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MODERN SYMMETRY

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Abstract—In previous studies [1, 2], the author—writing primarily as a geometric abstract artist—has attempted an approach to symmetry/asymmetry from a phenomenological point of view taking mathematics to be regarded as "... the theoretical phenomenology of structure" [3].

The main ideas of what is accepted as the mathematical treatment of symmetry have an extremely long history. It is not even clear that the study originates with the ancient Greeks; suffice to say it can be loosely regarded as a part of geometry and as such is therefore one of the earliest forms of "science" we can find. What we can say is that so far the earliest example of a "system" remains the books of Euclid, and it is generally accepted that the priority here was the methodology rather than the individual theorems. However, although research may change this, "Euclid" is a cornerstone in Western mathematics, and the "elements" include topics not restricted to quantitative and "metrical" geometry.

As a system or branch of mathematics the part we call plane geometry is closed, there are no more theorems to be discovered therein. However, this is not true of many topics we can regard as initiated by "Euclid" such as projective geometry and polytope theory. If one is interested in symmetry, the implications of the "classical concept", we find that mathematicians and also physical scientists—particularly chemists—were initiating a more abstract concept of symmetry which nevertheless could be seen as the result of contemplating fundamental features of very simple "structures". It is tempting to call this concept topological symmetry, but the term has not gained any currency despite the fact that the argument, if naive, is neither illogical nor a solecism. The notion will suggest that what is called symmetry and asymmetry can exist as features of a connected structure which remain invariant under certain simple deformations such that the feature then can be regarded as strictly qualitative and independent of quantitative considerations, thus belonging to the elastic geometry—"elastic lines" as in "rubber sheet" geometry, as topology is often described in the literature of popularizing (science and mathematics).

What are known as the five Platonic solids were generally conceived as literal "solids", the forms known as the tetrahedron, cube, octahedron, dodecahedron, icosahedron, most generally conceived as crystallographic or volumic "sculptural" forms. Essentially of course they can be seen as another set of forms, and perhaps the earliest well-known examples are the drawings of Leonardo da Vinci in which they appear as "skeletal" forms. These representations were the step which led to the geometry of the solids being represented "schematically" so that the prism-shaped "limbs" ("edges") of Leonardo's "closed lattices" could be replaced by the lines ("wires" in the case of a model) or the pencil lines on paper of the linear models. The final stage in this development had to wait for the spirit of the modern/abstract way of conceiving structures—we finally arrive at the Schlegel diagram in which the lengths of the lines and the area of the faces no longer carry over the "symmetry" of the figures represented. Finally it becomes immaterial whether the "lines" follow any regular feature, i.e. they need not be "straight", and in drawing them with "curves" these curved lines may in fact be uniquely different. We have "joined up" or connected a set of points just as we please, and what is drawn can still be regarded as pertaining to some form of physical structure; the wayward paths or "connectives" can be "seen" as elastic strings each of which seems to have undergone a unique deformation as if the "elasticity" of each line were intrinsically different or unique.

It is often pointed out as strange that the ancient Greeks did not notice the fundamental qualitative law which holds between the relation of the dimensional elements of a polyhedral structure, now referred to as points (zero-dimensional)—"corners"; lines (one-dimensional)—"edges"; and polygons (two-dimensional) or "faces". It seems Descartes was almost able to grasp the notion but it was Euler by whose name the famous theorem is known, not surprisingly since Euler was a founding father of topology and with his theorem a vast edifice of theorems having to do with connectivity was initiated.

If we look at the newer form of geometry as the study of structural features of "amorphous" or informal linear structures, schematic diagrams of degrees of connectivity, what we then say of the five Platonic structures is that each of the respective sets of elements is unidentifiable, interchangeable; we have a structure of utmost regularity—and redundancy—belonging to the set of regular coverings in two-dimensional space: plane tessellations either infinite—the three lattices made respectively from three-, four- and six-sided cells or polygons—or closed, as in the five closed systems which exhaust the possibilities for a closed system. When we ask: what else is there? the answer is that we can exhibit structures all of whose respective elements are distinguishable and permit no interchanging. Such structures are described as asymmetric. Finally we show that a structure may have some of its respective elements interchangeable while some remain identifiable; we don't call these "both symmetric and asymmetric" but say that they are symmetric by virtue of exhibiting some symmetries.

