

# ENERGY CRISIS OR A NEW SOLITON IN THE NONCOMMUTATIVE $CP(1)$ MODEL?

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## **Abstract:**

The Non-Commutative (NC)  $CP(1)$  model is studied from field theory perspective. Our formalism and definition of the NC  $CP(1)$  model differs crucially from the existing one [7].

Due to the  $U(1)$  gauge invariance, the Seiberg-Witten map is used to convert the NC action to an action in terms of ordinary spacetime degrees of freedom and the subsequent theory is studied. The NC effects appear as (NC parameter)  $\theta$ -dependent interaction terms. The expressions for static energy, obtained from both the symmetric and canonical forms of the energy momentum tensor, are *identical*, when only spatial noncommutativity is present. Bogomolny analysis reveals a lower bound in the energy in an unambiguous way, suggesting the presence of a new soliton. However, the BPS equations saturating the bound are not compatible to the full variational equation of motion. This indicates that the definitions of the energy momentum tensor for this particular NC theory, (the NC theory is otherwise consistent and well defined), are inadequate, thus leading to the "energy crisis".

A collective coordinate analysis corroborates the above observations. It also shows that the above mentioned mismatch between the BPS equations and the variational equation of motion is small.

Keywords:  $CP(1)$  model, Noncommutative field theory, Seiberg-Witten map.

## Introduction

Non-Commutative (NC) field theories have turned into a hotbed of research activity after its connection to low energy string physics was elucidated by Seiberg and Witten [1, 2]. Specifically, the open string boundaries, attached to D-branes [3], in the presence of a two-form background field, turns into NC spacetime coordinates [1]. (This phenomenon has been recovered from various alternative viewpoints [4].) The noncommutativity induces an NC D-brane world volume and hence field theories on the brane become NC field theories.

Studies in NC field theories have revealed unexpected features, such as UV-IR mixing [5], soliton solutions in higher dimensional scalar theories [6], to name a few. The inherent non-locality, (or equivalently the introduction of a length scale by  $\theta$  - the noncommutativity parameter), of the NC field theory is manifested through these peculiar properties, which are absent in the corresponding ordinary spacetime theories. Also, solitons in NC  $CP(1)$  model have been found [7], very much in analogy to their counterpart in ordinary spacetime. The present work also deals with the search for the solitons in the NC  $CP(1)$  model. The difference between our field theoretic analysis and the existing framework [7] is explained below. In fact, we closely follow the conventional field theoretic approach in ordinary spacetime [8]. The Seiberg-Witten map [1] plays a pivotal role in our scheme. The Bogomolny analysis of the static energy reveals a lower bound, protected by topological considerations. However, we encounter a small discrepancy between the BPS equations and the variational equation of motion. Although "small" in an absolute sense, the mismatch is conceptually significant for the reasons elaborated below. The above conclusions, drawn from field theoretic analysis, will be corroborated and quantified explicitly in a collective coordinate framework.

It appears natural to attribute the above mentioned problem to the definition of the energy functional of the NC  $CP(1)$  model in particular, and of the NC field theories in general, (since the BPS equations are derived directly from the energy of the system). As it is well known, there are complications in the definition of the Energy-Momentum (EM) tensor in NC field theory [9, 10]. In general, it is not possible to obtain a symmetric, gauge invariant and conserved EM tensor. There are two forms of EM tensor in vogue: a manifestly symmetric form [10], obtained from the variation of the action with respect to the metric, and the canonical form [9], following the Noether prescription. The former is covariantly conserved whereas the latter is conserved. Interestingly, we find that in the particular case that we are considering, that is NC  $CP(1)$  model in 2+1-dimensions, with only spatial noncommutativity, *expressions for the static energy, obtained from both the derivations, are identical.* Moreover the expression for energy is gauge invariant. We show that there is a Bogomolny like lower bound in the energy. However, the subsequently derived BPS equations (that saturate the lower bound), does not fully satisfy the equation of motion.

Let us put our work in its proper perspective. As such, our result in no way questions the consistency of the existing literature [7] on  $CP(1)$  solitons since our model differs crucially from the one considered in [7]. In particular, we have adopted a different NC generalization of the  $CP(1)$  constraint. (We have provided conceptual and technical reasons for allowing such a difference.) Thus it is not expected that the results of [7] will be reproduced. In fact, the energy profile of the localized structure that we uncover, is more sharply peaked and has an  $O(\theta)$  correction, with respect to the ordinary spacetime  $CP(1)$  soliton. (This will be made explicit in the collective coordinate analysis at the end.) Surprisingly, the BPS equations remain

unchanged, although the variational equation of motion is altered.

All the same, we emphasize that, we have provided a well defined NC gauge theory, which conforms to the expected features of such a system. Hence, the clash between the BPS equations and the equation of motion that is revealed here, can have a deeper bearing on the structure of a general NC gauge theory, indicating that the traditional lore of field theory in ordinary spacetime should be applied with greater care in the context of NC field theory.

