveals that, although an intriguing question, such a detailed task is not needed. Instead, to describe an equilibrium all that we need is to specify how the beliefs of the traders affect their subjectively perceived transition functions of all the state variables. Once these are specified, the Euler equations are fully specified and market clearing leads to equilibrium pricing. To carry out such a program we follow the structure developed in Kurz, Jin and Motolese (2005a), (2005b). We now outline this development for traders in our simple asset price model.

3.3 Market Belief as a State Variable: Diverse Opinions vs. Asymmetric Information

In markets without private information agents are willing to reveal their forecasts. Hence, in formulating our theory we now assume that market forecast data are public. The crucial difference between markets with and without private information is that when individual forecasts of a state variable are revealed in a market *without* private information, others do not see such forecasts as a source of new data and do not update their own beliefs about a parameter used to forecast that state variable. In such a market, a forecaster uses knowledge about the forecasts of others to alter his forecasts of endogenous variables since these depend upon the market belief. In short, the difference between an equilibrium with PI and an equilibrium without PI but with HB is that in the latter *agents do not learn from others and do not update their beliefs about state variables based on the opinions of others.* But then, how do we describe the individual and market beliefs?

The key analytical step we have taken (see Kurz (1994), Kurz (1997a), Kurz and Motolese (2001), Kurz, Jin and Motolese (2005a),(2005b)) is to treat individual beliefs as personal state variables, generated within the economy. That is, an individual belief about an economy's state variable are described with a personal state of belief which uniquely pins down the conditional probability or transition function of next period's economy's state variable. Hence, personal states of belief are analogous to other state variables in the decision problem of the agent, although iy can also be interpreted as defining the more familiar concept of a "type" of the trader. At date t the trader is not certain of his future belief type but his behavior (e.g. Bayesian updating) or procedural model and interpretation of current information determines the dynamics of the personal state of belief. The *distribution* of individual states of belief then becomes a central economy-wide dynamical force where the cross sectional average state of belief is simply the average of individual beliefs. As we indicated, the crucial fact is that the distribution of beliefs in the market is observable. In equilibrium, endogenous variables (e.g. prices) depend upon the economy's state variables, but in a large economy a trader's *"anonymity"* implies a personal state of belief has a negligible effect on prices. It turns out that with the utility function we use equilibrium endogenous variables depend only upon the *distribution* of market beliefs. Thus, as in any equilibrium, prices and other endogenous variables are functions of the economy's state variables and here these state variables include the distribution of personal beliefs. In our equilibrium the moments of the cross sectional distributions of belief are important economy state variables and their stochastic transition laws play a central role. Finally, since endogenous variables are functions of the market beliefs, it follows that future endogenous variables are forecasted by forecasting the market distribution of beliefs using the known equilibrium map. In short, to forecast future endogenous variables a trader must forecast the beliefs of others.

We thus introduce trader i's *state of belief* g_t^i. It describes his perception by pinning down his transition functions. Adding to "anonymity" we assume trader i knows his own g_t^i and the market *distribution* of g_t^k across k. As to past, he observes past distributions of the g_τ^k for all $\tau < t$ hence he knows past values of the moments of the distributions of the g_τ^k. We specify the dynamics of g_t^i by

(10) $$g_t^i = \lambda_Z g_{t-1}^i + \rho_t^{ig} \quad , \quad \rho_t^{ig} \sim N(0, \sigma_g^2)$$

where ρ_t^{ig} are correlated across i reflecting correlation of beliefs across individuals. The concept of an individual state of belief, with dynamics (10), is central to our development. Here we state (10) as a positive description of type heterogeneity but in Section 3.5 we prove (10) as a consequence of a Bayesian updating procedure. We postpone this demonstration in order to explain first the asset pricing theory implied by our model of HB. We note that in general g_t^i is used to express a trader's assessment of the *difference* between date t distribution of an observable state variable and the empirical distribution m. In the model of this paper the perception of trader i regarding Q_t at date t (denoted by Q_t^i) is described by using the belief state g_t^i as follows

(11a) $$Q_t^i = \lambda_Q Q_{t-1} + \lambda_Q^g g_t^i + \rho_t^{iQ} \quad , \quad \rho_t^{iQ} \sim N(0, \hat{\sigma}_Q^2).$$

The assumption that $\hat{\sigma}_Q^2$ is the same for all traders is made for simplicity. It follows that the state of belief g_t^i measures the deviation of his forecast from the empirical stationary forecast

(11b) $$E^i[Q_t^i | H_t, g_t^i] - E^m[Q_t | H_t] = \lambda_Q^g g_t^i.$$

Indeed, (11b) shows how to measure g_t^i in practice. For any state variable X_t, data on i's forecasts of X_t (in (11b) it is Q_t) are measured by $E^i[X_t^i | H_t, g_t^i]$. One then uses standard econometric techniques to construct the stationary forecast $E^m[X_t | H_t]$ with which one empirically constructs the difference in (11b). This construction and the data it makes available are at the core of the papers by Fan (2005) and Kurz and Motolese (2006). A trader type who believes the empirical distribution is the truth, is described by $g_t^i = 0$, hence he believes that $Q_t \sim N(\lambda_Q Q_{t-1}, \sigma_Q^2)$. Since belief heterogeneity is the result of dynamic non-stationarity of the economy, it should be clear that around 1900 the subjective assessments of the g_t^i were related to the development of electricity and the combustion engine, while around 2000 the belief g_t^i measured the impact of computers and information technology. Hence, success or failures of past g_τ^i do not really tell you anything what present day g_t^i should be. This issue is further explored in Section 3.5 and for additional details see Kurz (1997a).

Denote by Z_t the first moment of the cross sectional distribution of the g_t^i and we refer to it as "the average state of belief." It is observable. Due to correlation across traders, the law of large numbers is not operative and the average of ρ_t^{ig} over i does not vanish. We write it in the form

$$Z_{t+1} = \lambda_Z Z_t + \rho_{t+1}^Z . \tag{12}$$

The true distribution of ρ_{t+1}^Z is unknown. Correlation across agents exhibits non stationarity and this property is inherited by the { Z_t , t = 1, 2, ...} process. Since Z_t are observable, market participants actually have data on the joint process $\{(Q_t, Z_{t+1}), t = 1, 2, \ldots\}$. Traders are thus assumed to know the *joint empirical distribution* of these variables. For simplicity we assume that this distribution is described by the system of equations

$$\begin{aligned} &(13a) \qquad Q_t = \lambda_Q Q_{t-1} + \rho_t^Q \\ &(13b) \qquad Z_{t+1} = \lambda_Z Z_t + \rho_{t+1}^Z \end{aligned} \qquad \begin{pmatrix} \rho_t^Q \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_Q^2, & 0, \\ 0, & \sigma_Z^2 \end{bmatrix} = \Sigma \right), \quad \text{i.i.d.}$$

Now, a trader who does not believe that (13a)-(13b) is the truth for t, formulates his own model\belief. We have seen in (11a) how trader i's belief state g_t^i pins down his forecast of Q_t^i. We now broaden this idea to the trader's perception model of the two state variables (Q_t^i, Z_{t+1}^i). Keeping in mind that before observing Q_t trader i knows Q_{t-1} and Z_t, his belief takes the general symmetric form[9]

$$\begin{aligned} &(14a) \qquad Q_t^i = \lambda_Q Q_{t-1} + \lambda_Q^g g_t^i + \rho_t^{iQ} \\ &(14b) \qquad Z_{t+1}^i = \lambda_Z Z_t + \lambda_Z^g g_t^i + \rho_{t+1}^{iZ} \end{aligned} \qquad \begin{pmatrix} \rho_{t+1}^{iQ} \\ \rho_{t+1}^{iZ} \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \hat\sigma_Q^2, & \hat\sigma_{ZQ}, \\ \hat\sigma_{ZQ}, & \hat\sigma_Z^2 \end{bmatrix} = \Sigma^i \right),$$

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[9] Keep in mind the unnatural timing in the model. At date t trader i has a state of belief g_t^i about variables he does not know. These are: (i) Q_t to be announced at the end of date t, and (ii) Z_{t+1} to be revealed at the start of t+1. This peculiar timing is a consequence of the timing in the private information model presented in the Introduction according to which Q_t is revealed at the end of date t. Also, for simplicity we assume (14a)-(14b) is the same across traders: diversity is in the g_t^i.