It was only in the last century that the daunting task of enumerating all possible polyhedral structures attracted the attention of mathematicians. While much has been learnt between the first efforts and today, no-one is very confident that the problem is going to suddenly become easy and in due course solved. From the point of view of symmetry we discover that when we ask about the possibilities the answer is that polyhedra with any symmetry at all fade out of the "catalogue" as the "size" gets greater, i.e. as the number of vertices (points or corners), edges (or lines) and faces increases. So, nearly all polyhedra are asymmetric! There are then three sets of symmetric polyhedra: the famous Platonics, a mere five; the no less famous Aristotelian solids exhibiting an almost equally high degree of symmetry, of which there are 13, another set revered, and rightly so, by the ancient greeks; and lastly an infinite but diminishing set which struggle for existence, as one might put it, exhibiting various "degrees of symmetry".

By contrast we can construct (or "exhibit") a family of linear structures which, conversely, are linear all symmetric. This family did not "exist" until it was "invented" at the turn of the century. The family or form is known as a *tree* and we can confidently add that it was there all the time but had not been looked at mathematically.

A tree, like the Schlegel diagram and maps, is a topologically linear structure, one-dimensional since it consists of lines and points, but so connected that there are no closed areas—loops, polygons, circuits, faces . . . The enumeration of classes or families of trees has been solved although the overall problem continued to look rather intractable until recently. Amongst symmetric trees there is one special family or species which can be defined such that, however large, they will always be symmetric. Such trees, while being quite simple structures, have as yet no simple short name and are known as *homeomorphically irreducible*. The instruction goes like this: your tree must not contain points of degree two, which means that each point connects at least three lines or only one line, the latter being called the terminal points of the tree. For example, some capital letters of the Roman alphabet are trees of this kind—T, Y, X, G, H, K—while others are not, having points of degree two—A, E, F, L, M, N, V, W, Z.

Let us look at the following question of "pattern making"—it can be looked upon as purely mathematical or as belonging to gestalt theory or even aesthetics: we wish to partition circles with homeomorphically irreducible trees—we will call them *hitrees*. If we take each hitree in turn and

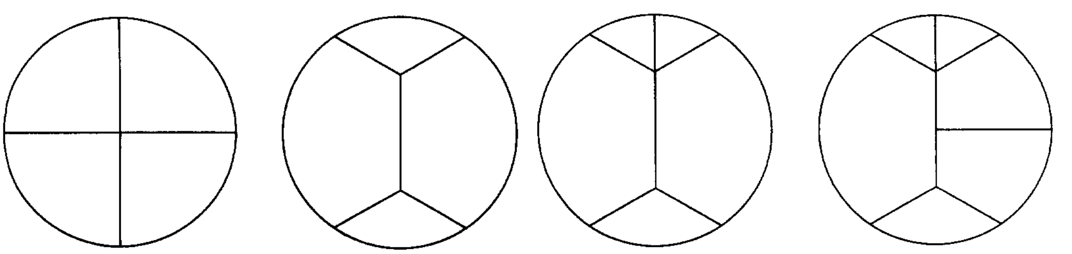

Fig. 1

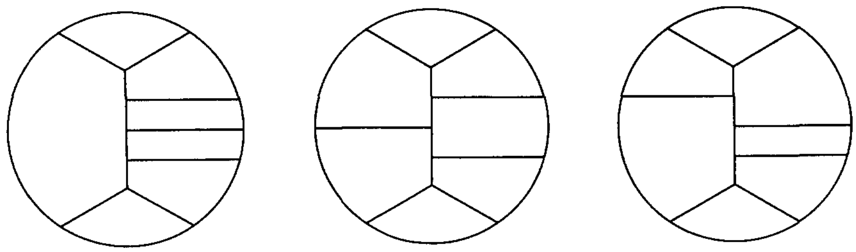

Fig. 2

use it as a form of partition we soon discover that we can obtain several distinguishable partititions from the same hitree. In mathematical terms we say that a tree may take a distinct number of embeddings and this is a feature of its symmetry. Thus a tree may permit only one embedding, as shown in Fig. 1 (and here we keep to hitrees). While others will clearly allow more than one (see Fig. 2).

If we were to exhaust all such patterns or partitions, up to say hitrees with 20 lines, we would soon discover that an increasing number of them would be asymmetric despite the fact that all of the hitrees employed and all that we may choose are symmetric.

