Expert Systems

Article

DOI: 10.1111/exsy.12022

Dominance-based rough set theory over interval-valued information systems

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Abstract: *This paper proposes a new generalization of classical real-valued information systems, that is, interval-valued information systems. By defining an interval-valued dominance relation on a condition attribute, a rough set model and attribute reduction are established over interval-valued information systems. Moreover, several interesting properties are investigated by constructive approach. Furthermore, knowledge reductions of consistent and inconsistent interval-valued dominance decision information systems are studied, respectively. Subsequently, some descriptive theorems of knowledge reduction are presented for interval-valued dominance decision information systems. Finally, the validity of the model and conclusions is verified by numerical example.*

Keywords: rough set, interval-valued dominance relation, knowledge reduction

1. Introduction

Rough set theory, introduced by Pawlak (1982), has become well established as an approach to studying information systems characterized by insufficient and incomplete information in a wide variety of applications. Various generalizations of rough approximations and applications have been made over the past years. Recently, rough set theory has become a popular mathematical tool in areas such as pattern recognition, image processing, feature selection (Pawlak & Skowron, 2007), conflict analysis (Pawlak, 1998), decision support (Tsumoto, 1998; Kim & Han, 2001;Tsumoto, 2004), data mining and knowledge discovery processes from large data sets (Lingras & Yao, 1998). The concept of rough set theory is founded on the assumption that every object of the universe of discourse is associated with some information. Objects characterized by the same information are indiscernible in view of their available information. This kind of generalization for indiscernibility relation forms the mathematical basis of rough set theory (Yao, 2003; Beynon *et al*., 2000; Skowron & Stepaniuk, 1996; Huynh & Nakamori, 2005).

Classical rough set adopts two definable sets, namely the lower and upper approximations, to deal with the approximation computation for any subset of universe. Then, potential knowledge will be revealed from information systems by lower and upper approximations, and a decision rule can be induced. One important usage of rough set theory is knowledge reduction for information systems. Knowledge reduction deletes unnecessary attributes from the original attribute set, while the same ability to classify remains (Kryszkiewicz, 1998; Ziarko, 2003; Xu *et al*., 2010). Many valuable conclusions have been drawn through the method of knowledge reduction in consistent information systems (Zhang *et al*., 2003a, 2003b). Meanwhile, more attention has been paid to knowledge reduction in inconsistent systems in research on rough set theory. Many types of knowledge reduction also have been proposed in this field (Zhang *et al*., 2006; Wang & Lin, 2006; Xu & Zhang, 2008; Xu *et al*., 2010).

Information systems or decision information systems transacting with Pawlak rough set theory are self-contained, and the value of the attributes is discrete or certain. At the same time, decision rule and knowledge discovery of information systems are acquired by attribute reduction based on equivalence relation (satisfying reflexive, symmetric and transitive properties). Recently, many authors have generalized equivalence relation to various forms such as similarity relation, dominance relation and covering relation in terms of Pawlak rough set theory. More detailed studies on generalized information systems follow from the perspective of knowledge reduction (Greco *et al*., 1999; Greco *et al*., 2001; Greco *et al*., 2002; Kazimierz, 2004; Zhu, 2007; Leung *et al*., 2008; Wu *et al*., 2005; Leung *et al*., 2006; Huang *et al*., 2012a, 2012b). Moreover, a large number of concepts of knowledge reduction such as generalized decision reduct, $\beta-$reduct (Kryszkiewicz, 2001; Zhang *et al*., 2006), probability reduct, dynamic reduct (Grzymala, 1997), entropy reduct and approximation entropy reduct (Slezak, 2002), distribution reduct and allocation reduct (Zhang *et al*., 2003a, 2003b, Zhang *et al*., 2006; Xu & Zhang, 2008; Xu *et al*., 2010) also have been developed for decision information systems.

However, the value of attribute may be symbolic-valued, set-valued, fuzzy-valued or fuzzy number-valued and continuous-valued for the existing database in practice. Therefore, Pawlak rough set has some limitations in handling these values of attributes in information systems. Moreover, apart from different definitions of the knowledge reduction established for classical real-valued information systems, various generalized information systems are proposed, such as set-valued information systems (Guan & Wang, 2006; Zhang *et al.*, 2006; Qian *et al.*, 2008, 2009; Wang & Zhang, 2006), ordered information systems (Xu & Zhang, 2008; Xu *et al.*, 2010), fuzzy information systems (Zhang *et al.*, 2003a, 2003b; Yang *et al.*, 2009), continuous-valued information systems (Ziarko, 2003; Zhang *et al.*, 2003a, 2003b), interval-valued information systems and interval-valued fuzzy information systems (Sun *et al.*, 2008; Gong *et al.*, 2008; Zhang *et al.*, 2009; Yang & Sheng, 2011; Huang, 2012; Dai *et al.*, 2012; Chen & Miao, 2011), interval-valued intuitionistic fuzzy information systems (Zhang, 2010; Huang *et al.*, 2012a, 2012b) and vague objective information systems (Feng *et al.*, 2010). Though there is different background in theory and application for these generalized information systems or decision information systems, these methods go along with classical real-valued information systems. Moreover, the concepts of knowledge reduction for classical real-valued information systems have been successfully generalized in these information systems. The terms with the reduction of aforementioned various generalized information systems also use the same concepts, such as the classical real-valued information systems (Zhang *et al.*, 2003a, 2003b; Zhang *et al.*, 2006).

