# Creating Mathematical Infinities: Metaphor, Blending, and the Beauty of Transfinite Cardinals

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The Infinite is one of the most intriguing ideas in which the human mind has ever engaged. Full of paradoxes and controversies, it has raised fundamental issues in domains as diverse and profound as theology, physics, and philosophy. The infinite, an elusive and counterintuitive idea, has even played a central role in defining mathematics, a fundamental field of human intellectual inquiry characterized by precision, certainty, objectivity, and effectiveness in modeling our real finite world. Particularly rich is the notion of *actual infinity*, that is, infinity seen as a "completed," "realized" entity. This powerful notion has become so pervasive and fruitful in mathematics that if we decide to abolish it, most of mathematics as we know it would simply disappear, from infinitesimal calculus, to projective geometry, to set theory, to mention only a few.

From the point of view of cognitive science, conceptual analysis, and cognitive semantics the study of mathematics, and of infinity in particular, raises several intriguing questions: How do we grasp the infinite if, after all, our bodies are finite, and so are our experiences and everything we encounter with our bodies? Where does then the infinite come from? What cognitive mechanisms make it possible? How an elusive and paradoxical idea such as the infinite structures an objective and precise field such as

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mathematics? Why the various forms of infinities in mathematics have the exact conceptual structure they have? These, of course, are not simple questions. Nor are they new questions. Some of them have been already approached in the fields of philosophy, philosophy of mathematics, and formal logic for centuries. The problem, however, is that historically, these disciplines developed quite independently of the natural sciences, and of the necessity of looking at real empirical data involving, for instance, real human reasoning and conceptual development. As a result, when dealing with the nature and structure of mathematical concepts they fail to consider important constraints imposed by findings in the contemporary scientific study of the human mind, the human language, and their biological underpinnings. In philosophy and logic the study of the nature and the foundation of mathematical entities is often ultimately reduced to discussions over formal proofs and axiomatization. The contemporary scientific study of the mind tells us that human reasoning and conceptual structures are far from functioning in terms of formal proofs and axioms. Therefore what we need in order to answer the above questions is to seriously take into account how the human mind works, and at the very least provide cognitively plausible answers, that eventually could be tested empirically.

In this article, I intend to accomplish three things. First, I want to provide answers to the above questions based on findings in Conceptual Metaphor and Blending Theories, building on the work I have done in collaboration with George Lakoff in the field of *Cognitive Science of Mathematics*. In the process I'll be using a technique we have called *Mathematical Idea Analysis* (Lakoff and Núñez, 2000). Second, I want to analyze a specific case of actual infinity, namely, *transfinite cardinals*, as conceived by one of the most imaginative and controversial characters in the history of mathematics, the $19^{th}$

century mathematician Georg Cantor (1845-1918). As we will see later, Cantor created a very precise and sophisticated hierarchy of infinities that opened up entire new fields in mathematics giving shape, among others, to modern set theory. Many celebrated counterintuitive and paradoxical results follow from his work. In this article I will try to explain the cognitive reasons underlying such paradoxes. Finally, I want to analyze what Lakoff and I called the BMI -- the Basic Metaphor of Infinity (Lakoff & Núñez, 2000)-- in terms of a conceptual blend (Fauconnier & Turner, 1998, 2002), more specifically, in terms of a double-scope blend. Lakoff and I have hypothesized that the BMI is a human everyday conceptual mechanism, originally outside of mathematics, that is responsible for the creation of all kinds of mathematical actual infinities, such as points at infinity in projective and inversive geometry, infinite sums, mathematical induction, infinite sets, infinitesimal numbers, least upper bounds, and limits[1]. We will see that, unlike in the domains of poetry, advertisement, music, and visual arts, in mathematics humans need to operate with very specific forms of conceptual mappings, which are highly normalized, precise, constrained, and remarkably stable. In this paper I will take the BMI to be a double-scope conceptual blend where "BMI" stands for the more generic term Basic *Mapping* of Infinity.

## What is special about Mathematics?

Mathematics is a very peculiar form of knowledge where the entities constituting the subject matter are not perceived through the senses, yet they are incredibly precise and amazingly stable. Before discussing mathematical infinities and their conceptual

[1] The analysis of how exactly the BMI gives the precise inferential structure observed in these mathematical concepts lies outside the scope of this article. For details see Lakoff and Núñez, 2000, chapters 8-14.

structure, it is important to step back for a moment, and make clear what is unique about mathematics in general, as a field of intellectual inquiry. In order to be precise and rigorous, our cognitive analysis will have to be constrained by the peculiarities of mathematics as a body of knowledge. This includes mathematical infinities.

Mathematics, distinguishes itself from other bodies of knowledge and human conceptual systems in that it is highly idealized and fundamentally abstract. No purely empirical methods of observation can be directly applied to mathematical entities. Think, for instance, of a point, the simplest entity in the Euclidean plane, which has only location but no dimension. How could you possibly test a conjecture about Euclidean points by carrying out an experiment with *real* points if they don't have dimension? And how could you empirically observe a line, if it only has length but no width? It is easy to see, how the same arguments apply to infinity. This, and other properties of mathematics give shape to the unique manner in which knowledge is gathered in this discipline. Unlike science, where knowledge increases largely via careful empirical testing of hypotheses, in mathematics knowledge increases via proving theorems, and by carefully providing formal definitions and axioms. Whereas in science after performing an experiment a scientist can declare that her hypothesis is confirmed with a 95% of confidence, a mathematician won't be taken seriously if she announces, for instance, that "with 95% of confidence number *X* is prime." The mathematician will need to prove with "absolute certainty" that number *X* is prime. An important consequence of proof-oriented deductive ways of gathering knowledge is that once a theorem is proved, *it stays proved* forever! This peculiar form of knowledge gathering in mathematics provides an amazingly *stable* conceptual system.

Besides stability, Lakoff and Núñez (2000) give a list of other basic properties of mathematics such as *precision*, *consistency* for any given subject matter, *universality* of fundamental results (such as "1 + 1 = 2" which doesn't change across time and communities), *symbolizability* of its subject matters via discrete well-defined signs, *calculability* and *generalizability* of results, and *discoverability* of new entailments (theorems)[2]. Any account of the nature of mathematics, philosophical, cognitive, or other, must take into account these properties. In our case, the task will be to show that there are ordinary human cognitive mechanisms, such as aspectual schemas, conceptual metaphors, and conceptual blends, which when combining in very specific ways and being highly normalized, constrained, and stable can give an account of transfinite numbers as a special case of mathematical actual infinity.

At this point, and in order to avoid any misunderstandings about the goal of this article and of the nature of mathematical idea analysis, it is very important to make clear that:

1. A cognitive analysis that takes into account the properties of mathematics described above, and
2. the bodily-grounded nature of human cognitive mechanisms such as conceptual metaphors and conceptual blends,

provides a *non-arbitrary* explanation of the nature of mathematics. This non-arbitrary approach radically differs from post-modern accounts, where mathematics is seen as an arbitrary social text or as a mere cultural artifact. The position we will endorse here

[2] Lakoff & Núñez (2000) also mention another very important property of mathematics, that of being *effective* as general tools for description, explanation, and prediction in a vast number of activities. This property, however, is less relevant when transfinite numbers are concerned since they were not meant to actually model real physical, chemical, or biological phenomena.

recognizes the importance of culture and history in the emergence and development of mathematical ideas, but explicitly rejects the claim that mathematics is arbitrarily shaped by history and culture *alone* (for details see Lakoff & Núñez, 2000, pp. 362-363). With this perspective in mind, we are now ready to approach infinity.

**Potential and Actual infinity**

Investigations, speculations, and debates about the Infinite go back to the Pre-Socratic philosophers. One of the first to rigorously invoke the infinite to deal with questions regarding the origin, nature, and limits of things in the universe was Anaximander (611-547 BC). He saw in απειρον (*apeiron*), which literally means "unlimited," the ultimate source of all things. Arguing in opposition to Thales of Miletus (ca.624-ca.547 BC), who had asserted that water was the basis of all things, Anaximander defended the idea that the enormous variety of things in the universe must come from something less differentiated than water. For him, this primary source was eternal, boundless, endless, from where even opposites such as cold and hot originated. This primary source was apeiron. It was subject to neither old age nor decay, perpetually generating fresh materials and dissolving them. Because of its very nature (i.e., unlimited) no limits could apply to apeiron, and therefore it was conceived as sourceless, without creation, and indestructible.

