applications in a manner analogous to the spinoffs of the switching and information theories. Indeed not every present-day systems engineer would have been able to cope directly with early Caldwell, Huffman, Shannon or Wiener; nevertheless he is now indirectly applying the results of those early investigations. It is hoped, however, that in its final form this prototype will become more accessible.

In warning the reader that his growth patterns should be seen as mathematical constructs rather than biological realities, Grenander quotes Rosen on biological morphogenesis: one investigates the *capability* of models. This reviewer has pointed out elsewhere that the similarity of patterns occurring at widely different scales is due to the fact that the specific nature of interactive forces is frequently superseded by the properties of three-dimensional space, which permit but a limited repertoire of patterns and connectivities. Therefore these mathematical constructs have a validity in equilibrium and steady-state systems, regardless of specific interactive forces.

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Spectral synthesis, by John J. Benedetto, Academic Press, Inc., New York, 1975, 278 pp., $27.50.

Let Φ be in $L^\infty(\mathbf{R})$. If Φ can be written as

$$\Phi(x) = \sum_{k=1}^{n} c_k \exp(ixy_k),$$

then the set of characters $\{\exp(ixy_k): k = 1, \ldots, n\}$ is called the spectrum of Φ and denoted $\operatorname{sp}\Phi$. The set of translates of Φ spans a finite-dimensional subspace $\mathfrak{T}_\Phi$ of $L^\infty(\mathbf{R})$, namely the linear span of $\operatorname{sp}\Phi$. In fact, $\operatorname{sp}\Phi$ is exactly the set of characters $\exp(ixy)$ belonging to $\mathfrak{T}_\Phi$. Thus the linear span of the translates of Φ is determined by its spectrum. The problem of spectral synthesis for bounded functions is to study suitable generalizations of this simple observation. That is, given Φ in $L^\infty(\mathbf{R})$, is the smallest translation-invariant subspace of $L^\infty(\mathbf{R})$ containing Φ and closed in some topology generated by the spectrum of Φ ? The problem has been studied with various topologies on $L^\infty(\mathbf{R})$, but for many purposes the most suitable is the weak-* topology. Also the setting is often generalized to a locally compact abelian group G with character group Γ. In our discussion above, $G = \mathbf{R}$ and $\Gamma = \{\exp(ixy): y \in \mathbf{R}\}$.

For the more general set-up, let Φ be in $L^\infty(G)$ and let $\mathfrak{T}_\Phi$ be the smallest weak-*-closed translation-invariant subspace of $L^\infty(G)$ containing Φ. For any weak-*-closed translation-invariant subspace $\mathfrak{T}$ of $L^\infty(G)$, we define its spectrum as $\mathfrak{T} \cap \Gamma$. And the spectrum of Φ is, by definition, the spectrum of $\mathfrak{T}_\Phi$. The spectrum is a closed subset of Γ and every closed subset E of Γ is the spectrum for at least one $\mathfrak{T}$. If there is exactly one $\mathfrak{T}$, i.e. if E determines $\mathfrak{T}$ in this sense, then E is called a spectral set.

The problem of synthesis in $L^\infty(G)$ can equally well be regarded as a problem in $L^1(G)$ since there is a one-to-one correspondence between weak-*-closed translation-invariant subspaces $\mathfrak{T}$ of $L^\infty(G)$ and the closed ideals I of $L^1(G)$. The ideal I corresponding to $\mathfrak{T}$ is simply $\mathfrak{T}^\perp = \{f \in L^1(G): \langle \Phi, f\rangle = 0 \text{ for all } \Phi \in \mathfrak{T}\}$. The spectrum of $\mathfrak{T}$ then turns out to be equal to the zero-set of I, namely $Z(I) = \{\gamma \in \Gamma: \hat{f}(\gamma) = 0 \text{ for all } f \in I\}$. Thus a closed set E in Γ is a spectral set provided it is the zero-set of exactly one closed ideal in $L^1(G)$. The largest closed ideal in $L^1(G)$ with zero-set E is $k(E) = \{f \in L^1(G): \hat{f} = 0 \text{ on } E\}$ and the smallest such closed ideal is the closure of $j(E) = \{f \in L^1(G): \hat{f} = 0 \text{ in a neighborhood of } E\}$.