Our methodology and its difference from the existing one [7] is explained below. There are two basic approaches in studying an NC field theory:

(i) The appropriate NC field theory is constructed in terms of NC analogue fields ( $\hat{\psi}$ ) of the fields ( $\psi$ ) with the replacement of ordinary products of fields ( $\psi\varphi$ ), by the Moyal-Weyl  $*$ -product ( $\hat{\psi} * \hat{\varphi}$ ),

$$\hat{\psi}(x) * \hat{\varphi}(x) = e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\sigma\mu}\partial_{\xi\nu}} \hat{\psi}(x+\sigma)\hat{\varphi}(x+\xi) |_{\sigma=\xi=0} = \hat{\psi}(x)\hat{\varphi}(x) + \frac{i}{2}\theta^{\rho\sigma}\partial_{\rho}\hat{\psi}(x)\partial_{\sigma}\hat{\varphi}(x) + O(\theta^2). \quad (1)$$

The *hatted* variables are NC degrees of freedom. We take  $\theta^{\rho\sigma}$  to be a real constant antisymmetric tensor, as is customary [1], (but this need not always be the case [11]). The NC spacetime follows from the above definition,

$$[x^{\rho}, x^{\sigma}]_* = i\theta^{\rho\sigma}. \quad (2)$$

Note that the effects of spacetime noncommutativity has been accounted for by the introduction of the  $*$ -product. For gauge theories the Seiberg-Witten Map [1] plays a crucial role in connecting  $\hat{\phi}(x)$  to  $\phi(x)$ . This formalism allows us to study the effects of noncommutativity as  $\theta^{\rho\sigma}$  dependent interaction terms in an ordinary spacetime field theory format. This is the prescription we will follow.

(ii) An alternative framework is to treat the NC theories as systems of operator valued fields and to directly work with operators on the quantum phase space, characterized by the noncommutativity condition (2). On the NC plane, the coordinates satisfy a Heisenberg algebra  $[x^1, x^2]_* = x^1 * x^2 - x^2 * x^1 = i\theta^{12} = i\epsilon^{12}\theta = i\theta$  which in the complex coordinates reduces to the creation annihilation operator algebra for the simple Harmonic Oscillator. Thus to a function in the NC spacetime, through Weyl transform, one associates an operator acting on the Hilbert space, in a basis of a simple Harmonic Oscillator eigenstates.

The investigations on the NC  $CP^1$  solitons carried out so far [7, 12] exploit the latter method. As it turns out, a major advantage is that structurally, the NC system with its dynamical equations, energy functionals etc., are similar to their ordinary spacetime counterpart. This happens because the  $*$ -products of (i) are replaced by operator products in (ii) and the spacetime integrals are replaced by trace over the basis states in the hilbert space.

In the present work, our aim is to study the NC  $CP^1$  solitons in the former field theoretic approach. From past experiences [13] we know this to be a perfectly viable formalism. Indeed, since the NC spacetime physics is not that much familiar or well understood, it is imperative that one explores different avenues to reach the same goal, to gain further insights. Also we would like to point out that since solitons are already present in the  $CP^1$  model at  $\theta = 0$  (i.e. ordinary spacetime), unlike the noncommutative solitons of the scalar theory [6] it is natural to analyze the fate of the solitons under a small perturbation, (which is a small value of  $\theta$  in the present case). The small  $\theta$ -results of [7, 12] are perfectly well defined.

The paper is organized as follows: Section **II** contains a short recapitulation of the  $CP(1)$  solitons. This will help us fix the notations and in fact, identical procedure will be pursued

in the NC theory as well. The detailed construction of our version of the NC  $CP(1)$  model is provided in section **III**. Section **IV** discusses the energy momentum tensor of the model. Section **V** consists of the Bogomolny analysis in the NC theory. Section **VI** is devoted to the collective coordinate analysis. The paper ends with a conclusion in Section **VII**.

## Section II - $CP(1)$ Soliton: a brief digression

Let us digress briefly on the BPS solitons of  $CP(1)$  model in ordinary spacetime. Later we will proceed with the NC theory in an identical fashion. The gauge invariant action,

$$S = \int d^3x [(D^\mu \phi)^\dagger D_\mu \phi + \lambda(\phi^\dagger \phi - 1)], \quad (3)$$

where  $D_\mu \phi = (\partial_\mu - iA_\mu)\phi$  defines the covariant derivative and the multiplier  $\lambda$  enforces the constraint, the equation of motion for  $A_\mu$  leads to the identification,

$$A_\mu = -i\phi^\dagger \partial_\mu \phi. \quad (4)$$

Since the "gauge field"  $A_\mu$  does not have any independent dynamics one is allowed to make the above replacement directly in the action. Obviously the infinitesimal gauge transformation of the variables are,

$$\delta\phi^\dagger = -i\lambda\phi^\dagger ; \quad \delta\phi = i\lambda\phi ; \quad \delta A_\mu = \partial_\mu \lambda. \quad (5)$$