Later, Aristotle (384-322 BC) referred back to Anaximander, but with a very different view. One of the crucial problems raised by Aristotle was the issue of exhaustion. Giving the example of the collection of numbers, Aristotle argued that the *totality* of numbers cannot be present in our thoughts. In generating, one by one, the list of numbers, we can't generate a *completed* list. There will always be a number that hasn't been considered before. In his *Physics* he argued that apeiron is not that thing outside of which there is nothing (i.e., exhausted), but outside of which there is always something (i.e., inexhaustible). Therefore apeiron, the "unlimited," cannot be seen as a *completed* totality. What is completed has an end, and the end is a limiting element. By its very meaning there is a lack of that limit in apeiron. With Aristotle then, apeiron takes a negative connotation, due to its inherent incompleteness and non-actualizable potential. Because it cannot actually be realized in a clearly defined form, apeiron became associated with the idea of "undefined." Moreover, in order to keep his fundamental tenet that the unknowable exists only as a potentiality, Aristotle rejected altogether the existence of the actual infinite: anything beyond the power of comprehension was seen as beyond the realm of reality. In his *Physics*, Aristotle stated quite clearly that infinity should be considered as something that "has potential existence" but never as an actual realized thing. Many analysts and historians consider this negative connotation to be the reason of the refusal of using actual infinity in Greek mathematics (for details see Zellini, 1996; and Boyer, 1949).

Since the time of Aristotle then, the infinite has been treated with extreme care. Many Greek thinkers considered the infinite as an undefined entity with no order, chaotic, unstructured. The infinite, therefore, was seen as an entity to be avoided in proper reasoning. This view dominated most of the debates (in Europe) involving the infinite all the way up to the Renaissance. In mathematics this was no exception and the distinction between *potential* and *actual* infinity has ever since been made, by readily accepting the former and by questioning or simply rejecting the latter.

Potential infinity is the kind of infinity characterized by an ongoing process repeated over and over without end. It occurs in mathematics all the time. For instance, it shows up when we think of the unending sequence of regular polygons with more and more sides (where the distance from the center to any of the vertices is constant). We start with a triangle, then a square, a pentagon, a hexagon, and so on. Each polygon in the sequence has a successor and therefore there is the potential of extending the sequence again and again without end (Figure 1). The process, at any given stage encompasses only a final number of repetitions, but as a whole doesn't have an end and therefore does not have a final resultant state.

[Insert Figure 1 about here]

But more than potential infinity, what is really interesting and mathematically fruitful is the idea of *actual* infinity, which characterizes an infinite process as a *realized* thing. In this case, even though the process is *in*-finite, that is, it does not have and end, it is conceived as being "completed" and as having a *final resultant state.* Following on the example of the sequence of regular polygons, we can focus our attention on certain aspects of the sequence and observe that because of the very specific way in which the sequence is built certain interesting things happen. After each iteration the number of sides grows by one, the sides become increasingly smaller, and the distance $r$ from the center to the vertices remain constant. As we go on and on with the process the perimeter and the area of the polygon become closer and closer in value to $2\pi r$ and to $\pi r^2$, respectively, which correspond to the values of the perimeter and the area of a circle.

Thinking in terms of actual infinity imposes an end *at infinity* where the entire infinite sequence *does have* a final resultant state, namely a circle *conceived as* a regular polygon with an infinite number of sides (see Figure 2). This circle has all the prototypical properties circles have (i.e., area, perimeter, a center equidistant to all points on the circle, $\pi$ being the ratio between the perimeter and the diameter, etc.) but conceptually it *is* a polygon.

[Insert Figure 2 about here]

It is the fact that there is a final resultant state that makes actual infinity so rich and fruitful in mathematics. But it is also this same feature that has made the idea of actual infinity extremely controversial because it has often lead to contradictions, one of the worst evils in mathematics. A classic example is the "equation" $k/0 = \infty$, where $k$ is a constant. This "equation" is based on the idea that (when finite values are concerned) as the denominator gets progressively smaller the value of the fraction increases indefinitely. So *at* infinity the denominator *is* 0 and the value of the fraction *is* $\infty$ (greater than any finite value). The problem is that accepting this result would also mean accepting that $(0 \cdot \infty) = k$, that is the multiplication of zero times infinity could be equal to any number. This, of course, doesn't make any sense. Becausee of contradictions like this one many brilliant mathematicians, such as Galileo (1564-1642), Carl Friedrich Gauss (1777-1855), Agustin Louis Cauchy (1789-1857), Karl Weierstrass (1815-1897), Henri Poincaré (1854-1912), among others had energetically rejected actual infinity. Gauss, for instance, in a letter to his colleague Heinrich Schumacher dated 1831 wrote (cited in Dauben, 1990, p. 120):

> *But concerning your proof, I must protest above all against your use of an infinite quantity as a completed one, which in mathematics is never allowed. The infinite is only a façon de parler, in which one properly speaks of limits.*

Here, limits are taken to be magnitudes to which certain ratios may approach as closely as desired when others are permitted to increase indefinitely, and are thus instances of potential infinity. Up to the 19th century there was a well-established consensus among mathematicians that at best actual infinity could provide some intuitive ideas when dealing with limits for instance (Gauss' *façon de parler*) but that no consistent and interesting mathematics could possibly come out of an infinity actually realized. George Cantor, following some preliminary work by Bernard Bolzano (1781-1848) and Richard Dedekind (1831-1916) radically challenged this view, seeing in actual infinity a genuine mathematical entity. His controversial, unconventional, and highly disputed work generated amazing new mathematics.

## Transfinite cardinals: The standard story

The 19th century was a very productive period in the history of mathematics, one that saw fundamental developments such as non-Euclidean geometries, and the so-called arithmetization of analysis. The latter, a program lead by Karl Weierstrass, Richard Dedekind, and others, intended to ban geometrical and dynamic intuition (thought to be the source of paradoxes) by reducing the whole field of calculus developed in the 17th century by Newton and Leibniz, into the realm of numbers. Counting and focusing on discrete entities, like numbers, became essential. It is in this *zeitgeist* that Georg Cantor, originally interested in the study of trigonometric series and discontinuous functions, was brought into his development of transfinite numbers, dispelling well-establish views that abolish the use of actual infinities in mathematics. Today, Cantor is best known for the creation of a mathematical system where numbers of infinite magnitude define very precise hierarchies of infinities with a precise arithmetic, giving mathematical meaning to the idea of some infinities being greater than others. His work was highly controversial, produced many counter-intuitive results and for most of his professional life Cantor had to struggle against heavy criticism (for an in depth analysis of Cantor's work and intellectual path, see Ferreirós, 1999, and Dauben, 1990).

A basic problem for Cantor was to determine the number of elements in a set (which he called *Menge*, aggregate). This is, of course, a trivial problem when one deals with finite sets, but when one deals with sets containing infinitely many elements, such as the set of counting numbers 1, 2, 3, … , (i.e., the set of so-called *natural numbers*) this is literally impossible. How do you count them if they are infinitely many? Cantor focused on the fact that when comparing the relative size of finite sets, not only we can count their elements, but we can also set up pairs by matching the elements of the two sets. When two *finite* sets have the same number of elements, a one-to-one correspondence between them can be established. And conversely, when a one-to-one correspondence between two *finite* sets can be established we can conclude that they have the same number of elements.