Although the belief state g_t^i was initially defined to be about the unknown value Q_t, (14a)-(14b) show that we use it also to pin down the transition of Z_{t+1}^i. We could have, instead, introduced a new variable g_t^{iZ} to express belief about future Z. We avoid this procedure for simplicity and in order to avoid an artificial problem of infinite regress. Hence, g_t^i expresses how the agent considers the present conditions to be different from the empirical distribution:

(14c)
$$E_t^i\begin{pmatrix} Q_t \\ Z_{t+1} \end{pmatrix} - E_t^m\begin{pmatrix} Q_t \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_Q^g g_t^i \\ \lambda_Z^g g_t^i \end{pmatrix}$$

The average market expectation operator is loosely defined by $\bar{E}_t(\bullet) = \int E_t^i(\bullet)\,di$. From (14c) it is

(14d)
$$\bar{E}_t\begin{pmatrix} Q_t \\ Z_{t+1} \end{pmatrix} - E_t^m\begin{pmatrix} Q_t \\ Z_{t+1} \end{pmatrix} = \begin{pmatrix} \lambda_Q^g Z_t \\ \lambda_Z^g Z_t \end{pmatrix}.$$

The perception models (14a)-(14b) explains why the average individual market belief is not a proper probability. To see this let $X = Q \times Z$ be a product space where (Q_{t-1}, Z_t) take their values and let G^i be the space of the g_t^i. Since i conditions on his own g_t^i, his unconditional probability is a measure on the space $((Q \times Z \times G^i)^\infty, \mathscr{F}^i)$ where $\mathscr{F}^i$ is i's sigma field. Hence, the average market conditional belief is an average of conditional probabilities, each conditioning on a *different* state variable. Hence, one cannot write down a probability space for the market belief and we have the following result:

Theorem 1: The average individual belief violates iterated expectations: $\bar{E}_t(Q_{t+1}) \neq \bar{E}_t\bar{E}_{t+1}(Q_{t+1})$.

Proof: From (14a)-(14b) we know that

$$E_t^i(Q_{t+1}) = \lambda_Q E_t^i(Q_t) + \lambda_Q^g E_t^i(g_{t+1}^i) = \lambda_Q[\lambda_Q Q_{t-1} + \lambda_Q^g g_t^i] + \lambda_Q^g \lambda_Z g_t^i.$$

It follows that

(15a)
$$\bar{E}_t(Q_{t+1}) = \lambda_Q^2 Q_{t-1} + \lambda_Q^g(\lambda_Q + \lambda_Z)Z_t.$$

On the other hand (14a) implies

$$\bar{E}_{t+1}(Q_{t+1}) = \lambda_Q Q_t + \lambda_Q^g Z_{t+1}$$

hence

$$E_t^i\bar{E}_{t+1}(Q_{t+1}) = \lambda_Q[\lambda_Q Q_{t-1} + \lambda_Q^g g_t^i] + \lambda_Q^g[\lambda_Z Z_t + \lambda_Z^g g_t^i]$$

and aggregating now to conclude that

(15b)
$$\bar{E}_t\bar{E}_{t+1}(Q_{t+1}) = \lambda_Q^2 Q_{t-1} + \lambda_Q^g(\lambda_Q + \lambda_Z + \lambda_Z^g)Z_t.$$

Comparison of (15a) and (15b) shows that $\bar{E}_t(Q_{t+1}) \neq \bar{E}_t\bar{E}_{t+}(Q_{t+1})$. ■

Belief and Information: Understanding what is Z_t. From the perspective of a trader, Z_t is a state variable like any other. News about Z_t are used to forecast prices and assess the time variability of market risk premia in the same way macroeconomic data such as GNP growth or Non Farm Payroll are used to assess the risk of a recession. Market belief may be wrong as it may forecast recessions that never occur. Market risk premia may fall just because traders are more optimistic about the future, not necessarily because there is any specific data which convinces everybody the future is bright. But then, how do traders update their beliefs when they observe Z_t? In sharp contrast with the PI theory, traders do not revise their own beliefs about the state variable Q_t; (14) specifically *does not* depend upon Z_t. Traders do consider Z_t as new information about Q_t since they know all used all available information. Without being a "signal" about unobserved private information, Z_t is not used to update beliefs about *exogenous* variables. The importance of Z_t is it's great value in forecasting future *endogenous* variables. Date t endogenous variables depend upon Z_t and future endogenous variables depend upon future market belief. Since market belief exhibits persistence, traders know that today's market belief is useful for forecasting future endogenous variables. How is this equilibrated? This is what we show now.

3.4 Combining the Elements: the Implied Asset Pricing Theory Under Diverse Beliefs

We now derive equilibrium prices under HB. Denote the conditional variance of Q_t (common to all traders) by σ_Q^2. By (8a)-(8b) we write the date t demand functions of the two type of traders as

(16a) $$D_t^{i2}(p_t) = \frac{\tau}{\sigma_Q^2}[E^i(Q_t + \mu \mid H_t) - p_t].$$

(16b) $$D_t^{i1}(p_t) = \frac{\tau}{G_1}[E^i(p_{t+1} \mid H_t) - p_t] + \frac{(G_2 - G_1)}{G_1}[E^i(D_{t+1}^{i2} \mid H_t)].$$

For an equilibrium to exist we need some stability conditions. To specify these we introduce the notation $\delta = 1 - G_2/(G_1 + \sigma_Q^2)$. Now we add:

<u>**Stability Conditions**</u>: We require that $0 < \lambda_Q < 1 \quad , \quad 0 < \lambda_Z + \lambda_Z^g < 1 \quad , \quad 0 < |\delta| < 1$.

The first requires $\{Q_t$, t = 1, 2, ...$\}$ to be stable and have an empirical distribution. The second is *a* stability of *belief* condition. It requires i to believe (Q_{t-1}, Z_t) is stable. To see why take expectations of (14b), average over the population and recall Z_t are market averages of the g_t^i. This implies that

$$\bar{E}_t[Z_{t+1}] = (\lambda_Z + \lambda_Z^g)Z_t.$$

Theorem 2: For the model with HB and under the specified stability conditions, there is a unique equilibrium price function which takes the form

$$p_t = a(Q_{t-1} + \mu) + bZ_t - cS.$$

Proof: Aggregating (16a) over all retiring traders and (16b) over all new traders at date t leads to

(17a) $$\bar{D}_t^2(p_t) = \frac{\tau}{\sigma_Q^2}[\bar{E}_t(Q_t + \mu) - p_t].$$

(17b) $$\bar{D}_t^1(p_t) = \frac{\tau}{G_1}[\bar{E}_t(p_{t+1} + Q_t + \mu) - p_t] + \frac{(G_2 - G_1)}{G_1}\frac{\tau}{\sigma_Q^2}[\bar{E}_t(Q_{t+1} + \mu) - \bar{E}_t p_{t+1}]].$$