From the EM tensor

$$T_{\mu\nu} = (D_\mu \phi)^\dagger D_\nu \phi + (D_\nu \phi)^\dagger D_\mu \phi - g_{\mu\nu} (D^\sigma \phi)^\dagger D_\sigma \phi, \quad (6)$$

the total energy can be expressed in the form,

$$E = \int d^2x (|D_0 \phi|^2 + |(D_1 \pm iD_2)\phi|^2) \pm 2\pi N, \quad (7)$$

where the last term denotes the topological charge

$$N \equiv \int d^2x n(x) = \frac{1}{2\pi i} \int d^2x \epsilon^{ij} (D_i \phi)^\dagger D_j \phi = \int d^2x \frac{1}{4\pi} \epsilon^{ij} F_{ij} = \int d^2x \frac{1}{2\pi} F_{12}, \quad (8)$$

corresponding to the conserved topological current. The Bogomolny bound follows from (7),

$$E \geq 2\pi |N|, \quad (9)$$

with the following saturation conditions (BPS equations) obeyed by the soliton,

$$|D_0 \phi|^2 = |(D_1 \pm iD_2)\phi|^2 = 0. \quad (10)$$

It can be checked that the solutions of the BPS equations belong to a subset of the full set of solutions, that satisfy the variational equation of motion.

## Section III - Construction of the NC $CP(1)$ model

Let us now enter the noncommutative spacetime. The first task is to generalize the scalar gauge theory (3) to its NC version, keeping in mind that the latter must be  $*$ -gauge invariant. The NC action is,

$$\hat{S} = \int d^3x (\hat{D}^\mu \hat{\phi})^\dagger * \hat{D}_\mu \hat{\phi} = \int d^3x (\hat{D}^\mu \hat{\phi})^\dagger \hat{D}_\mu \hat{\phi}, \quad (11)$$

where the NC covariant derivative is defined as

$$\hat{D}_\mu \hat{\phi} = \partial_\mu \hat{\phi} - i \hat{A}_\mu * \hat{\phi}.$$

Depending on the positioning of  $\hat{A}_\mu$  and  $\hat{\phi}$ , the covariant derivative can act in three ways ,

$$\begin{aligned} \hat{D}_\mu \hat{\phi} &= \partial_\mu \hat{\phi} - i \hat{A}_\mu * \hat{\phi} \\ &= \partial_\mu \hat{\phi} + i \hat{\phi} * \hat{A}_\mu \\ &= \partial_\mu \hat{\phi} - i(\hat{\phi} * \hat{A}_\mu - \hat{A}_\mu * \hat{\phi}) \end{aligned} \quad (12)$$

which are termed respectively as fundamental, anti-fundamental and adjoint representations. We have chosen the fundamental one.<sup>1</sup> Notice that for the time being we have not considered the target space (CP(1)) constraint. We will return to this important point later. The NC action (11) is invariant under the  $*$ -gauge transformations,

$$\hat{\delta} \hat{\phi}^\dagger = -i \hat{\lambda} * \hat{\phi}^\dagger ; \quad \hat{\delta} \hat{\phi} = i \hat{\lambda} * \hat{\phi} ; \quad \hat{\delta} \hat{A}_\mu = \partial_\mu \hat{\lambda} + i[\hat{\lambda}, \hat{A}_\mu]_* . \quad (13)$$

We now exploit the Seiberg-Witten Map [1, 16] to revert back to the ordinary spacetime degrees of freedom. The explicit identifications between NC and ordinary spacetime counterparts of the fields, to the lowest non-trivial order in  $\theta$  are,

$$\begin{aligned} \hat{A}_\mu &= A_\mu + \theta^{\sigma\rho} A_\rho (\partial_\sigma A_\mu - \frac{1}{2} \partial_\mu A_\sigma) \\ \hat{\phi} &= \phi - \frac{1}{2} \theta^{\rho\sigma} A_\rho \partial_\sigma \phi ; \quad \hat{\lambda} = \lambda - \frac{1}{2} \theta^{\rho\sigma} A_\rho \partial_\sigma \lambda. \end{aligned} \quad (14)$$

As stated before, the "hatted" variables on the left are NC degrees of freedom and gauge transformation parameter. The higher order terms in  $\theta$  are kept out of contention as there are certain non-uniqueness involved in the  $O(\theta^2)$  mapping. The significance of the Seiberg-Witten map is that under an NC or  $*$ -gauge transformation of  $\hat{A}_\mu$  by,

$$\hat{\delta} \hat{A}_\mu = \partial_\mu \hat{\lambda} + i[\hat{\lambda}, \hat{A}_\mu]_* ,$$

$A_\mu$  will undergo the transformation

$$\delta A_\mu = \partial_\mu \lambda.$$

Subsequently, under this mapping, a gauge invariant object in conventional spacetime will be mapped to its NC counterpart, which will be  $*$ -gauge invariant. This is crucial as it ensures that the ordinary spacetime action that we recover from the NC action (11) by applying the

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<sup>1</sup>This is the first difference between our model and [7] who use the anti-fundamental representation. In fact, in [7], it is difficult to proceed with the fundamental definition [14]. On the other hand, in the present work, the choice between the first and second definition is not very important as it affects the overall sign of  $\theta$  only. Similar type of situation prevails in [15, 13].