Cantor elaborated on the idea of one-to-one correspondence so it would apply also to infinite sets in a precise way. He addressed questions such as: Are there more natural numbers than even numbers? A similar question had already been asked in the first half of the 17th century by Galileo, who observed that it was possible to match, one-by-one *ad infinitum* the natural numbers with their respective squares, but because the squares are contained in the collection of natural numbers they form a smaller collection than the natural numbers. Facing this paradoxical situation Galileo concluded that attributes such as "bigger than," "smaller than," or "equal to" shouldn't be used to compare collections when one or both had infinitely many elements. In the 19th century Cantor could get around the "paradox" by building on the previous very creative though not well-recognized work by Bernard Bolzano and by Richard Dedekind. These two mathematicians were the first to conceive the possibility of matching the elements of an infinite set with one of its subsets as an *essential* property of infinite sets and not as a weird pathology. Dedekind in fact, for whom infinite sets constituted perfectly acceptable objects of thought, provided for the first time in history a definition involving the infinite in positive terms (i.e., not in negative terms such as *in*-finite or *non*-finite). He stated (in modern terminology) that a set *S* is infinite if and only if there exists a proper subset *S*'of *S* such that the elements of *S*' can be put into one-to-one correspondence with those of *S*. Only infinite sets have this important property (for historical and technical details, see Ferreirós, 1999).

[Insert Figure 3 about here]

With this background, Cantor had the way paved for answering his question regarding the "size" of the sets of natural and even numbers. He then declared that "whenever two sets—finite or infinite—can be matched by a one-to-one correspondence, they have the same number of elements" (Maor, 1991, p. 57). Because such a correspondence between natural and even numbers, can be established (Figure 3) he concluded, there are just as many even numbers as there are natural numbers. In this framework, the fact that all even numbers are contained in the natural numbers (i.e., they constitute a proper subset of) doesn't mean that the set of natural numbers is bigger. Following Dedekind's definition above, that fact simply shows a property of infinite sets.

And what about other kinds of infinite sets, with more challenging properties? Could such sets be put in one-to-one correspondence with the natural numbers? For instance, natural numbers and even numbers can be ordered according to magnitude such that every member has a definite successor. So what about say, rational numbers, which don't have this property? Rational numbers are *dense*, that is, between any two rational numbers, even if they are extremely close, we can *always* find another rational number. Rationals don't have successors. The set of rational numbers seems to have infinitely many more elements than the naturals because not only we can find infinitely many rationals bigger or smaller than a given number (i.e., towards the right or the left of the number line, respectively), but also we can find infinitely many rationals in any portion of the number line defined by two rationals. Is then the set of rationals bigger than the naturals?

In order to try to establish a one-to-one correspondence between the rationals and the naturals one needs, first of all, to display both sets in some organized way. In the case of even and natural numbers that organization was provided by order of magnitude. But because rationals are dense they can't be ordered by magnitude. Cantor, however, found a way of displaying *all* rationals, one by one, in a clever *infinite array*. Figure 4 shows such array, which displays all possible fractions. Fractions with numerator one are displayed in the first row, fractions with numerator two are in the second row, and so on. And similarly, fractions with denominator one are in the first column, fractions with denominator two in the second column, and so on. In 1874 Cantor was able to show, with this array, and against his own intuition (!), that it was possible to establish a one-to-one correspondence between the rationals and the naturals. All you need to do is to assign a natural number to each fraction encountered along the path indicated in Figure 4. The path covers all possible fractions going diagonally up and down ad infinitum[3].

[Insert Figure 4 about here]

When such a correspondence is established between two infinite sets, Cantor said that they have the same *power* (*Mächtigkeit*) or *cardinal number*. He called the power of the set of natural numbers, $\aleph_0$, the smallest transfinite number (denoted with the first letter of the Hebrew alphabet, aleph). Today, infinite sets that can be put in a one-to-one correspondence with the natural numbers are said to be *denumerable* or *countable*, having cardinality $\aleph_0$.

Cantor's next question was, are all infinite sets countable? Towards the end of 1873 he found out that the answer was no. He was able to provide a proof that the real

[3] A rational number can be expressed by different fractions. For the purpose of the one-to-one correspondence only the simplest fraction denoting a rational is considered. For example, 2/4, 3/6, 4/8, etc. are equivalent to 1/2, and therefore they are skipped.

numbers can't be put into one-to-one correspondence with the natural numbers: the set of real numbers is not denumerable. Later Cantor gave a different, simpler proof, known today as the famous proof by diagonalization. He started by assuming that a correspondence between the natural numbers and the real numbers between zero and one was possible. Since every real number has a unique non-terminating decimal representation he wrote down the correspondence as follows[4]:

$$\begin{array}{lll} 1 & \rightarrow & 0.a_{11}a_{12}a_{13}\ldots \\ 2 & \rightarrow & 0.a_{21}a_{22}a_{33}\ldots \\ 3 & \rightarrow & 0.a_{31}a_{32}a_{33}\ldots \\ \ldots & \rightarrow & \ldots\ldots \end{array}$$

The list, according to the original assumption includes *all* real numbers between 0 and 1. He then showed that he could construct a real number that wasn't included in the list, a number of the form $0.b_1b_2b_3\ldots$ where the first digit $b_1$ of this number would be different from $a_{11}$ (the first digit of the first number in the list), the second digit $b_2$ of the new number would be different from $a_{22}$ (the second digit of the second number in the list), and so on. As a result, the new number $0.b_1b_2b_3\ldots$, which is bigger than zero but smaller than 1, would necessarily differ from any of the numbers in the list in at least one digit. The digit $b_k$ (the $k$-th digit of the new number) will always differ from the digit $a_{kk}$ given by the diagonal (the $k$-th digit of the $k$-th number of the list). This leads to a contradiction since the original list was supposed to include *all* real numbers between 0 and 1, and therefore the one-to-one correspondence between the naturals and the reals in the interval (0, 1) can't be established. Since the naturals were a subset of the reals this means that the reals form a non-denumerable set which has a power higher than the naturals: A

[4] The proof requires that all real numbers in the list to be written as non-terminating decimals. For example, a fraction such as 0.3 should be written as 0.2999…

transfinite cardinal number bigger than $\aleph_0$. Cantor called it *c*, for the power of the continuum.

But Cantor's work went beyond these two transfinite numbers, $\aleph_0$ and *c*. He showed that in fact there is an entire infinite and very precise hierarchy of transfinite numbers. In order to do so Cantor elaborated on the idea of power set (i.e., the set whose elements are all the subsets of the original set, including the empty set and the original set itself), observing that for finite sets when the original set has *n* elements, its power set has exactly $2^n$ elements. Cantor extended this idea to infinite sets showing that the power (cardinality) of the power set of natural numbers was exactly $2^{\aleph_0}$. This new set in turn formed a power set whose cardinality was $2^{2^{\aleph_0}}$, and so on. This remarkable result defined a whole infinite hierarchy of transfinite cardinals holding a precise greater than relationship:

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \dots$$

Cantor was able to prove a further extraordinary result: The number of elements in the set of real numbers is the same as the number of elements in the power set of the natural numbers. In other words he proved the equation $c = 2^{\aleph_0}$ to be true, meaning that the number of points of the continuum provided by the real line had exactly $2^{\aleph_0}$ points. But Cantor didn't stop there. He was also able to show an extremely counter-intuitive result : Dimensionality of a space is not related with the numbers of points it contains. Any tiny segment of the real line has the same number of points as the entire line, and the same as in the entire plane, the entire three-dimensional space, and in fact in any "hyper-space" with a denumerable number of dimensions. Cantor added many more counter-intuitive and controversial results to his long list of achievements. He developed a very rich work on transfinite ordinals (see Dauben , 1983, and Sondheimer & Rogerson, 1981), and defined a precise arithmetic for tranfinite cardinals where unorthodox equations such as the following hold:

$$\aleph_0 + 1 = 1 + \aleph_0 = \aleph_0;$$

$$\aleph_0 + k = k + \aleph_0 = \aleph_0, \text{ for any natural number } k;$$

$$\aleph_0 + \aleph_0 = \aleph_0;$$

$$\aleph_0 \cdot k = k \cdot \aleph_0 = \aleph_0, \text{ for any natural number } k;$$

$$\aleph_0 \cdot \aleph_0 = \aleph_0;$$

$$(\aleph_0)^k = \aleph_0, \text{ for any natural number } k.$$

These equations represented an extraordinary improvement in approaching and studying the infinite when compared to the old and vague idea represented by the symbol ∞. With Cantor infinite numbers acquired a precise meaning, and constituted the corner stone of the development of extremely creative, and ingenious new mathematics. Realizing how deep and rich Cantor's work was, David Hilbert, one of the greatest mathematicians of the last couple of centuries said "No one shall drive us fro the paradise Cantor created for us."