The so-called abstract Wiener-Tauberian theorem asserts that the empty set is a spectral set. This tells us, for example, that if the Fourier transform $\hat{f}$ of a function f in $L^1(\mathbf{R})$ never vanishes, then the smallest closed ideal in $L^1(\mathbf{R})$ containing f is $L^1(\mathbf{R})$ itself. This result is called the Wiener-Tauberian theorem because Wiener used it to establish a number of theorems of Tauberian type. Incidentally, Wiener argued that it would be far more appropriate to call them Hardy-Littlewood theorems. In any case, the abstract Wiener-Tauberian theorem is now a simple result in harmonic analysis, but it has far-reaching consequences.

One of the major problems in spectral synthesis was whether spectral synthesis holds, i.e. whether every closed set in Γ is a spectral set. The answer is easily affirmative if Γ is discrete. The first example of a nonspectral set was given by Laurent Schwartz in 1948. His example is the surface S of the unit ball in $\mathbf{R}^n$ where $n \geqslant 3$. Hence there are two distinct closed ideals in $L^1(\mathbf{R}^3)$ whose zero-set is S, namely $k(S)$ and $\overline{j(S)}$. Actually, a result of Helson and others implies that there is a continuum of distinct closed ideals in $L^1(\mathbf{R}^3)$ whose zero-set is S. The problem remained open for such familiar groups as the circle group $\mathbf{T}$ and $\mathbf{R}$ until 1959 when Paul Malliavin proved the remarkable fact that every nondiscrete Γ contains a nonspectral set. A very interesting proof using tensor products was given by Varopoulos in 1965 with an improvement by Carl Herz. In 1972 Aharon Atzmon settled the related principal ideal problem by showing that if Γ is nondiscrete then $L^1(G)$ contains a closed ideal which is not finitely generated, much less singly generated.

Meanwhile, a good deal of energy was expended showing that many nice sets are spectral sets. Carl Herz (1960) showed that locally star-shaped bodies in $\mathbf{R}^n$ are spectral sets using ideas originating with Calderón (1956). In 1956 Herz showed that the Cantor set in $\mathbf{R}$ is a spectral set and in 1958 he showed that circles in $\mathbf{R}^2$ are spectral sets. In 1965 Varopoulos proved that all 0-dimensional Kronecker sets are spectral sets; Saeki later removed the need for 0-dimensionality. In 1970 Yves Meyer showed that a number of perfect sets in $\mathbf{R}$ defined by means of Pisot numbers are spectral sets.

Almost all books in harmonic analysis give some attention to spectral synthesis. They often obtain Malliavin's theorem and discuss examples. See, for example, Kahane and Salem [3], Katznelson [4] and Rudin [8]. Hewitt and Ross [2] deal with these matters and also give a fairly complete treatment on Wiener-Tauberian theorems. Reiter [7] omits Malliavin's theorem but gives a careful exposition on spectral synthesis, including his theorems concerning the relationship between a spectral set in G and in a closed subgroup or a quotient group. Meyer [6] devotes a chapter to synthesis with emphasis on his work on Pisot numbers. Lindahl and Poulsen [5] also have a brief chapter on Pisot numbers (written by Meyer), a chapter on Varopoulos's theorem that Kronecker sets in the circle group **T** are spectral sets, and a lot of other results on spectral synthesis. Benedetto [1] proves Malliavin's theorem.