Since $\bar{D}_t^2(p_t) + \bar{D}_t^1(p_t) = S$ we add (17a)-(17b) to conclude that

$$S = \frac{\tau}{\sigma_Q^2}[\bar{E}_t(Q_t + \mu) - p_t] + \frac{\tau}{G_1}[\bar{E}_t(p_{t+1} + Q_t + \mu) - p_t] + \frac{(G_2 - G_1)}{G_1}\frac{\tau}{\sigma_Q^2}[\bar{E}_t(Q_{t+1} + \mu) - \bar{E}_t[p_{t+1}]].$$

Hence

$$S = (\frac{\tau}{\sigma_Q^2} + \frac{\tau}{G_1})[\bar{E}_t(Q_t + \mu) - p_t] + (\frac{\tau}{G_1})[1 - \frac{(G_2 - G_1)}{\sigma_Q^2}]\bar{E}_t(p_{t+1}) + \frac{(G_2 - G_1)}{G_1}\frac{\tau}{\sigma_Q^2}[\bar{E}_t(Q_{t+1} + \mu)]].$$

Now use the perception models (14a)-(14b) about the state variables, average them over the population and use the definition of Z_t to deduce the following relationships which are the *key implications of treating individual and market beliefs as state variables*

(18a) $$\bar{E}_t(Q_t) = \lambda_Q Q_{t-1} + \lambda_Q^g Z_t$$

(18b) $$\bar{E}_t(Q_{t+1}) = (\lambda_Q)^2 Q_{t-1} + [\lambda_Q \lambda_Q^g + \lambda_Q^g \lambda_Z]Z_t$$

(18c) $$\bar{E}_t(Z_{t+1}) = (\lambda_Z + \lambda_Z^g)Z_t.$$

Now solve for date t price to deduce

(19) $$p_t = \bar{E}_t(Q_t + \mu) + \frac{(G_2 - G_1)}{G_1 + \sigma_Q^2}\bar{E}_t(Q_{t+1} + \mu) + \frac{\sigma_Q^2 + G_1 - G_2}{G_1 + \sigma_Q^2}\bar{E}_t(p_{t+1}) - \frac{\sigma_Q^2 G_1}{G_1 + \sigma_Q^2}[\frac{S}{\tau}].$$

Observe that (18a)-(18c) together with (19) imply that equilibrium price is the solution of the following difference equation

(20) $$p_t = A(Q_{t-1} + \mu) + BZ_t + \delta\bar{E}_t[p_{t+1}] - CS \quad , \quad \delta = \frac{\sigma_Q^2 + G_1 - G_2}{G_1 + \sigma_Q^2}$$

with

$$A = \lambda_Q + \frac{(G_2 - G_1)}{G_1 + \sigma_Q^2}\lambda_Q^2 \quad , \quad B = \lambda_Q^g + \frac{(G_2 - G_1)}{G_1 + \sigma_Q^2}[\lambda_Q\lambda_Q^g + \lambda_Q^g(\lambda_Z + \lambda_Z^g)] \quad , \quad C = (\frac{1}{\tau})\frac{\sigma_Q^2 G_1}{G_1 + \sigma_Q^2} .$$

(20) is a linear difference equation in the two state variables (Q_{t-1}, Z_t). Hence, a standard argument (see Blanchard and Kahn(1980), *Proposition*, page 1308) shows that the solution is

(21a) $$p_t = C_1(Q_{t-1} + \mu) + C_2 Z_t - C_3 S$$

with matching coefficients of

(21b) $$C_1 = [\frac{1}{1 - \delta\lambda_Q}]\frac{(G_2 - G_1)}{G_1 + \sigma_Q^2}\lambda_Q^2$$

(21c) $$C_2 = \frac{1}{1 - \delta(\lambda_Z + \lambda_Z^g)}\left\{\lambda_g^Q + \frac{(G_2 - G_1)}{G_1 + \sigma_Q^2}(\lambda_Q\lambda_Q^g + \lambda_Q^g(\lambda_Z + \lambda_Z^g)) + (\frac{\delta\lambda_Q^g}{1 - \delta\lambda_Q})\frac{(G_2 - G_1)}{G_1 + \sigma_Q^2}\lambda_Q^2\right\}$$

(21d) $$C_3 = (\frac{1}{1 - \delta})\frac{\sigma_Q^2 G_1}{\tau(G_1 + \sigma_Q^2)} .$$

The stability conditions ensure that (21a) - (21d) is the unique solution as asserted. ■

Finally, recall that the demand functions (8a)-(8b) were computed under the conjecture that prices are conditionally normally distributed. Theorem 2 provides the final confirmation of this conjectures.

3.5 Deducing the Markov Belief Process $g_t^i = \lambda_Z g_{t-1}^i + \rho_t^{ig}$ from Bayesian Inference

Our key analytical tool is the state of belief and we now justify the dynamics (10). Keeping in mind that we study asset pricing in a changing environment, our first justification is simplicity and analytic tractability as seen in the developments in Sections 3.1 - 3.4. In a changing environment there is no universal procedures to learn an unknown sequence of parameters. It is thus less important to explain why agents disagree and more important to be able to describe their diversity so that an equilibrium analysis is *tractable*. The description (10) of a state of belief in the form $g_{t+1}^i = \lambda_Z g_t^i + \rho_{t+1}^{ig}$ where $\rho_{t+1}^{ig} \sim N(0, \sigma_g^2)$ leads to a simple and useful description of equilibrium asset pricing with diverse beliefs. It also shows that in contrast with PI theories, it does not entail extraction of information from market prices. Instead, it requires different agents to have different state spaces for description of their uncertainty. It also requires the endogenous expansion of the economy's state space for a description of equilibrium pricing. We now explore the conditions under which the Markov dynamics (10) is proved as a *consequence of elementary principles of Bayesian inference*.

Before proceeding we formulate the problem in a setting which is more general than the one where an intrinsic value Q_t is revealed. Instead, we assume the observable variable is the dividend D_t paid by an asset before trading takes place. This will result in a more natural timing than in the case of the model developed earlier. The economy unfolds over an infinite horizon and agents have a long history of data, allowing statistical analysis which leads all to compute the same empirical moments and the same finite dimensional distributions of the observations. Hence they all deduce from the data a unique empirical probability on infinite sequences denoted by $\hat{m}$. It can be shown that $\hat{m}$ is stationary. We assume the data reveals that under $\hat{m}$ the $\{D_t, t = 1,2,...\}$ constitutes a Markov process where D_{t+1} is conditionally normally distributed with means $\mu + \lambda_d(D_t - \mu)$ and variance σ_d^2. Now we define $d_t = D_t - \mu$, hence $\{ d_t, t = 1, 2,... \}$ is zero mean with unknown true probability Π and an empirical probability m. We shall assume that under the true Π the $\{ d_t, t = 1, 2,... \}$ is a non-stationary process but under the assumptions made above the empirical probability of $\{ d_t, t = 1, 2,... \}$ is deduced from the data and is characterized by a first order Markov process with a stationary transition

(22) $$d_{t+1} = \lambda_d d_t + \rho^d_{t+1} \qquad , \qquad \rho^d_{t+1} \sim N(0, \sigma_d^2)$$

and we write $E^m[d_{t+1} | d_t] = \lambda_d d_t$. The probability m is merely an empirical average over an infinite sequence of regimes.We now turn to a Bayesian inference about the true probability Π.