So far, this is more or less a summarized version of the standard story about Cantor's transfinite cardinals as it is told in general books and articles on the history and philosophy of mathematics (see, for instance, Boyer, 1968; Dauben, 1983; Klein, 1972; Kramer, 1981; Maor, 1991, Sondheimer & Rogerson, 1981). Let us now try to understand what is, from a cognitive perspective, the conceptual structure underlying Cantor's ingenious work.

## BMI and conceptual blending: The birth of actual infinities

In order to understand the cognitive nature of actual infinity and the conceptual structure underlying transfinite cardinals, we need to refer to two main dimensions of human cognitive phenomena: One is *aspect* (Comrie, 1976), as it has been studied in cognitive semantics, and the other one is the BMI (originally described in Lakoff and Núñez, 2000, as the Basic Metaphor of Infinity) and here treated as the Basic Mapping of Infinity, a form of double-scope conceptual blend. Both, aspect and the BMI, being bodily-grounded phenomena of human cognition provide the elements to understand how the embodied mind (Johnson, 1987; Varela, Thompson and Rosch, 1991) makes the infinite possible.

### *Aspect*

In cognitive semantics, aspectual systems characterize the structure of event concepts. The study of aspect allows us to understand, for instance, the cognitive structure of iterative actions (e.g., “breathing,” “tapping”) and continuous actions (e.g., “moving”) as they are manifested through language in everyday situations. Aspect can tell us about the structure of actions that have inherent beginning and ending points (e.g., “jumping”), actions that have starting points only (e.g., “leaving”), and actions that have ending points only (e.g., “arriving”). When actions have ending points, they also have resultant states. For example, “arriving” (whose aspectual structure has an ending point) in *I arrive at my parents’ house*, implies that once the action is finished, I am located *at* my parents’ house. When actions don’t have ending points they don’t have resultant states. Many dimensions of the structure of events can be studied through aspect.

For the purpose of this article, the most important distinction regarding aspect is the one between *perfective aspect* and *imperfective aspect*. The former has inherent completion while the latter does not have inherent completion. For example, the prototypical structure of "jumping" has inherent completion, namely, when the subject performing the action touches the ground or some other surface. We say then that "jumping" has perfective aspect. "Flying," on the contrary, does not have inherent completion. The prototypical action of "flying" in itself does not define any specific end, and does not involve touching the ground. When the subject performing the action, however, touches the ground, the very act of touching puts an end to the action of flying but does not belong to "flying" itself. We say that "flying" has imperfective aspect.

Processes with imperfective aspect can be conceptualized both, as continuative (continuous) or iterative processes. The latter have intermediate endpoints and intermediate results. Sometimes continuous processes can be conceptualized in iterative terms, and expressed in language in such a way. For example, we can express the idea of sleeping continuously by saying "he slept and slept and slept." This doesn't mean that he slept three times, but that he slept uninterruptedly. This human cognitive capacity of conceiving something continuous in iterative terms turns out to be very important when infinity is concerned. Continuous processes without end (e.g., endless continuous monotone motion) can be conceptualized as if they were infinite iterative processes with intermediate endpoints and intermediate results (for details see Lakoff & Núñez, 2000).

With these elements we can now try to understand how human cognitive mechanisms bring *potential infinity* into being. From the point of view of aspect, potential infinity involves processes that may or may not have a starting point, but that explicitly *deny the possibility of having an end point*. They have no completion, and no final resultant state. We arrive then to an important conclusion:

- Processes involved in potential infinity have imperfective aspect.

*BMI, the Basic Mapping of Infinity*

Now let's analyze actual infinity, which is what we really care about in this article. It is here where the BMI becomes crucial. The BMI is a general conceptual mapping which is described in great detail elsewhere (Lakoff & Núñez, 2000). It occurs inside and outside of mathematics, but it is in the precise and rigorous field of mathematics that it can be best appreciated. Lakoff and Núñez have hypothesized that the BMI is a single human everyday conceptual mechanism that is responsible for the creation of all kinds of mathematical actual infinities, from points at infinity in projective geometry, to infinite sums, to infinite sets, and to infinitesimal numbers and limits. When seen as a double-scope conceptual blend[5] the BMI has two input spaces. One is a space involving Completed Iterative Processes (with perfective aspect). In mathematics, these processes correspond to those defined in the finite realm. The other input space involves Endless Iterative Processes (with imperfective aspect), and therefore it characterizes processes involved in potential infinity. In the blended space what we have is the emergent inferential structure required to characterize processes involved in actual infinity. Figure 5 shows the correspondences between the input spaces and the projections towards the blended space.

[5] Details of conceptual blending can be seen in other contributions in this volume, and in the original work of Fauconnier and Turner (1998, 2002).

[Insert Figure 5 about here]

It is important to see that the richness and peculiarity of the BMI is its organization and structure as a double-scope blend (Fauconnier & Turner, 2002). The correspondence between the two input spaces involves all the elements with the exception of the very last one, the single element that distinguishes in a fundamental way a finite process from a potentially infinite process. This provides a major conflict: a clash between a characterization of a process as explicitly *having an end* and a *final resultant state*, and one as explicitly characterizing the process as *being endless* and with *no final resultant state*. Often these conflicts lead to paralysis, where no blended space is formed at all, leaving the original input spaces as they were with their own local inferential structure. The history of science and mathematics provide many such examples. My interpretation, for instance, is that this is in part what occurred to Galileo when he observed that natural numbers and even numbers could be put in one-to-one correspondence, but failed to make any conclusions that would have required *completing* and *endless* process[6]. Rather than paralysis, a double-scope blend handles the conflict in a creative way providing fundamentally new inferential structure in the blended space. In the BMI, this is what occurs:

- From the Completed Iterative Process Input (with perfective aspect) the fact that the process *must have an end* and *a final resultant state* is profiled and projected to the blended space, ignoring the clause that the *process must be finite*.

[6] As we will see later, another important component contributing to his paralysis was, of course, that at that time he wasn't able to operate with the conceptual metaphor SAME NUMBER IS PAIRABILITY, which is Cantor's metaphor.

- And from the Endless Iterative Processes (with imperfective aspect), the fact that the process *has no end* is profiled and projected into the blended space, ignoring the clause that the processes *does not have a final resultant state.*
- As a result, in the blended space there is now new inferential structure, which provides *an endless process* with *an end* and *a final resultant state.*

As Lakoff & Núñez (2000) have pointed out, a crucial entailment of the BMI is that the final resultant state is *unique* and *follows every nonfinal state.* The uniqueness comes from the input space of completed processes, where for any completed process the final resultant state is unique. The fact that the final resultant state is indeed *final*, means that there is no earlier final state. That is, there is no distinct previous state within the process that both follows the completion stage of the process yet precedes the final state of the process. Similarly, there is no later final state of the process. That is, there is no other state of the process that both results from the completion of the process and follows the final state of the process.