The book under review is the first book known to the reviewer that is devoted solely to spectral synthesis. The book consists of three chapters. In the first chapter the author establishes notation and terminology and, implicitly, his points of view. One of these is that all the action is really on the character group Γ. Thus $L^1(G)$ is replaced by $A(\Gamma) = \{\hat{f}: f \in L^1(G)\}$; $A(\Gamma)$ is given the norm $\|\hat{f}\|_A = \|f\|_1$ and hence is isometrically isomorphic with $L^1(G)$. The conjugate space $A'(\Gamma)$ of $A(\Gamma) \cong L^1(G)$ is of course isometrically isomorphic with $L^\infty(G)$ but is written as $A'(\Gamma)$ to emphasize the functional point of view and its relationship with Γ. Since $A(\Gamma) \subseteq C_0(\Gamma)$ [by the Riemann-Lebesgue lemma], $A'(\Gamma)$ contains all the finite regular Borel measures on Γ. Just as with Radon measures and distributions, the support $\operatorname{supp} T$ can be defined for any element T in $A'(\Gamma)$; $\operatorname{supp} T$ is a closed subset of Γ. Accordingly, the elements T of $A'(\Gamma)$ are called pseudomeasures on Γ. If T is the pseudomeasure in $A'(\Gamma)$ corresponding to the L^∞-function Φ on G, then the spectrum of Φ is exactly the support of T. For a closed set E in Γ, let $A'(E) = \{T \in A'(\Gamma): \operatorname{supp} T \subseteq E\}$ and let $A'_S(E) = \{T \in A'(E): \langle T, \varphi\rangle = 0 \text{ for all } \varphi \in k(E)\}$. [Note that henceforth we regard $k(E)$ and $j(E)$ as ideals in $A(\Gamma)$.] Then $A'(E) = \overline{j(E)}^\perp$ and $A'_S(E) = k(E)^\perp$ so that E is a spectral set if and only if $A'(E) = A'_S(E)$. The pseudomeasures in $A'_S(E)$ are called synthesizable pseudomeasures (supported by E). These and other elementary facts and definitions are set down in Chapter 1. Throughout this chapter the author weaves in historical comments and relates the theory to other areas. In fact, the chapter includes discussions on integral equations, the theory of light, the theory of tides, cubism, the theory of filters, music, interferometers, and the Heisenberg principle.

The third chapter is concerned with the more difficult and technical aspects of spectral synthesis. The first section deals with nonsynthesis and contains a thorough discussion of both Malliavin's and Varopoulos's proof of Malliavin's theorem. The second section concerns the synthesizability of pseudomeasures and of functions φ in $A(\Gamma)$: φ is synthesizable provided $\langle T, \varphi\rangle = 0$ for all pseudomeasures supported by the zero-set $Z\varphi$ of φ, i.e. φ belongs not only to $k(Z\varphi)$ but to $\overline{j(Z\varphi)}$. This section contains a proof of the Beurling-Pollard theorem, which asserts that functions in $\operatorname{Lip}_\alpha(\mathbf{T}) \cap A(\mathbf{T})$, $\alpha \geqslant 1/2$, are synthesizable, and Katznelson's theorem, which asserts that functions in $A(\mathbf{T})$ of bounded variation are synthesizable.