In a standard Bayesian environment an agent faces data generated under a stationary structure but with an unknown and fixed parameter. The agent starts with a prior on the parameter and then uses Bayesian inference for retrospective updating of his belief. The term "retrospective" stresses that inference is made *after* data is observed. In real time the agent must use the prior to forecast all variables while learning can only improve *future* forecasts of these variables. Our model has some parameters fixed and others that change over time. The fixed parameters are known as they are deduced from the empirical frequencies. The time varying parameters, reflecting the non stationarity of the economy, are modeled by the fact that under the true probability Π the value d_t has a transition function of the form

$$d_{t+1} - \lambda_d d_t = b_t + \rho_{t+1}.$$

The sequence of parameters b_t is an exogenous, time varying mean value function. Agents know λ_d

but not the "regimes" b_t. This formulation includes economies with slow changing regimes, each lasting a long time or fast changing regimes. The mere fact that they change limits the validity of Bayesian updating. To see why observe that at date t our agent has a prior belief about b_t with which he forecasts d_{t+1}. After observing d_{t+1} he updates his prior to have a sharper posterior estimate of b_t. But when date t+1 arrives he needs to forecast d_{t+2} and for that *he needs a prior on b_{t+1}*. Agents do not know if and when a parameter changes. If the b_t change slowly, a sharp posterior estimate of b_t (given d_{t+1}) may serve also as a prior belief about b_{t+1}. Indeed, if the agent knew that $b_t = b_{t+1}$ the *updated posterior of b_t is the best prior of b_{t+1}.* In the absence of such knowledge, agents would believe that $b_t = b_{t+1}$ is only one possibility. They would, then, seek additional information and use subjective interpretation of other public data to arrive at alternative subjective estimates of b_{t+1} to supplement the Bayesian posterior they have. Such subjective interpretation of public data arises naturally from the fact that public quantitative data is always provided together with a vast amount of *qualitative* information which is an important source of subjective interpretation of data.

3.5.1 Qualitative Information and Subjective Interpretation of Public Information

Bayesian inference is only possible with *quantitative* measures. The fact is that quantitative data like d_t are always accompanied with much *qualitative* public information about usual or unusual conditions. For example, data on inflation are interpreted with reports on normal or abnormal productivity features, conditions of the labor markets, assessment of the price of energy, political environment, etc. If d_t are profits of a firm then d_t is just one number extracted from a detailed financial report of the firm, the industry, the technology or the products involved. If d_t are profits of the S&P500 then qualitative information includes general business conditions, monetary policy, political environment, prospective tax reform, trends in productivity and other macroeconomic conditions. Qualitative information cannot, in general, be compared over time and does not constitute conventional "data." For example, when a firm announces a new research into something that did not exist before, no past data is available for comparison. When a new product alters the nature of an industry, it is a unique event. Financial markets pay a great deal of attention to qualitative announcements which are often the focus of diverse opinions of investors.

There is little modeling of deduction from qualitative information. Saari (2006) uses qualitative information in a competitive model of market shares. Toukan (2006) is a second example.

Here we provide a simple formalization of the use of qualitative information. Thus, qualitative information consist of *statements about the future*. Let date t statements be $A_t = (A_{t1}, A_{t2}, ..., A_{tK_t})$, each with quantitative measures in some units. The list may change with t and K_t varies with time. The activity in a statement may turn out to impact $(d_{t+1} - \lambda_d d_t)$ or not. The effects may be desirable or not. A realization at t+1 is a vector $\varphi_{t+1} = (\varphi_{t+1,1}, \varphi_{t+1,2}, ..., \varphi_{t+1,K_t})$ of numbers which are 0 or 1. A 0 means the activity turns out to have no effect and 1 means it has an effect. These can be interpreted as "success" or "failure." There are 2^{K_t} possible vectors of outcomes $\varphi_{t+1}(k)$, $k = 1, 2, ..., 2^{K_t}$. Next, agent i has a subjective map from φ_{t+1} to an expected value $\Phi^i(\varphi_{t+1})$ of $(d_{t+1} - \lambda_d d_t)$. This is an independent estimate by agent i on how different he expects $(d_{t+1} - \lambda_d d_t)$ to be from the stationary forecast conditional upon the success or failure of the statements. But, keep in mind, the quantitative estimate $\Phi^i(\varphi_{t+1})$ depends upon the statement A_t. For example, a research plan with \$1 million budget would be expected to have a smaller impact than a plan with a \$1 billion budget. Finally, conditional on A_t, agent i attaches probabilities $(a_1^i, a_2^i, \ldots, a_{2^{K_t}}^i)$ to the vectors $\varphi_{t+1}(k)$. The result of this is that agent i has an alternate subjective estimate of $(d_{t+1} - \lambda_d d_t)$ based only on public data A_t:

$$\Psi_t^i(A_t) = \sum_{k=1}^{2^{K_t}} a_k^i \Phi^i(\varphi_{t+1}^i(k)).$$

By (22) the long term average of $(d_{t+1} - \lambda_d d_t)$ is zero. Hence, rationality requires that the Ψ_t^i are zero mean random variables. Although public *data* consist only of d_t, the procedure outlined shows that in a complex world agents endogenously create subjective quantitative measures which reflect their beliefs. We incorporate such a measure in the Bayesian procedure below.

3.5.2 A Bayesian Inference: Beliefs are Markov State Variables

Start by assuming that d_{t+1} has a true transition of the form

$$d_{t+1} - \lambda_d d_t = b_t + \rho_{t+1}^d \quad , \quad \rho_{t+1}^d \sim N(0, \frac{1}{\beta}).$$

Agents do not know b_t but β is known. At first decision date t (say, t = 1) an agent has two pieces of information. He knows d_t and observes qualitative information $(A_{(t)1}, A_{(t)2}, ..., A_{(t)K_t})$ with which to assess Ψ_t^i. Assume that without Ψ_t^i the prior subjective mean at t = 1 is b but to start the process he uses *both sources* to form, as yet unspecified, a prior belief $E_t^i(b_t | d_t, \Psi_t^i)$ about b_t (used to forecast d_{t+1}). The changing parameter b_t leads to a problem. When $d_{t+1} - \lambda_d d_t$ is observed, agent i updates his belief about the same parameter b_t to $E^i_{t+1}(b_t|d_{t+1},\Psi^i_t)$ [10] in a standard Bayesian inference but before knowing the assessment Ψ^i_{t+1}.The point is that the agent needs an *estimate of* b_{t+1}, not of b_t. Hence, his problem is how to go from $E^i_{t+1}(b_t|d_{t+1},\Psi^i_t)$ to a prior of b_{t+1}? Without new information his belief about b_{t+1} is unchanged and he would use $E^i_{t+1}(b_t|d_{t+1},\Psi^i_t)$ as a prior of b_{t+1}. This is surely true if the b's change very slowly or when $b_{t+1}=b_t$. Hence, if agent i believes $b_{t+1}\neq b_t$ a new prior is needed. To that end he uses the qualitative information $(A_{(t+1)1}, A_{(t+1)2},\ldots,A_{(t+1)K_{t+1}})$ released publicly before trading at t+1. These lead to an alternate subjective estimate Ψ^i_{t+1} of b_{t+1}. Now the agent has two independent sources for belief about b_{t+1}: the last posterior $E^i_{t+1}(b_t|d_{t+1},\Psi^i_t)$ is to be used if $b_{t+1}=b_t$ and Ψ^i_{t+1} if $b_{t+1}\neq b_t$. With a Bayesian approach we assume:

Assumption (*): Agent i uses a subjective probability μ to form date t+1 prior belief which is then

(23a) $$E^i_{t+1}(b_{t+1}|d_{t+1},\Psi^i_{t+1}) = \mu E^i_{t+1}(b_t|d_{t+1},\Psi^i_t) + (1-\mu)\Psi^i_{t+1} \qquad 0\le\mu<1.$$

At t=1 it was assumed an initial posterior b, hence for consistency, if Ψ^i_1 is Normal then

(23b) $$b_1 \sim N(\mu b + (1-\mu)\Psi^i_1, \frac{1}{\alpha}) \quad \text{for some } \alpha.$$

This assumption is the new element that permits the posterior $E^i_{t+1}(b_t|d_{t+1},\Psi^i_t)$ of b_t to be upgraded to a date t+1 prior belief $E^i_{t+1}(b_{t+1}|d_{t+1}, \Psi^i_{t+1})$ about b_{t+1}, before d_{t+2} is observed. We then have:

Theorem 3 : Suppose $\Psi^i_t \sim N(0, \frac{1}{\gamma})$, i.i.d. and Assumption (*) holds. Then for large values of t, the prior belief $E^i_t(b_t|d_t,\Psi^i_t)$ is a Markov state variable such that if we define $g^i_t = E^i_t(b_t|d_t,\Psi^i_t)$ and $\mu\kappa = \lambda_Z$ for some $0<\kappa<1$ then the dynamics (10) holds: (23a) implies (10).