In order to illustrate how the BMI works, let's take the example mentioned earlier of the sequence of regular polygons (Figure 2). As Lakoff & Núñez (2000, Chapter 8) point out, in order to get from the BMI as a general cognitive mechanism to special cases of actual infinity, one needs to *parameterize* the mapping. That is, one must characterize precisely what are the elements under consideration in the iterative process. In our example the first input space (located on the left in Figure 5) provides a finite process with perfective aspect. The process is a specific sequence of regular polygons where the distance from the center to any of the vertices is kept constant. The process starts with a triangle, then a square, a pentagon, and so on, all the way to a polygon with a finite number of sides, say 127 sides. At each stage, we have specific values for the perimeter and area of each polygon in the sequence, which get closer to $2\pi r$, and $\pi r^2$, respectively (where $r$ is the distance from the center to the vertices). The perimeter and the area of the final resultant state in this first input space (i.e., polygon with 127 sides) has the closest values to $2\pi r$, and $\pi r^2$, respectively. The second input space (located on the right in Figure 5), involves the sequence shown in Figure 1, that is, an endless sequence of regular polygons (which has imperfective aspect). At each stage we obtain specific values for the perimeter and area of each polygon in the sequence, which get *endlessly* closer to $2\pi r$, and $\pi r^2$, respectively. The distance from the center to any of the vertices is always constant, namely, $r$. There is no final resultant state in this second input space.

In the blend, all the corresponding elements are projected, which gives us the sequence of regular polygons with a triangle, a square, a pentagon, and so on. The conflict between the final resultant state of a finite sequence of polygons (i.e., polygon of 127 sides) and the endless nature of the sequence is handled by the double-scope blend to give an endless sequence of regular polygons with a final resultant state (with infinitely many sides). At this final resultant state no difference in terms of perimeter, area, and distance from center to vertices can be detected between the "final" polygon obtained via the BMI and a circle. For the circle the values of the perimeter, the area, and the radius are precisely $2\pi r$, $\pi r^2$ and $r$, respectively. Therefore, when parameterized in this manner, the final resultant state is conceived as an actual unique polygon-circle: A very peculiar kind of polygon with an infinite number of sides, a distance from center to vertices equal to $r$, a perimeter equal to $2\pi r$, and an area equal to $\pi r^2$. The BMI guarantees that this figure is unique and that is indeed the final resultant state. No polygon comes after the polygon-circle in the process.

## Transfinite cardinals: The cognitive story

It is now time to come back to Cantor's work by looking closely to one of his classic texts. Here is Cantor himself at the beginning of his *Contributions to the Founding of the Theory of Transfinite Numbers*:

> Every aggregate M has a definite "power," which we will also call "cardinal number." We will call by the name "power" or " cardinal number" of M the general concept which, by means of our faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements *m* and of the order in which they are given (p. 86).

By "aggregate" (*Menge*) Cantor means "any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought" (p.85) (For the purpose of this discussion, aggregates can be seen as collections or sets). The objects he refers to are the "elements" of M. Cantor then defines the crucial concept that will allow him to build the notion of transfinite number, which as we will see is a metaphorical extension of cardinal numbers of finite aggregates. This is the idea of *equivalence*:

> We say that two aggregates M and N are "equivalent" … if it is possible to put them, by some law, in such a relation to one another that to every

> element of each one of them corresponds one and only one element of the other (p. 86).

With these extremely simple but powerful notions (i.e., aggregate, element, cardinal number, equivalence), Cantor wants to build an entirely new theory of numbers that would encompass at the same time his transfinite numbers and the usual counting (natural) numbers. These are his own words:

> We will next show how the principles which we have laid down, and on which later on the theory of actually infinite or transfinite cardinal numbers will be built, afford also the most natural, shortest, and most rigorous foundation for the theory of finite numbers (p. 97-98).

This is a crucial passage. Cantor is explicitly telling us that he is not only concerned with actual infinity. But, why someone interested in infinity would attempt to give rigorous foundations to finite numbers as well? The reason is simple. He has a more ambitious goal: to generalize in a rigorous way the very notion of number in itself! He wants to do this by building transfinite *and* finite numbers, using the *same principles.* Cantor proceeds by building the natural numbers from scratch:

> To a single thing $e_0$, if we subsume it under the concept of an aggregate $E_0 = (e_0)$, corresponds as cardinal number what we call "one" and denote by 1 … Let us now unite with $E_0$ another thing $e_1$, and call the union-aggregate $E_1$, so that $E_1 = (E_0, e_1) = (e_0, e_1)$. The cardinal number of $E_1$ is called "two" and is denoted by 2 … By addition of new elements we get the series of aggregates $E_2 = (E_1, e_2)$, $E_3 = (E_2, e_3)$, ... which give us

successively, in unlimited sequence, the other so-called “finite cardinal numbers” denoted by 3, 4, 5, … (p. 98).

Notice that each finite aggregate $E_n = (E_{n-1}, e_n)$, and it has cardinality $(n + 1)$. Because $E_n$ is formed via the union of the predecessor $E_{n-1}$ with the “thing” $e_n$, this implies that $E_{n-1}$ is contained in (using modern terminology we would say, “is a proper subset of”) $E_n$. The series of aggregates is constructed exhibiting a nested structure (under the relation of being a proper subset of) where the difference between any two consecutive aggregates $E_n$ and $E_{n-1}$ is just the “thing” $e_n$, being their cardinal numbers $(n + 1)$ and $n$, respectively. Cantor then is able to state a (apparently naïve but) very important theorem:

- The terms of the unlimited series of finite cardinal numbers 1, 2, 3, …, ν… are all different from one another, “that is to say, the condition of equivalence established [earlier] is *not fulfilled* for the corresponding aggregates” (p. 99; our emphasis).

When finite natural numbers are concerned this theorem is innocuous and it seems to be totally irrelevant. Basically, it says that the number 1 is different from the number 2, and that they are different from the number 3, and so on. The theorem is simply telling us that the cardinal number of two finite sets are different if the sets cannot be put into one-to-one correspondence (i.e., they are not equivalent). This is a simple fact, but it has profound consequences: From this conceptualization, the number 2 and the number 3, for instance, are different, not because the latter is the result of *counting* an aggregate with “three” elements, while the former is the result of *counting* an aggregate with only “two,” but because the aggregates from which they are cardinal numbers of cannot be put in a one-to-on correspondence (i.e., they do not fulfill the condition of equivalence). The real power of the notion of equivalence and of this theorem becomes visible when he finally introduces the smallest transfinite cardinal number Aleph-Zero:

> The first example of a transfinite aggregate is given by the totality of the finite cardinal numbers $\nu$; we call its cardinal number “Aleph-zero” and denote it by $\aleph_0$ … That $\aleph_0$ is a *transfinite* number, that is to say, is not equal to any finite number $\mu$, follows from the simple fact that, if to the aggregate $\{\nu\}$ is added a new element $e_0$, the union aggregate $(\{\nu\}, e_0)$ is equivalent to the original aggregate $\{\nu\}$. For we can think of this reciprocally univocal correspondence between them: to the element $e_0$ of the first corresponds the element 1 of the second, and to the element $\nu$ of the first corresponds the element $\nu+1$ of the other. … we thus have $\aleph_0 + 1 = \aleph_0$ (p. 103-104)

This is another crucial passage. It constitutes one of the first moments in history in which a (rather unorthodox but) well-defined equation involving infinite quantities is established. Extending the nested construction of the series of finite aggregates, where $E_n = (E_{n-1}, e_n)$, he now builds the aggregate $(\{\nu\}, e_0)$ by adding the element $e_0$ to $\{\nu\}$, the infinite aggregate containing *all* finite cardinal numbers. But an important difference with the finite cases emerges: Although $\{\nu\}$ is *contained* in $(\{\nu\}, e_0)$ (i.e., it is a proper subset of), now these two aggregates do fulfill the conditions of equivalence (i.e., they can be put in a one-to-one correspondence). Some fundamental entailments follow:

1. A finite number $k$ and the *transfinite* number $\aleph_0$ are different, not because the latter is the result of *counting* the aggregate of *all* finite cardinal numbers, while

the former is the result of *counting* an aggregate with only "*k*" elements, but because the aggregates from which they are cardinal numbers of cannot be put in a one-to-on correspondence (i.e., they do not fulfill the condition of equivalence).