Seiberg-Witten Map will be gauge invariant. Thus the NC action (11) in ordinary spacetime variables reads,

$$\hat{S} = \int d^3x [(D^\mu \phi)^\dagger D_\mu \phi + \frac{1}{2} \theta^{\alpha\beta} \{F_{\alpha\mu} ((D_\beta \phi)^\dagger D^\mu \phi + (D^\mu \phi)^\dagger D_\beta \phi) - \frac{1}{2} F_{\alpha\beta} (D^\mu \phi)^\dagger D_\mu \phi\}] \quad (15)$$

The above action is manifestly gauge invariant. The equation of motion now satisfied by  $A_\mu$  is,

$$\begin{aligned} & i(-2iA_\mu \phi^\dagger \phi + \phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi) (1 - \frac{1}{2} \theta^{\alpha\beta} F_{\alpha\beta}) \\ & + \frac{1}{2} \theta_{\alpha\mu} [\partial^\alpha \{(D^\beta \phi)^\dagger D_\beta \phi\} - \partial_\beta \{(D^\alpha \phi)^\dagger D^\beta \phi + (D^\beta \phi)^\dagger D^\alpha \phi\}] \\ & - \frac{1}{2} \theta_{\alpha\beta} [\partial^\alpha \{(D^\beta \phi)^\dagger D_\mu \phi + (D_\mu \phi)^\dagger D^\beta \phi\} + iF_\mu^\alpha (\phi^\dagger D^\beta \phi - (D^\beta \phi)^\dagger \phi)] = 0 \end{aligned} \quad (16)$$

Remember that so far we have not introduced the  $CP^1$  target space constraint in the NC spacetime setup. *Let us assume the constraint to be identical to the ordinary spacetime one, i.e.,*<sup>2</sup>

$$\phi^\dagger \phi = 1. \quad (17)$$

The reasoning is as follows. Primarily, after utilizing the Seiberg-Witten Map, we have returned to the ordinary spacetime and its associated dynamical variables and the effects of noncommutativity appears only as additional interaction terms in the action. Hence it is natural to keep the  $CP^1$  constraint unchanged. Alternatively, the above assumption can also be motivated in a roundabout way. Remember that the the  $CP(1)$  constraint has to be introduced in a \*-gauge invariant way. In order to introduce the  $CP^1$  constraint directly in the NC action (11) or (15), the constraint term  $\int \lambda(\phi^\dagger \phi - 1)$  has to be generalized to a \*-gauge invariant one by the application of the (inverse) Seiberg-Witten Map. This is quite straightforward but needless because as soon as we apply the Seiberg-Witten Map to the \*-gauge invariant constraint term, we recover the earlier ordinary spacetime constraint.

This allows us to write,

$$A_\mu = -i\phi^\dagger \partial_\mu \phi + a_\mu(\theta) \quad (18)$$

with  $a_\mu$  denoting the  $O(\theta)$  correction, obtained from (16,17). For  $\theta = 0$ .  $A_\mu$  reduces to its original form. Note that  $a_\mu$  is *gauge invariant*. Thus the  $U(1)$  gauge transformation of  $A_\mu$  remains intact, at least to  $O(\theta)$ . Keeping in mind the constraint  $\phi^\dagger \phi = 1$ , let us now substitute(18) in the NC action (15). Since we are concerned only with the  $O(\theta)$  correction, in the  $\theta$ -term of the action, we can use  $A_\mu = -i\phi^\dagger \partial_\mu \phi$ . However, in the first term in the action, we must incorporate the full expression for  $A_\mu$  given in (18). Remarkably, the constraint condition conspire to cancel the effect of the  $O(\theta)$  correction term  $a_\mu$ . Finally it boils down to the following: the action for the NC  $CP^1$  model to  $O(\theta)$  is given by (15) with the identification  $A_\mu = -i\phi^\dagger \partial_\mu \phi$  and  $\phi^\dagger \phi = 1$ .

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<sup>2</sup>This is the second difference between our model and that of [7], where  $\theta$ -correction terms are present in the  $CP(1)$  constraint. This is a serious difference as it drastically alters the structures of the model in [7] from ours. Apart from the conceptual reasoning given above, there also appears a technical compulsion. We would like to obtain perturbative  $\theta$ - corrections to the ordinary spacetime  $CP(1)$  model. Incorporating  $\theta$ - corrections in the  $CP(1)$  constraint as in [7] in our system will lead to a differential equation for the multiplier  $\lambda$ , instead of an algebraic one as in the ordinary spacetime case. This will change the  $\phi$ - equation of motion in a qualitative way. We stress that our model is a perfectly well defined NC theory which, incidentally, is distinct from the existing NC  $CP(1)$  model [7].