Proof: Pick a starting date t = 1. Data d_t is known and the agent generates a subjective measure of Ψ^i_t. He then forms a prior on b_t, which by Assumption (*) is $b_t \sim N(\mu b + (1-\mu)\Psi^i_t, \frac{1}{\alpha})$. Now we move on to t+1 and d_{t+1} is observed. The agent updates the prior in a standard Bayesian manner:

$$E^i_{t+1}(b_t|d_{t+1},\Psi^i_t) = \frac{\alpha(\mu b + (1-\mu)\Psi^i_t) + \beta[d_{t+1} - \lambda_d d_t]}{\alpha + \beta}, \qquad 0\le\mu\le 1.$$

[10] **Note our notation**. We use the notation of $E^i_t(b_t|d_t,\Psi^i_t)$ for date t *prior* belief about the parameter b_t used to forecast d_{t+1}. We then use the notation $E^i_{t+1}(b_t|d_{t+1},\Psi^i_t)$ for the posterior belief *about the same* b_t given the observation of d_{t+1} but without forming the estimate of Ψ^i_t. Assumption (A) will use this posterior belief as a building block in the formation of the new prior $E^i_{t+1}(b_{t+1}|d_{t+1}, \Psi^i_{t+1})$ about the new parameter b_{t+1}.

But before date t+1 trading he generates the subjective measure Ψ^i_{t+1} of qualitative data. By Assumption (*) the expected parameter b_{t+1} under the new prior at t+1 is

$$E_t^{\,i}(b_{t+1}|d_{t+1},\Psi^i_{t+1}) \;=\; \mu\, E_t^{\,i}(b_t|d_{t+1},\Psi^i_t) + (1-\mu)\Psi^i_{t+1} \quad , \quad 0 \le \mu \le 1 .$$

Denote by $\zeta = \dfrac{1}{\mu^2}$ and $\xi = \dfrac{1}{(1-\mu)^2}$. Then the prior is

$$b_{t+1} \sim N\Big(E_t^{\,i}(b_{t+1}|d_{t+1},\Psi^i_{t+1})\,,\, \frac{1}{\zeta(\alpha+\beta)+\xi\gamma}\Big).$$

It is used to forecast $d_{t+2}-\lambda_d d_{t+1}$. Moving on to t+2, the agent observes $d_{t+2}-\lambda_d d_{t+1}$ and based on this observation he uses Bayesian inference to deduce a posterior belief about b_{t+1}

$$E^{\,i}_{t+1}(b_{t+1}|d_{t+2},\Psi^i_{t+1}) = \frac{(\zeta(\alpha+\beta)+\xi\gamma)[\mu E_t^{\,i}(b_t|d_{t+1},\Psi^i_t)+(1-\mu)\Psi^i_{t+1}] + \beta[d_{t+2}-\lambda_d d_{t+1}]}{\zeta(\alpha+\beta)+(\xi\gamma+\beta)}.$$

Before the start of date t+2 trading, the agent generates a new value Ψ^i_{t+2} leading to t+2 prior belief about the unobserved parameter b_{t+2}

$$E^{\,i}_{t+2}(b_{t+2}|d_{t+2},\Psi^i_{t+2}) \;=\; \mu\, E^{\,i}_{t+1}(b_{t+1}|d_{t+2},\Psi^i_{t+1}) + (1-\mu)\Psi^i_{t+2} \quad , \quad 0 \le \mu < 1$$

When $d_{t+3}-\lambda_d d_{t+2}$ is observed the posterior belief about b_{t+2} is then

$$E^{\,i}_{t+2}(b_{t+2}|d_{t+3},\Psi^i_{t+2}) = \frac{[\zeta^2(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{1}\zeta^n-\beta][\mu E^{\,i}_{t+1}(b_{t+1}|d_{t+2},\Psi^i_{t+1})+(1-\mu)\Psi^i_{t+2}]+\beta[d_{t+3}-\lambda_d d_{t+2}]}{\zeta^2(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{1}\zeta^n}.$$

Next the agent generates a new value Ψ^i_{t+3} leading to a t+3 prior belief $E^i_{t+3}(b_{t+3}|d_{t+3},\Psi^i_{t+3})$ about b_{t+3}.
By forward iteration we conclude that after N rounds the prior takes the form

$$E^{\,i}_{t+N}(b_{t+N}|d_{t+N+1},\Psi^i_{t+N}) = \frac{[\zeta^{N-1}(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n-\beta]}{\zeta^N(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n}[\mu E_t^{\,i}(b_{t+N-1}|d_{t+N},\Psi^i_{t+N-1})+(1-\mu)\Psi^i_{t+N}] + $$
$$+\; \frac{\beta[d_{t+N+1}-\lambda_d d_{t+N}]}{\zeta^N(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n}.$$

Now take the limit. Since $\zeta > 1$, as N increases $\zeta^N \to \infty$ hence we find that

$$\lim_{N\to\infty}\frac{[\zeta^{N-1}(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n-\beta]}{\zeta^N(\alpha+\beta)+(\xi\gamma+\beta)\sum_{n=0}^{N-1}\zeta^n} = \frac{(\alpha+\beta)+(\xi\gamma+\beta)(\frac{\zeta}{\zeta-1})}{\zeta(\alpha+\beta)+(\xi\gamma+\beta)(\frac{\zeta}{\zeta-1})} \equiv \kappa$$

Since all terms are positive it is clear that $0 < \kappa < 1$. Hence we have that for large t

$$E^{i}_{t+1}(b_{t+1}|d_{t+2}, \Psi^{i}_{t+1}) = \kappa[\mu E_t^{\,i}(b_t|d_{t+1}, \Psi^{i}_{t}) + (1-\mu)\Psi^{i}_{t+1}]$$

But by definition we have

(24) $$E^{i}_{t+1}(b_{t+1}|d_{t+1}, \Psi^{i}_{t+1}) = \mu E_t^{\,i}(b_t|d_{t+1}, \Psi^{i}_{t}) + (1-\mu)\Psi^{i}_{t+1}$$

We conclude that for large t, the contribution of each new observation of dividends is negligible hence

$$E_t^{\,i}(b_t|d_{t+1}, \Psi^{i}_{t}) = \kappa E_t^{\,i}(b_t|d_t, \Psi^{i}_{t}).$$

Inserting this last equation again into (24) we finally have the desired conclusion that for large t

(25) $$E^{i}_{t+1}(b_{t+1}|d_{t+1}, \Psi^{i}_{t+1}) = \mu\kappa E_t^{\,i}(b_t|d_t, \Psi^{i}_{t}) + (1-\mu)\Psi^{i}_{t+1}.$$

Now identify $g_t^{\,i} = E_t^{\,i}(b_t|d_t, \Psi^{i}_{t})$, $(1-\mu)\Psi^{i}_{t+1} = \rho^{ig}_{t+1}$ and $\mu\kappa = \lambda_Z$ to see that (25) is actually (10). ■

The Theorem shows that as the d_t data set increases, there is nothing new to learn. The posterior does not converge but *the law of motion* of the posterior converges to a time invariant stochastic law of motion defined by (25). The posterior fluctuates forever, providing the basis for the dynamics of market belief but the fluctuations follow a simple Markov transition. New data d_t and Ψ^{i}_{t} alter the conditional probability of the agent, but these do not change the dynamic law of motion of $g_t^{\,i}$.