2. The *transfinite* numbers $\aleph_0$ and $\aleph_0 + 1$ are equal, not because the latter is the result of *counting* the aggregate of *all* finite cardinal numbers, while the former is the result of *counting* the same aggregate plus one element, but because the aggregates from which they are cardinal numbers of can be put in a one-to-on correspondence (i.e., they do fulfill the condition of equivalence).
3. These criteria for discriminating finite numbers from transfinite ones, are consistent with Richard Dedekind's revolutionary definition of infinite sets mentioned earlier: A set *S* is infinite if and only if there exists a proper subset *S*'of *S* such that the elements of *S*' can be put into one-to-one correspondence with those of *S*.

What has Cantor done with this new "generalized" notion of number? A could such a generalization apply in a consistent way to finite and infinite numbers? What are the major implicit cognitive steps he has gone through? With the cognitive tools described in the previous section we can now analyze the conceptual structure underlying Cantor's transfinite cardinals.

1) Cantor's Metaphor: SAME NUMBER AS IS PAIRABILITY

In order to characterize his notion of power (*Mächtigkeit*), or cardinal number, for infinite sets (or aggregates, *Mengen*), Cantor makes use of a very important conceptual metaphor: SAME NUMBER AS IS PAIRABILITY (EQUIVALENCE) (for details, see Lakoff & Núñez, 2000). This metaphor allows him to create the conceptual apparatus for giving a precise metaphorical meaning to the comparison of number of elements (i.e., power, cardinality) of infinite sets. This is how this works.

The everyday notions of "same numbers as" and "more than" are, of course, based on the experience we have with finite –not infinite– collections. The following are everyday (non-formal) characterizations of these finite notions:

- **Same Number As**: A (finite) collection (or aggregate) *A* has the same number of elements as (a finite) collection *B* if, for every member of *A*, you can take away a corresponding member of *B* and not have any members of *B* left over.
- **More Than**: A (finite) collection (or aggregate) *B* has more elements than (a finite) collction *A* if, for every member of *A*, you can take away a member of *B* and still have members left in *B*. If collection *A* happens to be contained in (is a proper subset of) *B*, the sub-collection of elements *left over* after the matching is equal to the sub-collection of elements in *B* that are not in *A*.

There is nothing uncontroversial about these everyday notions, to the point that we totally take them for granted. In fact, decades ago, the Swiss psychologist Jean Piaget described in detail how these fundamental notions get organized quite early in children's cognitive development without explicit goal-oriented education (Piaget, 1952, Núñez, 1993). So, if we extend the *left-over* idea to infinite cases, and approach the question "Are there more natural numbers than even numbers?" equipped exclusively with the ordinary notions of "same number as" and "more than," the answer is straightforward. We can match the elements of both sets as shown in Figure 6 and arrive to the conclusion that there are indeed more natural numbers, because there are the odd numbers *left over*. Following the previous characterization of “more than,” the collection of even numbers is contained in (is a proper subset of) the collection of natural numbers, and therefore what is *left over* after the matching corresponds to the sub-collection of elements in the natural numbers that are not in the collection of even numbers. This is nothing other than the sub-collection of odd numbers. In this sense an answer based on a natural notion of “more than” is unambiguous.

[Insert Figure 6 about here]

But, it is true that the two sets, if arranged properly, are also pairable (equivalent) in the sense that we can put them in a one-to-one correspondence as shown earlier in Figure 3. Pairability and “same number as,” however, are two very different ideas. They do have the same extension for finite collections, as Cantor carefully pointed out when constructing the natural numbers from scratch (i.e., they cover the same cases giving the same results). However, they are cognitively different and their inferential structures differ in important ways. In his investigations into the properties of infinite sets, Cantor used the concept of *pairability* (equivalence) in place of our everyday concept of *same number as.* In doing so, and by implicitly dropping the “left-over” idea, he established a conceptual metaphor, in which one concept (same number as) is conceptualized in terms of the other (pairability). Figure 7 shows the mapping of Cantor’s simple but crucial conceptual metaphor.

[Insert Figure 7 about here]

It is very important to understand that this new conception of number is metaphorical in nature. By simply being able to establish pairability one doesn't get too far. As we said earlier, this is exactly what happened to the brilliant Galileo two centuries before Cantor. In order to be able to extend the notion of cardinality from finite sets, which we can literally count, to infinite sets, which we cannot literally count, we do need to actively and fully ignore the "left over" clause embedded in the ordinary notion of "more than." Only then we can go on with the metaphorical extension to conceive cardinality for infinite sets.

We often see in mathematics books, textbooks, and articles statements like "Cantor proved that there are *just as many* positive even integers as natural numbers." According to a cognitive account of our ordinary notion of "As Many As" Cantor proved no such thing. What Cantor did was simply to prove that the sets were *pairable* (assuming, via the BMI, that you can pair *all* of the natural numbers with their corresponding even integers). It is only via Cantor's metaphor that it makes sense to say that he "proved" that there are, metaphorically, "just as many" even numbers as natural numbers. Unfortunately, many mathematics texts ignore the metaphorical nature of Cantor's new meaning given to the idea of pairability, ascribing to it a kind of transcendental truth, and failing to see its truth as derived from a very human conceptual metaphor. As a consequence, they often conclude that there is something fundamentally wrong with human intuition when dealing with infinity. Consider for instance the following citation:

> "Would it be possible, for example, to match on a one to one basis the set of all counting numbers with the set of all even numbers? At first thought this seems impossible, since there seem to be *twice as many* counting numbers as there are even numbers. And yet, if we arrange all the even numbers in a row according to their magnitude, then this very act already shows that such a matching is possible … So our intuition was wrong!" (Maor, 1991, p. 56, our emphasis).

The same applies to views within mathematics regarding the role of everyday language. Consider the following citation concerning the problem of comparing similar infinite sets: "The confusions and apparent paradoxes in this subject arise from the transfer of everyday language, acquired from experience with finite collections, to infinite sets where we must train ourselves to work strictly with the mathematical rules of the game even though they lead to surprising results." (Sondheimer & Rogerson, 1981, p. 149).

Our cognitive analysis shows that there is nothing wrong with our "intuition" per se. And there is nothing wrong with "everyday language" either. Extensive work in cognitive linguistics shows that conceptual metaphor and conceptual blending are not mere linguistic phenomena, but they are about thought and cognition. In the practice of mathematics what is often called "intuition" or naïve ideas expressed by "everyday language" are in fact very well organized conceptual structures based on bodily-grounded systems of ideas with very precise inferential structures. But in mathematics, often what counts as primary are the "strict" and rigorous "mathematical rules" (which from a cognitive perspective need to be explained as well!). "Intuition" and "everyday language" are seen as vague and imprecise (for further discussion of this and its implications for formal programs in mathematics see Núñez & Lakoff, 1998; Núñez & Lakoff, submitted).

Consider this other statement: “[Cantor concluded,] there are just as many even numbers as there are counting numbers, just as many squares as counting numbers, and just as many integers (positive and negative) as counting numbers” (Maor, 19, p. 57). In our ordinary conceptual system, this is not true. Not because our intuition is wrong, or because our everyday language is imprecise and vague, but because it is an inference made within a different conceptual structure with a different inferential structure. According to our ordinary notion of “more than” there are indeed more natural numbers than there are positive even integers or squares. And there are more integers than there are natural numbers. There is a precise cognitively-structured logic underlying this inference, which we can make as rigorous as we want. Lack of rigor, then, is not the issue.

This of course doesn’t lessen Cantor’s brilliant results. Cantor’s ingenious metaphorical extension of the concept of pairability and his application of it to infinite sets constitutes an extraordinary conceptual achievement in mathematics. What he did in the process was create a new technical mathematical concept—pairability (equivalence)—and with it, new mathematics.This new mathematics couldn’t have been invented only with our everyday ordinary notions of “same number as” and “more than.” But, as we saw above in Cantor’s original text (and presumably for ideological and philosophical reasons), Cantor also intended pairability to be a *literal generalization* of the very idea of number. An *extension* of our ordinary notion of “same number as” from finite to infinite sets (for historical details see Ferreirós, 1999). There Cantor was mistaken. From a cognitive perspective, it is a metaphorical rather than literal extension of our very precise everyday concept.