## Section IV - Energy-momentum tensor for the NC $CP(1)$ model

Our aim is to study the possibility of soliton solutions for the action (15). Let us try to derive the Bogolmony bound and BPS equations in the present case. The first task is to compute the EM tensor.

We follow [10] in computing the symmetric form of the EM tensor by coupling the model with a weak gravitational field and get,

$$T_{\mu\nu}^S = (1 - \frac{1}{4}\theta^{\alpha\beta}F_{\alpha\beta})[(D_\mu\phi)^\dagger D_\nu\phi + (D_\nu\phi)^\dagger D_\mu\phi - g_{\mu\nu}(D^\sigma\phi)^\dagger D_\sigma\phi] + \frac{1}{2}\theta^{\alpha\beta}[F_{\alpha\mu}((D_\beta\phi)^\dagger D_\nu\phi + (D_\nu\phi)^\dagger D_\beta\phi) + F_{\alpha\nu}((D_\beta\phi)^\dagger D_\mu\phi + (D_\mu\phi)^\dagger D_\beta\phi) - g_{\mu\nu}F_{\alpha\sigma}((D_\beta\phi)^\dagger D^\sigma\phi + (D^\sigma\phi)^\dagger D_\beta\phi)]. \quad (19)$$

The  $T_{\mu\nu}^S$  stands for the symmetric form of the EM tensor. In the static situation,

$$\dot{\phi} = 0 \quad \rightarrow \quad A_0 = F_{0i} = D_0\phi = 0$$

and the static energy density simplifies to,

$$T_{00}^S = (1 + \frac{1}{2}\theta^{12}F_{12})(D^i\phi)^\dagger D^i\phi, \quad (20)$$

where  $F_{\mu\nu}$  is also expressible in the form

$$F_{\mu\nu} = -i[(D_\mu\phi)^\dagger D_\nu\phi - (D_\nu\phi)^\dagger D_\mu\phi] = -i[(\partial_\mu\phi)^\dagger \partial_\nu\phi - (\partial_\nu\phi)^\dagger \partial_\mu\phi].$$

Now we discuss the canonical form of the EM tensor. Remembering that the indices of adjacent  $\phi$ 's are summed, the expanded form of the Lagrangian is,

$$\hat{L} = (D^\mu\phi)^\dagger D_\mu\phi + \lambda(\phi^\dagger\phi - 1) - \frac{i}{2}\theta^{\alpha\beta}[2\partial_\mu\phi^\dagger\partial_\beta\phi\partial_\alpha\phi^\dagger\partial^\mu\phi + 2\partial_\mu\phi^\dagger\partial_\beta\phi\phi^\dagger\partial_\alpha\phi\phi^\dagger\partial^\mu\phi - 2\partial_\beta\phi^\dagger\partial_\mu\phi\phi^\dagger\partial_\alpha\phi\phi^\dagger\partial^\mu\phi - \partial_\alpha\phi^\dagger\partial_\beta\phi\partial^\mu\phi^\dagger\partial_\mu\phi - \partial_\alpha\phi^\dagger\partial_\beta\phi\phi^\dagger\partial^\mu\phi\phi^\dagger\partial_\mu\phi]. \quad (21)$$

The canonical energy-momentum tensor is,

$$\begin{aligned} T_{\mu\nu} &= \frac{\delta\hat{L}}{\delta(\partial^\mu\phi^\dagger)}\partial_\nu\phi^\dagger + \frac{\delta\hat{L}}{\delta(\partial^\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\hat{L} \\ &= (1 - \frac{1}{4}\theta^{\alpha\beta}F_{\alpha\beta})(D_\mu\phi^\dagger D_\nu\phi + (\mu \leftrightarrow \nu)) \\ &\quad - i\theta^{\alpha\beta}[\partial_\nu\phi^\dagger\partial_\beta\phi(\partial_\alpha\phi^\dagger\partial_\mu\phi + \phi^\dagger\partial_\alpha\phi\phi^\dagger\partial_\mu\phi) + (\mu \leftrightarrow \nu) - \partial_\beta\phi^\dagger\partial_\nu\phi\phi^\dagger\partial_\alpha\phi\phi^\dagger\partial_\mu\phi - (\mu \leftrightarrow \nu)] \\ &\quad - i\theta_{\mu\alpha}[(-\frac{1}{2}\partial_\nu\phi^\dagger\partial^\alpha\phi + \frac{1}{2}\partial^\alpha\phi^\dagger\partial_\nu\phi)(D^\mu\phi)^\dagger D_\mu\phi - \partial_\sigma\phi^\dagger\partial_\nu\phi(\partial^\alpha\phi^\dagger\partial^\sigma\phi + \phi^\dagger\partial^\alpha\phi\phi^\dagger\partial^\sigma\phi) \\ &\quad + \partial_\sigma\phi^\dagger\partial^\alpha\phi(\partial_\nu\phi^\dagger\partial^\sigma\phi + \phi^\dagger\partial_\nu\phi\phi^\dagger\partial^\sigma\phi) + \partial_\nu\phi^\dagger\partial_\sigma\phi\phi^\dagger\partial^\alpha\phi\phi^\dagger\partial^\sigma\phi - \partial^\alpha\phi^\dagger\partial_\sigma\phi\phi^\dagger\partial_\nu\phi\phi^\dagger\partial^\sigma\phi] - g_{\mu\nu}\hat{L}. \quad (22) \end{aligned}$$