Now it should come as no surprise that each pattern (or partition) is in fact a representation of a polyhedron, if we replace the circle by a polygon so that the terminal points of the hitress are joined by straight lines or edges we have changed nothing but the new "diagram" or "figure" can be taken as a Schlegel diagram. Thus, there exists a species of polyhedron generated in this manner: mathematically they would be described as having a *homeomorphically irreducible spanning tree*.

Returning to our plane patterns or partitions (or maps) we could ask that the circle be replaced by a rectangle—for example a square; let us also specify that the lines of the tree partitioning our square are to be parallel with its sides. Our pattern consists of horizontal and vertical lines only, thus the cells or areas of the partition will be "rectangles". In this special family of partitions the recognizable (but not topologically) distinct partitions of the circle with "orthogonally embedded" trees produces many more possibilities and these are of course increased when the circle is replaced by the square (see Fig. 3).

Whereas we can find only two for one of our trees when the surround is a circle, as soon as the circle is replaced by the square it becomes obvious that the number is more than double and the reader can see that it is not hard to reproduce the other four.

Elsewhere I have discussed the fact that these orthogonal partitions constitute the most characteristic compositional schemes of the abstract paintings of Piet Mondrian, certainly between the years 1918–44. The computer scientist Frieder Nake [4] and I were able to propose how to enumerate all possible "Mondrians" in the rectangular format. This can be extended to deal with the lozengical format which Mondrian frequently adopted, and this in turn comes up with some surprising results.

The example on the right-hand side is in fact the scheme chosen for what must be one of Mondrian's most strikingly simple compositions as the painting consists of just the two intersecting black lines (or in the hitree) on a white lozengical format. Of course what the viewer is confronted

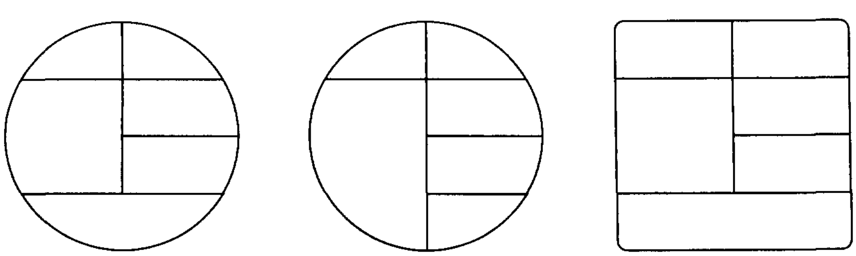

Fig. 3

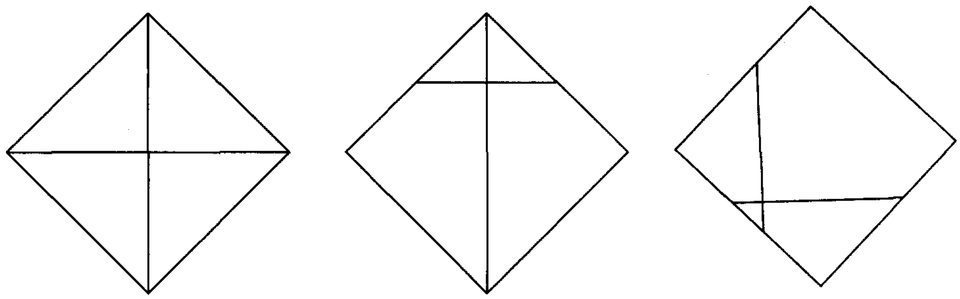

Fig. 4

with contains many other features, the lozenge appears not to be a perfect square, the lines are not of exactly the same thickness, and when we examine the resulting polygons it is clear that even if the lines were of the same thickness the arrangement has no metrical symmetry, the two areas adjacent to the two sides of the triangular area are not of the same size, and it follows that the remaining area is not "symmetric". Bisecting the triangle and extending the "axis" helps one see this to be the case. One need hardly add that depending where the point of intersection of the lines (the four degree node of the tree) is placed the artist could choose a great number of possibilities by which he could in this manner partition the lozenge.