2) The BMI and the proof of rationals denumerability

As we saw earlier, Cantor provided a very simple, elegant, and powerful proof of the possibility of establishing a one-to-one correspondence between the natural numbers and a dense set such as the rational numbers. What is rarely mentioned in mathematics texts (to say the least) is that this proof makes implicit use of human cognitive mechanisms such as conceptual metaphor and blending. Consider Cantor's infinite array of fractions shown in Figure 4. There the BMI is used over and over, implicitly and unconsciously, in comprehending the diagram. It is used in each row of the array, for assuring that *all* fractions are included. First, the BMI is used in the first row for assuring that a*ll* fractions with numerator one are included in a *completed* collection, without missing a single one. Then, the BMI is used to assure that *all* fractions with numerator two are *actually* included, and so on. In the same way, the BMI is implicitly used in each column of the array to assure that *all* fractions with denominator one, two, three, and so on, are actually included in this infinite array providing completion to it. Finally, the BMI is used in conceptualizing the endless arrow covering a *completed* path. The arrow covers every single fraction in the array assuring, via the BMI, the possibility of the one-to-one correspondence between *all* rationals and naturals. The BMI together with Cantor's metaphor discussed earlier validate the diagram as a proof that the natural numbers and the rational numbers can be put into one-to-one correspondence and therefore have the same power—that is, the same cardinality.

3) The BMI in Cantor's diagonal proof of the non-denumerability of real numbers

Cantor's celebrated diagonal proof also makes implicit use of the BMI. First, there is the use of the special case of the BMI for infinite decimals. Each line, is of the form $0.a_{j1}a_{j2}a_{j3}$…., where $j$ is a natural number denoting the number of the line. Thanks to the BMI each of these unending lines can be conceived as being completed. It is important to remind that Cantor's diagonal proof requires that all real numbers in the list to be written as non-terminating decimals, which provide another name for the same number. It is the BMI that allows a fraction like 0.5 (with terminating decimals) to be conceived and written as 0.4999… a non-terminating –yet completed-- decimal. Second, there is the use of the special case of the BMI for the set of *all* natural numbers. Each row corresponds to a natural number, and *all* of them must be there. This provides the conditions for testing the assumed denumerability of the real numbers between zero and one. Third, the proof (which works by reductio ad absurdum) assumes that *all* real numbers between zero and one are included in the list. This provides the essential condition for the success of the proof because it guarantees that there is a contradiction if a number is constructed that is not included in the originally assumed *completed* list. This is indeed the case of the new constructed number $0.b_1b_2b_3$…. Fourth, there is the sequence along the diagonal formed by the digits of the form $a_{jk}$ where $j = k$. It, too, is assumed to include *all* such digits on the diagonal. The fact that all real numbers must be written as non-terminating decimals guarantees that a digit $a_{jk}$ when $j = k$ (on the diagonal) is not a part of an endless sequences of zeroes (i.e., an endless sequence of zeroes for digits $a_{jk}$ when $j < k$, which would be the case of a fraction such as 0.5000…). This is another implicit use of the BMI. And finally, there is the process of constructing the new number $0.b_1b_2b_3$… by replacing each digit $a_{jk}$ (with $j = k$ on the diagonal) with another digit. The process is unending, but must cover the *whole* diagonal, and must create the new real number, not included in the original list, written as a non-terminating—yet complete--decimal. Another implicit special case of the BMI.

## Conclusion

In this article I have briefly introduced one aspect of George Cantor's creative work--transfinite cardinals--and I have analyzed some of his celebrated counterintuitive and paradoxical results. Counter-intuitive ideas and paradoxes are very interesting and fertile subject matters for cognitive studies because they allow us to understand human abstraction through conflicting conceptual structures. From the point of view of cognitive science, especially from cognitive linguistics and Mathematical Idea Analysis, it is possible to clarify what makes Cantor's results counterintuitive. These analyses show also that, contrary to many mathematicians' and philosophers of mathematics' beliefs, the nature of potential and actual infinity can be understood not in terms of transcendental (or platonic) truths, or in terms of formal logic, but in terms of human *ideas*, and *human cognitive mechanisms*. Among the most important mechanisms for understanding the cognitive nature of transfinite cardinals and actual infinities are:

- Aspectual systems; with iterative and continuative processes, perfective and imperfective structures with initial states, resultant states, and so on.
- Conceptual metaphors, such as Cantor's Metaphor SAME NUMBER AS IS PAIRABILITY.

- Conceptual blending, such as the multiple implicit uses of the BMI, the Basic Mapping of Infinity, in Cantor's proofs.

These mechanisms are not mathematical in themselves. They are human embodied cognitive mechanisms, realized and constrained by the peculiarities of human bodies and brains.

Transfinite cardinals are the result of a masterful combination of conceptual metaphor and conceptual blending done by the extremely creative mind of Georg Cantor, who worked in a very prolific period in the history of mathematics. These ideas and the underlying cognitive mechanisms involved in Cantor's work, are bodily-grounded and *not arbitrary*. That they are not arbitrary is a very important point that often gets confused in the mathematical and sometimes the philosophical communities where human-based mechanisms are often taken to be mere "social conventions." What is ignored is that species-specific bodily-based phenomena provides a biological ground for social conventions to take place. This ground, however, is not arbitrary. It is in fact constrained by biological phenomena such as morphology, neuroanatomy, and the complexity of the human nervous system (Varela, Thompson, and Rosch, 1991; Thelen & Smith, 1994; Núñez & Freeman, 1999). Abundant literature in conceptual metaphor and blending tells us that source and target domains, input spaces, mappings, and projections are realized and constrained by bodily-grounded experience such as thermic experience, visual perception and spatial experience (Johnson, 1987; Lakoff, 1987). In the case of transfinite numbers these constraints are provided by container-schemas for understanding (finite) collections and their hierarchies, genetically-determined basic quantity-discrimination mechanisms (e.g., subitizing), visual and kinesthesic experience involved in size comparison and the matching of elements, correlates between motor control and aspect, and so on (for details see Lakoff & Núñez, 2000, Chapter 2). The strong biological constraints operating on these mechanisms provide very specific inferential structures which are very different from non (or weakly) constrained "social conventions" like the color of dollar bills or the font used in stop signs. These non-arbitrary cognitive mechanisms, which are essential for the understanding of conceptual structures, can be studied empirically and stated precisely, and cognitive science techniques such as Mathematical Idea Analysis can serve this purpose.

In this article, I mainly referred to transfinite cardinals, as an example of a very rich and interesting case of actual infinity. But this is only one case. Lakoff and Núñez (2000) have shown that there are many other instantiations of actual infinity in mathematics realized via the BMI, such as points at infinity in projective and inversive geometry, infinite sets, limits, transfinite ordinals, infinitesimals, and least-upper bounds. What is important to make clear about these mathematical infinities is the following:

1. They belong to completely different fields within mathematics.
2. They have, from a purely mathematical point of view, their existence guaranteed by very specific tailor-made axioms in various fields. In set theory, for instance, one can make use of infinite sets simply because there is a specific axiom, the *axiom of infinity*, that grants their existence. The existence of other mathematical actual infinities in other fields is guaranteed by similar axioms.

With this in mind, we can now see the relevance of the BMI:

- It explains with a single mechanism cases of actual infinity occurring in different non-related mathematical fields. Whereas in mathematics actual infinities are characterized by different sets of axioms in different fields, cognitively, they can be characterized by a single cognitive mechanism: the BMI.

- It provides a cognitively plausible explanation of the nature of actual infinity that is constrained by what is known in the scientific study of human cognition, human conceptual structures, human language, and the peculiarities of the human body and brain.

Mathematical axioms, don't have to comply any constraints of this kind, because they only operate within mathematics itself. Therefore, axioms can't provide explanations of the nature of transfinite cardinals, actual infinities, or, for that matter, of mathematical concepts in general. The BMI, along with other cognitive mechanisms, such as conceptual metaphors and the use of aspect, allows us to appreciate the beauty of transfinite cardinals, and to see that the portrait of infinity has a human face.

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Figure 1.