Note that  $\theta^{\alpha\beta}$  part is symmetric. In the energy density  $T_{00}$  the contribution coming from the non-symmetric parts in the  $\theta$ -contribution drop out if only space-space noncommutativity

is assumed, i.e.  $\theta^{0i} = 0$ . For this special case, in the static limit, the above  $\theta$ -contribution completely drops out and the energy density reduces to

$$T_{00}^N = -\hat{L} = (1 + \frac{1}{2}\theta^{12}F_{12})(D^i\phi)^\dagger D^i\phi. \quad (23)$$

Clearly this is identical to the static energy (20) obtained from the symmetric form  $T_{\mu\nu}^S$ . Indeed, it is satisfying that in this particular case, both the canonical and symmetric forms of the EM tensor lead to the same expression of the static energy, which is manifestly gauge invariant and conserved (as it comes from the canonical form).

Interestingly to  $O(\theta)$ , the noncommutativity effect factors out from the ordinary spacetime result. Also notice that in the two spatial dimensions that we are considering, the  $\theta$ -term in the energy density is proportional to the topological charge density  $n(x)$  in (8). This is because the expression for the topological current remains unchanged since the dynamical variables as well as the  $CP(1)$  constraint is unaltered in our model. We specialize to only space-space noncommutativity,  $\theta^{ij} = \theta\epsilon^{ij}$ ,  $\theta^{0i} = 0$ , and find,

$$T_{00}^{(\theta)} = \pi\theta n(x)(D^i\phi)^\dagger D^i\phi, \quad (24)$$

where the superscript  $S$  or  $N$  is dropped.

## Section V - Analysis of the Bogomolny bound

In order to obtain the Bogomolny bound, we follow the same procedure as that of the  $CP^1$  model in ordinary spacetime and rewrite the static energy functional in the following form,

$$\begin{aligned} E &= \int d^2x (1 + \pi\theta n) [| (D_1 \pm iD_2)\phi |^2 \pm 2\pi n] \\ &= \int d^2x [(1 + \pi\theta n) | (D_1 \pm iD_2)\phi |^2 \pm 2\pi n \pm 2\pi^2\theta n^2] \\ &= \int d^2x (1 + \frac{1}{2}\pi\theta n)^2 | (D_1 \pm iD_2)\phi |^2 \pm 2\pi N \pm 2\pi^2\theta \int d^2x n^2(x) + O(\theta^2). \end{aligned} \quad (25)$$

Now individually all the terms in the energy expression are positive definite. Hence we obtain the Bogomolny bound to be

$$E \geq N + 2\pi^2\theta \int d^2x n^2(x) \quad (26)$$

and the saturation condition is

$$(1 + \frac{1}{2}\pi\theta n)^2 | (D_1 \pm iD_2)\phi |^2 = 0. \quad (27)$$

The BPS equation turns out to be,

$$D^1\phi = \pm iD^2\phi. \quad (28)$$

Thus we find that the BPS equation remains unchanged and there is a  $O(\theta)$  correction in the static energy of the soliton. Note that both of the above results do not agree with [7, 12]. But this is not unexpected since as we have mentioned before, the defining conditions of the NC  $CP(1)$  models are different. However, we repeat that *a priori* there is nothing inconsistent in our NC model.



Finally, we are ready to discuss the curiosity. It appears that the BPS equations (28) do not satisfy the equation of motion for  $\phi$ . A straightforward computation yields the dynamical equation for  $\phi$ ,

$$D^\mu[(1 - \frac{1}{4}\theta^{\alpha\beta}F^{\alpha\beta})D_\mu\phi] + \frac{1}{2}\theta^{\alpha\beta}[iD_\alpha\{(D^\sigma\phi)^\dagger D_\sigma\phi D_\beta\phi\} + D_\beta\{F_{\alpha\mu}D^\mu\phi\} + D^\mu\{F_{\alpha\mu}D_\beta\phi\} - iD_\alpha\{(D_\beta\phi)^\dagger D^\mu\phi + (D^\mu\phi)^\dagger D_\beta\phi\}D_\mu\phi] + iD_\mu\{(D_\beta\phi)^\dagger D^\mu\phi + (D^\mu\phi)^\dagger D_\beta\phi\}D_\alpha\phi] - \lambda\phi = 0. \quad (29)$$