4. Contrasting the Models: PI vs. HB

We discussed in Section 1 the natural objections to an excessive use of the assumption of PI. We now complete the comparison between the two theories and their empirical implications based on the *analytic results* of the two models. We also return to the notation of the earlier model.

4.1 Sharp Differences in Asset Pricing Implications

The striking difference between the two theories are revealed by their equilibrium price maps. Hence they lead to different characteristics of all phenomena which depend upon price dynamics, such as market volatility, risk premia, etc. We thus examine the difference between the maps (1a)-(1c) under PI and (21a)-(21d) under HB. Prices (1a)-(1c) under PI have infinite memory, a generic property of any learning. This follows from a theorem which says that if some components of a Markov process are unobserved, the process without full observability becomes one of infinite memory[11]. Since learning in a noisy REE with PI is driven by unobserved supply shocks, the inference utilizes all past prices which are

[11] Some (e.g. Bacchetta and van Wincoop (2005b)) bypass this theorem by not carrying out the full inference and instead making the arbitrary assumption that the information structure in a noisy REE is of finite memory.

proxies for these shocks. The infinite memory of prices arises despite the fact that the exogenous shocks have no memory at all. In addition, since private signals are independent they are averaged out and the average equals Q, the true value. Hence under PI price dynamics, risk premia and all other endogenous phenomena which depend upon prices are driven only by market "fundamentals": changes in Q_t and past supply shocks $S^t = (S_1, S_2, \ldots, S_t)$, all of which are unobservable. Indeed, noisy REE under PI leads to the peculiar result that prices depend on variables *which nobody observes*.

In contrast, the asset pricing theory under HB leads to an invariant price map which is defined over the economy's state variables, including the market state of belief, all of which are observable. Hence, the pricing process is non-stationary only to the extent that state variables are non stationary. Under HB the price does not reflect the unknown intrinsic value of the object since no one knows it and p_t forever fluctuate around their "fundamental" values. But there is a deeper principle involved here which is orthogonal to any Rational Expectations thinking. In a market with HB agents never learn the true structure of the economy and this leads to a simple principle. In an equilibrium with HB there is one true stochastic law of motion of state variables but traders hold diverse beliefs about this dynamics. *Hence, most traders are wrong most of the time*. Hence, under HB prices are determined by the distribution of forecasting mistakes of the traders. Indeed, prices are functions of both the observed exogenous variables as well as the market's belief about the future. But the market belief is the aggregation of individual assessments, including all mistaken assessments. As a result, under HB the price space is larger than under PI and price volatility is greater than the volatility implied by exogenous shocks. Kurz (1974), (1997a) and Kurz and Wu (1996) call this component of market risk "Endogenous Uncertainty." Samuelson expressed the intuition of this formal result by noting that "the market has predicted ten of the last six recession." Recall under HB agents do all the learning they can from past data and past data is ample, hence both heterogeneity and price volatility are persistent.

4.2 Difference in Beauty Contest Implications

A great deal has been written about the Keynesian Beauty Contest metaphor. In the context of the PI model with N rounds of trading one can rewrite (1c) in the form

$$p_1 = \bar{E}_1 \bar{E}_2 \ldots \bar{E}_N(Q) - \frac{Var_1(p_2)}{\tau} S_1$$

Allen, Morris and Shin (2003) associate this equation with the Beauty Contest since in a noisy REE the price today reflects tomorrow's (i.e. next round) average market forecast of the *fundamental value Q*.

This is much too narrow interpretation of the "Beauty Contest." An examination of this idea, as explained by Keynes (see Keynes (1936), page 156), shows that the crux of Keynes's conception is that there is little merit in the idea of using fundamental values as a yardstick for market valuation. Hence what matters for the market pricing of an asset is what the market believes the future price of that asset will be rather than what the intrinsic value of the asset will be. Moreover, Keynes insists future price depends upon future market beliefs which *may be right or wrong* but have no necessary relation to fundamental values. Hence, one must interpret the "Beauty Contest" as Keynes' statement that the price today is determined by today's market belief about the forecasts of the market's investors tomorrow, when such forecasts may be "right" or "wrong." The Allen, Morris and Shin (2003) interpretation does not rise to this level of subtlety required of the "Beauty Contest" idea. Finally, the idea of "trading rounds" is a modeling construct and as the number of rounds increases the private information market leads to full revelation hence, with time, $p = Q$. When this is the case, traders do not engage in any "Beauty Contest"at all, which makes any "Beauty Contest" temporary.

Another line of thinking in the literature about the Beauty Contest often stresses the role of *higher order* expectations. This is entirely misleading since higher order expectations are *intrinsic mathematical properties* of a probability measure over future sequences. Indeed, conditions (1a)-(1b) imply iterated higher order expectations and this is true with and without PI or HB (see also Townsend (1978),(1983)). The idea that higher order beliefs "are important" in some sense is no more than the statement that investors hold probability beliefs about future sequences.

We now examine the HB perspective of the Beauty Contest, keeping in mind the fact that *the unknown value Q_t is announced at the end of each date*. To enable full comparison with the PI model suppose for the moment that we discard the hedging demand and assume our traders are short lived. In that case demand function (1c) would apply and we would write it in the dynamic context as\

(26a) $$p_t = \bar{E}_t(p_{t+1}) - \frac{G_1}{\tau}S .$$

Insert into (25a) the corresponding equilibrium map under HB which would remain of the functional form $p_{t+1} = \hat{C}_1(Q_t + \mu) + \hat{C}_2 Z_{t+1} - \hat{C}_3 S)$ to have

(26b) $$p_t = \hat{C}_1(\bar{E}_t[Q_t] + \mu) + \hat{C}_2\bar{E}_t[Z_{t+1}] - (\hat{C}_3 + \frac{G_1}{\tau})S .$$

Compare (26b) with the price maps (1a)-(1b) under PI which depends upon Q and the supply shocks

$$p_1 = \kappa_1(\lambda_1 y + \mu_1 Q - S_1)$$
$$p_2 = \hat{\kappa}_2(\hat{\lambda}_2 y + \hat{\mu}_2 Q - S_2 + \psi S_1).$$

In (26b) the price at t is a function of the market expectation of Q_t and of Z_{t+1}, the market belief state at date t+1. The crux of the HB theory says that the root causes of Endogenous Uncertainty are the mistakes markets make in pricing assets too high or too low, leading to excess volatility. The term $\bar{E}_t[Z_{t+1}]$ in the price map (26b) says *the price today depends upon today's risk perception of the mistaken assessment the market may collectively make tomorrow.* This risk is central to any theory with HB. This, in our view, is the essence of the Keynesian Beauty Contest.

In short, the difference between the PI and the HB perspectives of the Beauty Contest is very sharp. Under PI the price is a function of the true fundamental value which is unknown to anyone. Under HB, the price depends upon the market expectation of Q_t and and of Z_{t+1}. But market belief may be right or wrong hence it may cause the asset to be overpriced or underpriced. This fear of future price volatility induced by future market beliefs, which we call "Endogenous Uncertainty," is therefore at the heart of the "Beauty Contest" phenomenon.