A case of potential infinity: the sequence of regular polygons with *n* sides, starting with *n* = 3 (assuming that the distance from the center to any of the vertices is constant). This is an unending sequence, with no polygon characterizing an ultimate result.

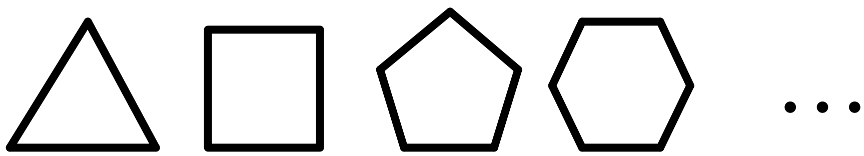

Figure 2.

A case of actual infinity: the sequence of regular polygons with *n* sides, starting with *n* = 3 (assuming that the distance from the center to any of the vertices is constant). The sequence is endless but it is conceived as being completed. The final resultant state is a very peculiar entity, namely, a circle conceived as a polygon with infinitely many sides of infinitely small magnitude.

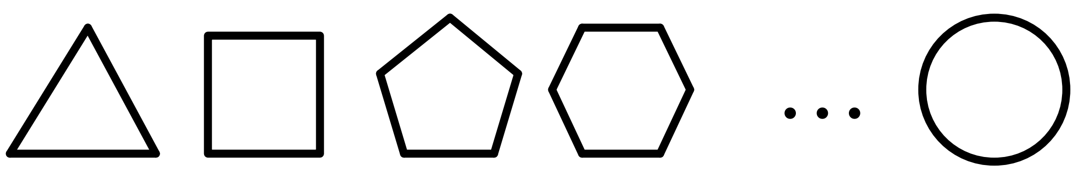

Figure 3.

A mapping establishing the one-to-one correspondence between the sets of natural and even numbers.

| | | | | | |
|---|---|---|---|---|---|
| Natural numbers | 1 | 2 | 3 | 4 | … |
| | ↕ | ↕ | ↕ | ↕ | |
| Even numbers | 2 | 4 | 6 | 8 | … |

Figure 4.

Cantor’s infinite array of rational numbers conceived for the proof of their denumerability. Each fraction covered by the arrow can be mapped to a natural number thus establishing a one-to-one correspondence between the natural and the rational numbers.

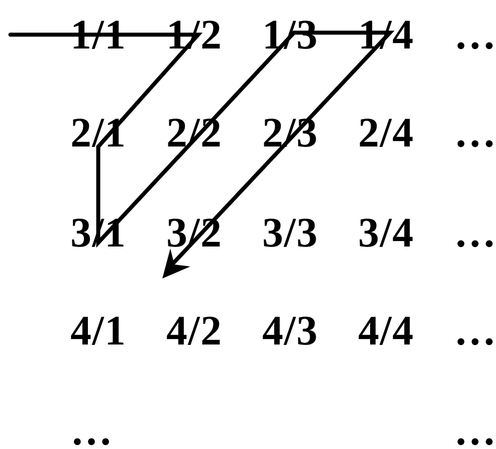

Figure 5.

The BMI, the Basic Mapping of Infinity, as a double-scope conceptual blend.

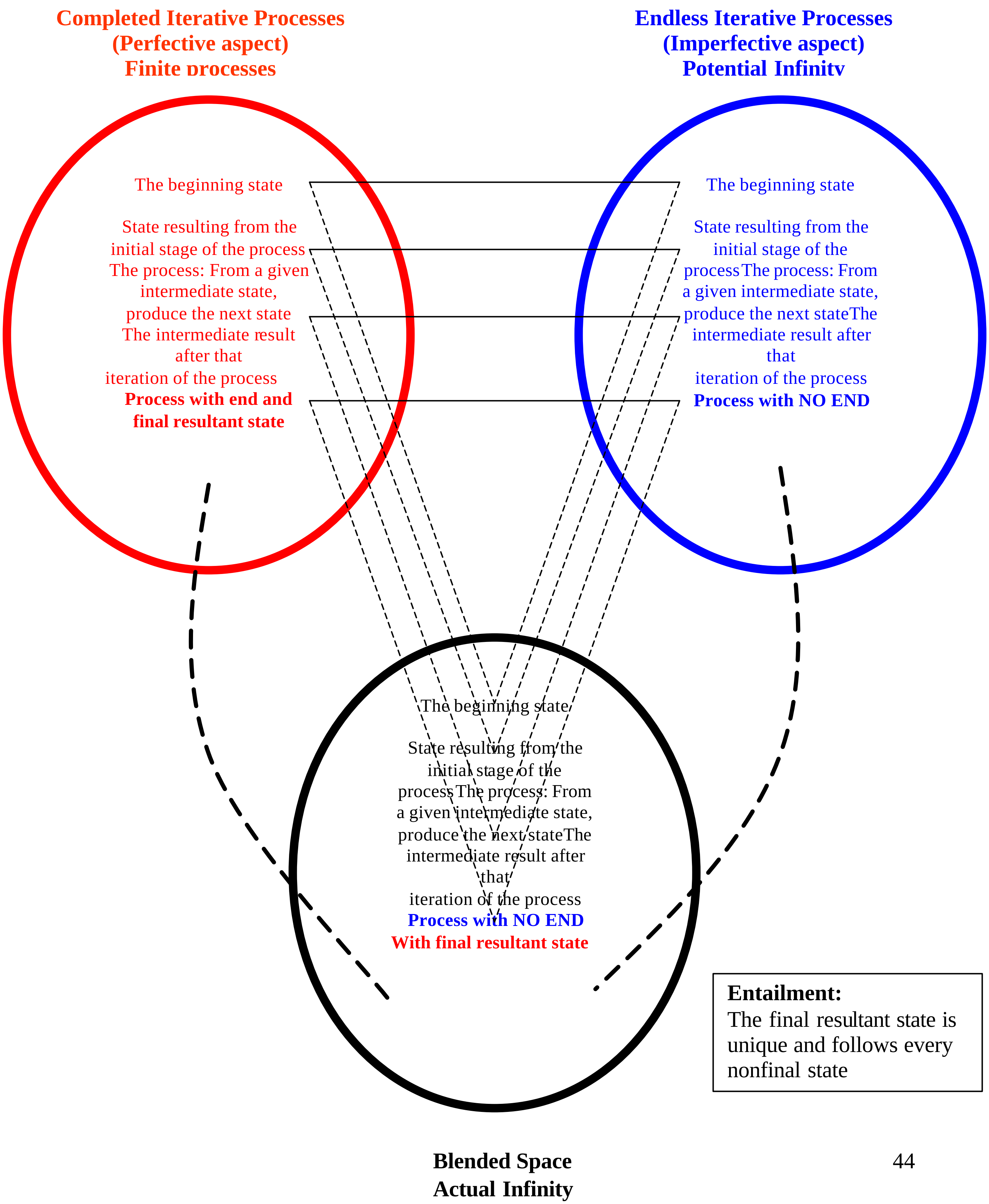

Figure 6.

A mapping between the natural and even numbers based on the ordinary notion of "same as" and "more than." The mapping shows that one can pair elements of the two collections and have the odd numbers left over (shown with a circle). The entailment of this natural mode of reasoning is that there are more natural numbers than even numbers.

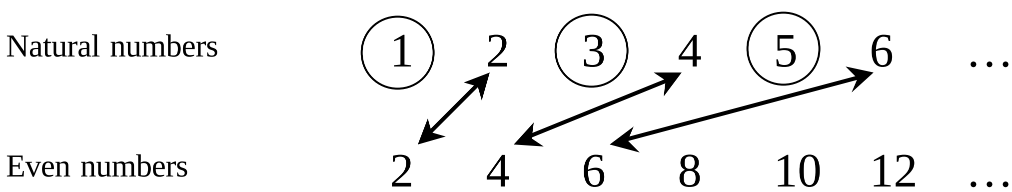


Figure 7.

Georg Cantor's fundamental conceptual metaphor SAME NUMBER AS IS PAIRABILITY. This simple but ingenious metaphor is at the core of transfinite numbers and modern set theory.