(Details of the derivation are provided in the appendix.) To get  $\lambda$ , contract by  $\phi^\dagger$  and use  $\phi^\dagger\phi = 1$ . For the time being, instead of writing the full equation of motion, we want to check the consistency of the programme only, that is whether the solution of the BPS equation satisfies the equation of motion, which they should. Since the BPS equation remains unchanged here, the  $\theta$ -term in the equation of motion should vanish for those solutions that satisfy the BPS equation as well. So we consider the equation of motion in a simplified setting where the BPS equation is satisfied and only  $\theta^{12}$  is non-zero and obtain for  $\lambda$

$$\lambda = \phi^\dagger D^i D_i \phi + \frac{1}{2}\theta^{12}\phi^\dagger D^i (F_{12}D_i\phi). \quad (30)$$

Putting  $\lambda$  back in the equation of motion, we get

$$D^i D_i \phi - \phi^\dagger D^i D_i \phi \phi + \frac{1}{2}\theta^{12}[D^i (F_{12}D_i\phi) - \phi^\dagger D^i (F_{12}D_i\phi)\phi] = 0. \quad (31)$$

This equation can be rewritten as

$$D^i D_i \phi - \phi^\dagger D^i D_i \phi \phi + \frac{1}{2}\theta^{12}[\partial^i F_{12}D_i\phi - \phi^\dagger \partial^i F_{12}D_i\phi \phi + F_{12}(D^i D_i \phi - \phi^\dagger D^i D_i \phi \phi)] = 0. \quad (32)$$

Clearly the last term, that is  $\frac{1}{2}\theta^{12}F_{12}(D^i D_i \phi - \phi^\dagger D^i D_i \phi \phi) \approx O(\theta^2)$  and can be dropped. The term  $\theta^{12}\phi^\dagger \partial^i F_{12}D_i\phi \phi = \theta^{12}\partial^i F_{12}\phi^\dagger D_i\phi \phi = 0$ . However the remaining  $O(\theta)$ -term,  $\frac{1}{2}\theta^{12}\partial^i F_{12}D_i\phi$  does not vanish. This is the purported mismatch between the BPS equations and the full equation of motion. This brings us to the last part - the collective coordinate analysis, where we can check explicitly the above conclusions in a simplified setup.

## Section VI - Collective coordinate analysis

We consider the topological charge  $N = 1$  sector. As we have discussed before, expression for the topological current and subsequently the charge remains same (in our NC  $CP(1)$  model) as that of the ordinary spacetime  $CP(1)$  model. This means that we can use the same parameterizations as before [8] to introduce the collective coordinates. As a first approximation, only the zero mode arising from the global  $U(1)$  invariance is being quantized. In the  $O(3)$  nonlinear sigma model, the  $N = 1$  sector is characterized by [8]

$$n^a = \{\hat{r}\sin(g(r)), \cos(g(r))\}; a = 1, 2, 3$$

with the constraint  $n^a n^a = 1$  and the boundary conditions  $g(0) = 0; g(\infty) = \pi$ . Keeping in mind the  $O(3) - CP(1)$  duality and the Hopf map  $n^a = \phi^\dagger \sigma^a \phi$ , the soliton profile in the  $CP(1)$  variables is of the form,

$$\phi \equiv \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} = \begin{pmatrix} \cos(\frac{g}{2}) \\ \sin(\frac{g}{2})e^{i(\varphi+\alpha(t))} \end{pmatrix} \quad (33)$$

where the gauge  $(\phi^1)^* = \phi^1$  has been used and  $r, \varphi$  refer to the plane polar coordinates.  $\alpha(t)$  is the collective coordinate. Substituting the above choice (33) in the static energy expression in (20) leads to,

$$E(r) = (1 + \theta \frac{\sin(g)g'}{2r})[(g')^2 + \frac{\sin^2(g)}{r^2}], \quad (34)$$

where  $g' = \frac{dg}{dr}$ . In *figure* (1) the effect of the  $\theta$ - correction is shown where the following simple form of  $g(r)$  is considered,

$$g(r) \approx \pi(1 - e^{-\mu r}). \quad (35)$$

One can clearly see that with typical values of the parameters, ( $\theta = 1$ ,  $\mu = 1$ ) the energy density for the NC case is more sharply peaked. (In reality,  $\theta$  should be smaller.) This assures us of the rationale of our previous Bogolmony analysis. Next we look in to the equation of motion.