4.3 Difference in Rationality Assumptions and Restrictions: Rational Beliefs

Up to now we compared the PI model with the HB model without specifying any restrictions on information or requiring restrictions on beliefs. We criticized the PI theory for permitting arbitrary, unobservable, private signals and introducing contrived random supply shocks. These make it possible to prove anything with a model with PI. One may make a similar argument against HB, claiming that with HB one can prove anything. This last argument is false for two reasons. First, since the distribution of market belief is observable, hypotheses about the impact of market beliefs are testable. Second, we have defined individual beliefs to be about *deviations from the empirical distribution* and this, by itself, places restrictions on beliefs. Indeed, it is essential that we seek additional a priori restrictions on beliefs in order to further narrow down the empirical implications of the theory given the specified information. Recall that at t trader i knows his own (g_t^i, g_{t-1}^i) but he does not observe any past values of g_τ^k for all k. He does observe the distribution of g_τ^k for all dates up to t. This is an assumption of *anonymity*.

The theory of Rational Belief (in short, RB) due to Kurz (1994), (1997a) proposes natural restrictions on beliefs. In a sequence of papers the theory has been applied to various markets (e.g. Kurz (1996), (1997a), (1997b), Kurz and Schneider (1996), Kurz and Wu (1996), Kurz, Jin and Motolese (2005b), Motolese (2001), (2003) Nielsen (1996), (2003), Wu and Guo (2003), (2004)). In relation to the equity risk premium, Kurz and Beltratti (1997), Kurz and Motolese (2001), and Kurz, Jin and Motolese (2005a) explain the equity premium by asymmetry in the distribution of beliefs.

A belief is a Rational Belief *if it is a probability model of the observed market variables which, if simulated, reproduces the known empirical distribution*. An RB is thus a model which cannot be rejected by the empirical evidence. We specified beliefs with the perception models (14a) -(14b). For these to be RB they must induce the assumed empirical distribution (13a)-(13b). But this requires that

(27) The empirical distribution of $\begin{pmatrix} \lambda_Q^g g_t^{\,i} + \rho_t^{iQ} \\ \lambda_Z^g g_t^{\,i} + \rho_{t+1}^{iZ} \end{pmatrix}$ = the distribution of $\begin{pmatrix} \rho_t^Q \\ \rho_{t+1}^Z \end{pmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_Q^2, & 0, \\ 0, & \sigma_Z^2 \end{bmatrix}\right)$, i.i.d.

To compute the moments of the empirical distribution of the market variables (Q_{t-1}, Z_t) implied by the model (14a) -(14b), one treats the $g_t^{\,i}$ symmetrically with other random variables. From (10), the unconditional variance of $g_t^{\,i}$ is

$$\mathrm{Var}(g^{\,i}) = \frac{\sigma_g^2}{1 - \lambda_Z^2} .$$

Hence, we have the following rationality conditions on the moments, which follow from (27):

$$\text{(i)} \quad \frac{(\lambda_Q^g)^2 \sigma_g^2}{1 - \lambda_Z^2} + \hat{\sigma}_Q^2 = \sigma_Q^2 \qquad \text{(ii)} \quad \frac{(\lambda_Z^g)^2 \sigma_g^2}{1 - \lambda_Z^2} + \hat{\sigma}_Z^2 = \sigma_Z^2 \qquad \text{(iii)} \quad \frac{\lambda_Q^g \lambda_Z^g \sigma_g^2}{1 - \lambda_Z^2} + \hat{\sigma}_{ZQ} = 0$$

$$\text{(iv)} \quad \frac{(\lambda_Q^g)^2 \lambda_Z \sigma_g^2}{1 - \lambda_Z^2} + \mathrm{Cov}(\hat{\rho}_t^{iQ}, \hat{\rho}_{t-1}^{iQ}) = 0 \qquad \text{(v)} \quad \frac{(\lambda_Z^g)^2 \lambda_Z \sigma_g^2}{1 - \lambda_Z^2} + \mathrm{Cov}(\hat{\rho}_t^{iZ}, \hat{\rho}_{t-1}^{iZ}) = 0 .$$

The first three conditions pin down the covariance matrix in (14a)-(14b). The last two pin down the serial correlation of the two terms $(\hat{\rho}_t^{iQ}, \hat{\rho}_t^{iZ})$. An inspection of (14a)-(14b) reveals the only choice left for a trader are the two free parameters $(\lambda_Q^g, \lambda_Z^g)$. But under the RB theory these are not completely free either. The requirements that $\hat{\sigma}_Q^2 > 0$, $\hat{\sigma}_Z^2 > 0$ place two strict conditions on $(\lambda_Q^g, \lambda_Z^g)$:

$$|\lambda_Q^g| < \frac{\sigma_Q}{\sigma_g}\sqrt{1 - \lambda_Z^2} \qquad |\lambda_Z^g| < \frac{\sigma_Z}{\sigma_g}\sqrt{1 - \lambda_Z^2}.$$

Next, to ensure the covariance matrix in (14a)-(14b) is positive definite one must impose an additional condition. A condition such as

$$\frac{1 - \lambda_Z^2}{\sigma_g^2} > \frac{(\lambda_Z^g)^2}{\sigma_Z^2} + \frac{(\lambda_Q^g)^2}{\sigma_Q^2}$$

is sufficient. Finally, if one accepts the Bayesian procedure in Section 3.5 as the basis for (10), one can also insist on the restriction $\lambda_Q^g = 1$, *but this restriction does not flow from the basic RB condition.* Within the Bayesian framework we would also insist that the time average of the distinct valuations under qualitative assessment be zero as they would measure only deviations from the stationary forecast. We then see that the "free" parameters $(\lambda_Q^g, \lambda_Z^g)$ are restricted to a rather narrow range.

4.4 Difference in Testable Implications

In Section 1 we explored the assumptions of the PI theory. Since neither the private signals nor the supply noise are observable, the PI theory lacks testable implications. The model's key implication is the price map which results from the informational assumptions which are contrived and implausible.

In contrast, the model under HB is entirely testable since all central components of the theory are observable, including the average market belief. Data on the variables Z_t are constructed exactly as required by averaging (11). In a standard asset pricing equilibrium one can then write down the Euler equations of the agents, aggregate them and use the market data on returns, asset prices and market beliefs for a full identification. Recent examples of work where this has been done include Fan (2005) and Kurz and Motolese (2006). These papers show that with data on asset returns and market belief an asset pricing theory leads to specific testable restrictions. Moreover, by studying the excess returns on different categories of assets (i.e. stocks, bonds, etc.) one derives sharp estimates of the market risk premia of different assets and the effect of market beliefs on such premia. Clearly, the empirical evidence is the decisive factor to reveal which of the two theories discussed here is superior.

References

Allen, F., Morris, S., Shin, H.S. (2003) : "Beauty Contests, Bubbles and Iterated Expectations in Asset Markets." Cowles Foundation Discussion paper No. 1406. (Forthcoming in the *Review of Financial Studies,* **19**, 2006).

Amato, J. and Shin, H.S. (2002): "Imperfect Common Knowledge and Economic Stability." Working Paper, London School of Economics and the IBS.

Bacchetta, P. and van Wincoop, E. (2005a): "Can Information Heterogeneity Explain the Exchange Rate Determination Puzzle?" Working paper, Study Center Gerzensee, (forthcoming in the *American Economic Review*).

Bacchetta, P. and van Wincoop, E. (2005b): "Higher Order Expectations in Asset Pricing" Working paper, Study Center Gerzensee.