The most distinctive chapter in the book is Chapter 2 on Tauberian theorems. As mentioned before, the direct connection between spectral synthesis and Tauberian theorems is the fact that the empty set is a spectral set. The history of the subject isn't so direct and Chapter 2 indicates how the theory developed, especially how Wiener developed it. In the late 1920's Wiener was led to the spectral analysis of L^∞-functions on $\mathbf{R}$, equivalently of pseudomeasures. Since these objects are usually not L^1-functions or L^2-functions, the ensuing analysis was termed generalized harmonic analysis. Chapter 2 begins by introducing the "Wiener spectrum" and proving Wiener's Tauberian theorem. Beurling's theorem relating the spectrum of a bounded uniformly continuous function Φ on $\mathbf{R}$ with the Beurling spectrum is proved using Wiener's Tauberian theorem. It is interesting to note, as Benedetto does, that Beurling's original proof did not use the Tauberian theorem and that the Tauberian theorem can be deduced from Beurling's theorem. Next a vast array of classical Tauberian theorems is given. Wiener's proof of the prime number theorem based on his Tauberian theorem is also given. Next we find Wiener's original proof of his famous inversion theorem [if f in $A(\mathbf{T})$ never vanishes, then $1/f$ is in $A(\mathbf{T})$] and the more general Wiener-Levy theorem. The latter theorem was later generalized to commutative Banach algebras with unit by Gelfand (1941) from which Wiener's original inversion theorem is immediate. The last section of Chapter 2 is more modern in flavor and includes Ditkin's theorem, a brief study of Calderón sets and Herz sets, and concludes with a proof of Varopoulos's theorem that Kronecker sets E are spectral sets, in fact they support no "true pseudomeasures" (meaning "truly pseudo" measures), i.e. $A'(E)$ consists exactly of the regular Borel measures supported by E.

The book contains a wealth of material, though much of it can be found in other books and journals. The book is well motivated and well documented. Unfortunately, in my opinion the book is not well organized. It is easy to find some useful item someplace and then lose it forever. The index is complete as far as definitions of terms is concerned but not of much use in finding references to an item that may well be sprinkled throughout the book. For instance, an entry on examples of nonsynthesis sets would have been helpful. The index refers to sections, not pages, and this is a hindrance since the same number may be used for a section, proposition, theorem, formula and exercise. Section 1.4.1 contains formulas 1.4.1–1.4.8 and Propositions 1.4.1 and 1.4.2, while Theorem 1.4.1 shows up later in Section 1.4.5. It is also difficult to find a result by a frequently quoted author; Beurling is referred to in 36 sections and exercises!

The question remains: For whom is the book suitable? The specialist in spectral synthesis will of course find the book invaluable, but may become frustrated chasing down useful tidbits. I would not recommend the book to the casual nonspecialist who wants to learn a little about the subject. For the person who wants to learn commutative harmonic analysis I would recommend several less specialized books instead. And for the student interested in spectral synthesis I would recommend Benedetto's book in conjunction with other books on the subject such as [**8**], [7], [**2**], [**5**] or [**1**]; incidentally [**1**] seems better organized than the present text. A lot can also be learned by going back to the original sources, for instance [**9**].

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The theory of approximate methods and their application to the numerical solution of singular integral equations, by V. V. Ivanov, Noordhoff International Publishing, Schuttersveld 9, P. O. Box 26, Leyden, The Netherlands, xvii + 330 pp., price Dfl. 70,--.

The main theme of the book is the numerical solution of singular integral equations with Cauchy kernels. The following set up is typical. Let γ denote the unit circle in the complex plane and consider the equation:

$$K\phi \equiv K^0\phi + \gamma k\phi = f, \tag{1}$$

where the dominant operator K^0 and the operator k are given by

$$K^0 \equiv a(t)\phi(t) + (\pi i)^{-1}b(t)\int_\gamma \phi(\tau)(\tau - t)^{-1}d\tau, \quad k\phi \equiv \int_\gamma k(\tau, t)\phi(\tau)d\tau,$$

with the first integral having its Cauchy principal value. In the classical theory (cf. [**1**]) the solution ϕ is sought in the Hölder class $H(\alpha,\gamma)$ $(0 < \alpha \leqslant 1)$, and the coefficient functions together with $(k\phi)(t)$ are assumed to be Hölder continuous on γ. There is also an L_p theory in which these restrictions on the coefficient functions and the kernel $k(\tau, t)$ are relaxed somewhat.

There is a good case and a bad case. The good case is when $a^2 - b^2 \neq 0$ on γ, and bad is "not good".

In the good case, K is normally solvable, there is a simple formula for the