It is straightforward check that the profile (33) satisfies the BPS equation as well as the equation of motion for the ordinary spacetime situation,  $\theta = 0$ . For  $\theta \neq 0$ , the BPS equations are once again satisfied since they remain unaltered. So we concentrate only on the problem term  $\frac{1}{2}\theta^{12}\partial^i F_{12}D_i\phi$  in the equation of motion (32). With the particular form of  $g(r)$  in (35) we obtain

$$\frac{1}{2}\theta^{12}\partial^i F_{12}D_i\phi = \frac{\theta g'}{4}[\frac{g''}{2r}\sin(\frac{g}{2}) - \frac{g'}{2r^2}\sin(\frac{g}{2}) + \frac{(g')^2}{4r}\cos(\frac{g}{2})] \begin{pmatrix} \sin(\frac{g}{2}) \\ -\cos(\frac{g}{2})e^{i\varphi} \end{pmatrix} \equiv \begin{pmatrix} X(r) \\ Y(r) \end{pmatrix}. \quad (36)$$

In *Figure 2* we again use the  $g(r)$  given in (35) and compare the magnitude of the above expression with a typical term,

$$\partial^i \partial_i \phi \equiv \begin{pmatrix} S(r) \\ T(r) \end{pmatrix},$$

occurring in the equation of motion (32). For consistency, the expressions in (36) should have vanished. However, even for  $\theta = 1$ , (which is quite a large value), the mismatch term is small in an absolute sense.

## Section VII - Conclusions

In this paper, we have attempted to recover the soliton solutions in the non-commutative  $CP(1)$  model, discovered earlier [7]. The  $U(1)$  and NC  $U(1)$  gauge invariances in the  $CP(1)$  and NC  $CP(1)$  models respectively, requires the use of the Seiberg-Witten map, to convert the NC action to an action comprising of ordinary spacetime dynamical variables. The effects of noncommutativity are manifested as interaction terms. For theoretical as well as technical reasons, we found it convenient to keep the  $CP(1)$  constraint unchanged, *i.e.* without any  $\theta$  correction.

From the above action, we construct both the symmetric and canonical forms of the energy momentum tensor. For only spatial noncommutativity, both the above forms reduce to an identical (gauge invariant) expression for the static energy. The Bogomolny analysis yields a lower bound in the energy, hinting at the presence of a new type of soliton. However, the resulting BPS equations do not match completely with the full variational equation of motion. The present model is otherwise a perfectly well defined NC field theory with the expected features. Hence we conclude that *inadequacy in the definitions of the energy momentum tensor*

in an NC field theory is responsible for this failure. The above phenomena are nicely visualized in a collective coordinate framework. The above awkward situation clearly demands further study.

Finally, as a future work we mention that inclusion of the Hopf term, (in the form of Chern-Simons term in  $CP(1)$  variables), in the NC theory would indeed be interesting. The Hopf term was introduced [8, 17] to impart anyonic behavior to the  $CP(1)$  solitons. The exact form of the NC version of the Chern Simons term is known [15] - it is a "non-abelian" generalization of the Chern Simons term. In our formalism, application of the Seiberg-Witten map will reduce it to the ordinary Chern Simons term [15] but there will appear  $O(\theta)$  correction terms since the gauge field of the Chern Simons term is actually a non-linear combination of the  $CP(1)$  variables.

Another interesting problem is the reconstruction of the NC  $CP(1)$  model of [7] in our framework.

**Appendix:** To get the  $\phi$ -equation of motion, we consider variation of  $\phi^\dagger$ . and exploit the relations,

$$(D^\mu \phi)^\dagger \phi = \phi^\dagger D^\mu \phi = 0, \tag{37}$$

$$\begin{aligned} \delta(D^\mu \phi)^\dagger &= \delta(\partial^\mu \phi^\dagger + (\phi^\dagger \partial^\mu \phi) \phi^\dagger) = \partial^\mu (\delta \phi^\dagger) + (\delta \phi^\dagger \partial^\mu \phi) \phi^\dagger + (\phi^\dagger \partial^\mu \phi) \delta \phi^\dagger \\ \delta(D^\mu \phi) &= \delta(\partial^\mu \phi - (\phi^\dagger \partial^\mu \phi) \phi) = -(\delta \phi^\dagger \partial^\mu \phi) \phi. \end{aligned} \tag{38}$$

In the action the terms are products of the generic form  $\int (D^\mu \phi)^\dagger (D^\nu \phi) X(x)$ . The variation of  $(D^\mu \phi)$  will reproduce

$$\int (D^\mu \phi)^\dagger \delta(D^\nu \phi) X = - \int (D^\mu \phi)^\dagger \phi (\delta \phi^\dagger \partial^\mu \phi) X = 0,$$

by using (37). Similarly, the variation of  $(D^\mu \phi)^\dagger$  will yield

$$\int \delta(D^\mu \phi)^\dagger (D^\nu \phi) X = \int (\partial^\mu (\delta \phi^\dagger) + (\delta \phi^\dagger \partial^\mu \phi) \phi^\dagger + (\phi^\dagger \partial^\mu \phi) \delta \phi^\dagger) D^\nu \phi X = - \int \delta \phi^\dagger D^\mu (D^\nu \phi X)$$

by partial integration and using (37). the above identities simplifies the computations considerably and leads to the equation (29).

**Acknowledgements:** It is a pleasure to thank Professor Hyun Seok Yang for fruitful correspondence. Also I thank Professor Avinash Khare for a helpful discussion. Lastly I am indebted to Bishwajit Chakraborty for a free access to his notes on  $CP(1)$  model.

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