Batchelor, R. and Dua, P. (1991): "Blue Chip Rationality Tests." *Journal of Money, Credit and*

Banking, **23**, 692 - 705.
Blanchard, O. J., and Kahn, C. M. (1980): "The Solution of Linear Difference Models Under Rational Expectations." *Econometrica*, **48**, 1305 - 1311.
Brown, D. and Jennings, R. (1989): "On Technical Analysis." *Review of Financial Studies*, **2**, 527-551.
Detemple, J., Murthy S.(1994): "Intertemporal Asset Pricing with Heterogeneous Beliefs." *Journal of Economic Theory* **62**, 294-320
Diamond, D. and Verrecchia, R. (1981): "Information Aggregation in a Noisy Rational Expectations Equilibrium." *Journal of Financial Economics*, **9**, 221 - 235
Fan, M., (2005): "Heterogeneous Beliefs, the Term Structure and Time-Varying Risk Premia." Doctoral dissertation submitted to the Department of Economics, Stanford University.
Grundy, B. and McNichols, M. (1989): "Trade and Revelation of Information Through Prices and Direct Disclosure." *Review of Financial Studies*, **2**, 495-526.
Harris, M., Raviv, A. (1993) : "Differences of Opinion Make a Horse Race." *Review of Financial Studies* **6**, 473-506
Harrison, M. and Kreps, D.(1978): "Speculative Investor Behavior in a Stock Market with Heterogenous Expectations." *Quarterly Journal of Economics* **92**, 323-336.
He, H. and Wang, J. (1995): " Differential Information and Dynamic Behavior of Stock Trading Volume." *Review of Financial Studies*, **8**, 914 - 972.
Hellwig, C. (2002) : "Public Announcements, Adjustment Delays and the Business Cycle." Working Paper, Department of Economics, UCLA.
Judd, K. L., and Bernardo, A.E. (2000) : "Asset Market Equilibrium with General Tastes, Returns, and Informational Asymmetries", *Journal of Financial Markets*, **1**, 17- 43
Judd, K. L., and Bernardo, A.E.(1996) :"Volume and Price Formation in an Asset Trading Model with Asymmetric Information,", Working Paper, Stanford University.
Lucas, R. E. (1972): "Expectations and the Neutrality of Money." *Journal of Economic Theory*, **4**, 103 - 124.
Keynes, J. M. (1936): *The General Theory of Employment, Interest and Money*. Macmillan: London.
Kurz, M. (1974): "The Kesten-Stigum Model and the Treatment of Uncertainty in Equilibrium Theory." In Balch, M.S., McFadden, D.L., and Wu, S.Y. (ed.), *Essays on Economic Behavior Under Uncertainty*. Amsterdam: North-Holland, 389 - 399.
Kurz, M. (1994): "On the Structure and Diversity of Rational Beliefs." *Economic Theory* **4**, 877 - 900 . (An edited version appears as Chapter **2** of Kurz, M. (ed.) (1997a)).
Kurz, M. (ed) (1997a): *Endogenous Economic Fluctuations: Studies in the Theory of Rational Belief*. Studies in Economic Theory, No. **6**, Berlin and New York: Springer-Verlag.
Kurz, M. (1997b): "On the Volatility of Foreign Exchange Rates." Chapter 12 in Kurz, M. (ed.) (1997a) *Endogenous Economic Fluctuations: Studies in the Theory of Rational Belief*. Studies in Economic Theory, No. **6**, Berlin and New York: Springer-Verlag, 317 -352.
Kurz, M., Jin, H., Motolese, M. (2005a): "Determinants of Stock Market Volatility and Risk Premia." *Annals of Finance*, **1**, 109-147.
Kurz, M., Jin, H., Motolese, M. (2005b): "The Role of Expectations in Economic Fluctuations and the Efficacy of Monetary Policy." *Journal of Economic Dynamics and Control*, **29**, 2017 - 2065.
Kurz, M., Motolese, M. (2001): "Endogenous Uncertainty and Market Volatility." *Economic Theory*,

17, 497 - 544.
Kurz, M., Motolese, M. (2006): "Risk Premia, Diverse Beliefs and Beauty Contests." Working paper, Stanford University, September 19, 2006.
Kurz, M., Schneider, M.(1996): Coordination and Correlation in Markov Rational Belief Equilibria. *Economic Theory* **8**, 489 - 520.
Kurz, M., Wu, H.M. (1996): "Endogenous Uncertainty in a General Equilibrium Model with Price Contingent Contracts." *Economic Theory*, **8**, 461 -488. (Appears as Chapter **2** of Kurz, M. (ed.) (1997a))
Lucas, R. E. (1972): "Expectations and the Neutrality of Money." *Journal of Economic Theory*, **4**, 103 - 124.
Morris, S., Shin, H.S. (2002) : "Social Value of Public Information." *American Economic Review*, **92,** 1521 - 1534.
Motolese, M. (2001): "Money Non-Neutrality in a Rational Belief Equilibrium with Financial Assets." *Economic Theory*, **18**, 97 - 16.
Motolese, M. (2003): "Endogenous Uncertainty and the Non-Neutrality of Money." *Economic Theory*, **21**, 317 - 345.
Nielsen, K. C. (1996): "Rational Belief Structures and Rational Belief Equilibria." *Economic Theory*, **8**, 339 - 422 .
Nielsen, K. C. (2003): "Floating Exchange Rates vs. A Monetary Union Under Rational Beliefs: The Role of Endogenous Uncertainty." *Economic Theory*, **21**, 347 - 398.
Phelps, E.(1970): " Introduction : The New Microeconomics in Employment and Inflation Theory." In *Microeconomic Foundations of Employment and Inflation Theory*, New York: Norton, 1970.
Romer, C. D., Romer, D. H. (2000): "Federal Reserve Information and the Behavior of Interest Rates." *American Economic Review*, June, pp. 429-457.
Singleton, K.(1987): "Asset Prices in a Time-Series Model with Disparately Informed, Competitive Traders." In, Barnet, W. and Singleton, K. (ed.) *New Approaches to Monetary Economics, Proceedings of the Second International Symposium in Economic Theory and Econometrics.* Cambridge: Cambridge University Press.
Saari, D. (2006): **"**Parts, Whole, and Evolution", Lecture notes, University of California at Irvine.
Swanson, E. (2006): "Have Increases in Federal Reserve Transparency Improved Private Sector Forecasts of Short Term Interest Rates? *Journal of Money, Credit and Banking* (forthcoming).
Toukan, A. (2006): "Privately Held or Publicly Owned? Evolutionary Game Theoretic Analysis." Working Paper, University of California at Irvine.
Townsend, R. (1978): "Market Anticipations, Rational Expectations and Bayesian Analysis." *International Economic Review*, **19**, 481 - 494.
Townsend, R. (1983): "Forecasting the Forecasts of Others." *Journal of Political Economy*, **91**, 546 - 588.
Wang, J. (1994): "A Model of Competitive Stock Trading Volume." *Journal of Political Economy*, 102, 127 - 168.
Varian, H.R. (1985): "Divergence of Opinion in Complete Markets: A Note." *Journal of Finance* **40**, 309-317
Varian, H.R. (1989) : "Differences of Opinion in Financial Markets." In *Financial Risk: Theory, Evidence and Implications*, Proceeding of the 11th Annual Economic Policy Conference of the Federal Reserve Bank of St. Louis, Stone, C.C. ed. Boston: Kluwer Academic Publishers.

Wu, H.M., Guo, W.C. (2003): “Speculative Trading with Rational Beliefs and Endogenous Uncertainty.” *Economic Theory*, **21** , 263 292.
Wu, H.M., Guo, W.C. (2004): “Asset Price Volatility and Trading Volume with Rational Beliefs.” *Economic Theory*, **23** , 461- 488
Woodford, M. (2003) : "Imperfect Common Knowledge and the Effect of Monetary Policy." Chapter 1 in Aghion, P., Frydman, R., Stiglitz, J., Woodford, M. (ed.) *Knowledge, Information and Expectations in Modern Macroeconomics, in Honor of Edmund S. Phelps* Princeton: Princeton University Press.