Despite his importance for modern art as one of the most respected "geometric" abstract painters and a founder of abstract art, Mondrian took no interest or inspiration in any aspect of mathematics, not the time-honoured golden section nor any other formula; the idea of calculation was inimical to him; his works, although extremely rigorous and perfectionist, resemble more the free toccata than the canonical fugue. It is often stressed in art historical exegesis that he belonged to the mystical and spiritual stream; whatever truth there may be in that his essential importance is that of a radical plastician, perhaps the very last *painter*, a constructive painter no longer relating to *La Belle Peinture* and the continuous rhetoric of painting as it had been from Lascaux to Van Gogh. His ideas of space, time, surface, structure came out of cubism and took painting—as with other great innovators of the time—to the position of the tableau object, the autonomous plastic art work. For the initiated modern artist there is no turning back from this arrival point: it is the watershed.

To the artists who followed in the wake of neo-plasticism (Mondrian and the group known as De Stijl)—pioneer constructivism, and the less messianic formalism inaugurated by Jakobson and his school, it is indeed the sciences and mathematics, although not exclusively, which provide a continuing inspiration and link with the scientific ethos as opposed to movements which seek for an identity in such areas a "automatism", the mystic, the unconscious, all somehow part of the modernist thrust along with the ubiquitous expressionism, not to mention the stereotypic "humanism" which is set to optimize the conventional image and icon which characterize the work of the work-a-day artist.

Essentially modernism is not wholly identified with the formalist direction, it indeed recognizes its complement viz. the irrational, the subversive and anarchistic as best demonstrated by the iconoclastic dadaists and the montage pieces by the constructivists. It sees an end to the slothful conventions whereby art is to be equated with what is generally accepted as being such if only because it is done in artists' studios, old conventions jacked up by the inclusion of some modern terms of plastic grammar and syntax, the attempt to project a modern art in the terms of the old—all of this can safely be abandoned. The modern artist has within his grasp modern science, modern mathematics, modern concepts due to various other disciplines, and by relating to these things—although this ensures no guarantee—he continues, paradoxically if you like, the tradition whereby the artist while being an individual (perhaps even a solipsist) works in a context a large part of which parallels and reflects the thinking of the new age. No longer is he the servant of the Church, nor need he replace Church by State (Marxism), but equally no longer is he trapped in the labyrinth of egotistic romanticism which leads to excesses such as self-expressionism, nor of course need he mock the artists of the past by peddling an uncountable variety of "modernistic" formulations of the secular traditions in art; portraiture, "landscape", "still life"—all destined to be a decadent charade which all too easily finds admirers.

Some abstract artists have favoured symmetry and made great works which espouse the notion; one thinks of Brancusi's *Endless Column*. Others, like Mondrian, have strenuously avoided symmetry. They may not have known of the words of the celebrated French biologist Claude Bernard which can be rendered as: "It is the asymmetric that creates life."

The idea of asymmetry has too often been relegated to the areas of the non-important, the non-beautiful, even if it is understood that in nature perfect symmetry is never to be found, only some form of approximation to the mathematical ideal.

Clearly the artist is free to choose—there is both symmetry and asymmetry. However, let us end by stating that in asymmetry the idea of an approximation is rather meaningless. To put it technically, if a connected structure has an automorphism group characterized as being the identity class this means that it has the feature that all its elements are distinguishable; this is a special and most fundamental condition—certainly if regarded from the point of view of information theory, it represents an "absolute" as soon as this can be demonstrated. What is fascinating is that no algorithms exist for determining whether a given structure is asymmetric or not. In the case of large structures it becomes necessary to painfully apply the procedures to determine the automorphism number. Significantly it is in chemistry that such information may prove crucial, and much effort has gone into the research, also undertaken by mathematicians. To that extent the problem remains one of a series that have yet to be solved; the answer will have practical rewards and no less in mathematics a most profound step will also have been taken.

Which subsumes which, the demonstrable metric concept of symmetry/asymmetry or the "abstract" concept of the automorphism group? The latter is equally demonstrable and in order to furnish a proof one is forced to state a piecemeal account of the "neighbourhood situation" of every point and this has to be done by a sequence of simple observations; it is a giant piece of "micro-checking", as one might put it, although one is not dealing in the real micro-world, just the zero-dimensional implications of a strucutre of at most three dimensions. *Generalization* being one of the key strategies in mathematics—"generalize it up to the next dimension"—one can quickly grasp that the concept of symmetry/asymmetry poses many difficult questions. Some we may require for solving recondite problems, others remain more like nightmare chess problems. It is unlikely that art can contribute to this daunting area, as it once did in the Renaissance, but there is no "logical" reason why not since the notions of symmetry/asymmetry belong, in a sense, to both science and art.

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