So far, most research has been conducted on interval-valued information systems. Actually, the interval value of an attribute is a closed interval of the universe of real number for existing interval-valued information systems (Huang *et al.*, 2006; Leung *et al.*, 2008; Qian *et al.*, 2008; Yang *et al.*, 2009). There are many useful applications based on these interval-valued information systems, such as pattern recognition and automated knowledge acquisition. However, it would be of great value to understand what restricts the value of an attribute on the closed interval of [0,1] where the uncertainty decision or comprehension evaluation is concerned. Therefore, research on the information systems with the value of an attribute in closed interval [0,1] is a necessity. In this paper, we tentatively discuss the rough set model and knowledge reduction in interval-valued information systems.

This paper studies interval-valued information systems by defining an interval-valued dominance relation over attribute set, that is, we establish a rough set model on interval-valued information systems. Like the Pawlak rough set, several properties of this model are also studied. Meanwhile, we also study interval-valued decision information systems by defining an interval-valued dominance relation over condition attribute and an equivalence relation over decision attribute, respectively. The attribute reduction of interval-valued information systems is investigated in a manner similar to classical real-valued information systems (Zhang *et al.*, 2003a, 2003b; Mi *et al.*, 2004; Wu *et al.*, 2005; Zhang *et al.*, 2006; Wu, 2008; Xu & Zhang, 2008; Xu *et al.*, 2010). Furthermore, knowledge reduction of both consistent and inconsistent interval-valued decision information systems is discussed. In fact, all conclusions presented in this paper are generalization of the corresponding conclusions of classical real-valued decision information systems (Zhang *et al.*, 2003a, 2003b; Mi *et al.*, 2004; Wu *et al.*, 2005; Zhang *et al.*, 2006; Xu & Zhang, 2008; Xu *et al.*, 2010).

However, there also exist some differences with the corresponding conclusions of classical real-valued information systems because the interval values in closed interval [0,1] have some specialized properties. Generally speaking, all similar research focuses on the definition of a binary relation on condition and decision attributes. The existing research related to our work focuses on the following aspects. One is to define two dominance relations on both condition and decision attributes, as in ordered information systems (Xu & Zhang, 2008; Xu *et al.*, 2010). Another is to define two equivalence relations over both condition and decision attributes, as in real-valued information systems (Zhang *et al.*, 2003a, 2003b; Zhang *et al.*, 2006; Mi *et al.*, 2004). Some others define a binary relation or partial relation on condition and decision attributes, as in set-valued or language-valued information systems (Guan & Wang, 2006; Wang & Zhang, 2006; Wang & Lin, 2006; Qian *et al.*, 2008; Qian *et al.*, 2009). Therefore, some conclusions reached in the existing related work could not hold in the interval-valued (decision) information systems studied in this paper.

The rest of this paper is organized as follows. Section 2 briefly introduces some important notions of the classical Pawlak rough set theory and the concept of interval-valued systems. Section 3 establishes the rough set model over interval-valued information systems. Furthermore, attribute reduction is studied in detail for interval-valued information systems. In Section 4, knowledge reduction for both consistent and inconsistent interval-valued dominance decision information systems is discussed, respectively. A numerical example is employed to test our conclusions. We conclude our research and set further future research directions in Section 5

2. Preliminary

In this section, we present some concepts and conclusions that will be used in this paper (Pawlak & Skowron, 2007; Gong *et al.*, 2008; Sun *et al.*, 2008; Zhang, 2010).

Let U be a non-empty finite set, which is called universe. Let R be an equivalence relation on U. That is, R is a binary relation and satisfies reflexive, symmetric and transitive properties. Then, we call (U,R) an approximation space. Moreover, binary relation R divides the universe U into piecewise disjoint classes such that any two different elements x and y belong to the same class if and only if $(x,y) \in R$.

Furthermore, $U/R = \{X_1, X_2, \ldots, X_m\}$ denotes all equivalence classes in universe U induced by equivalence relation R. If both of any two different elements x and y of U belong to $X_i \in U/R$, then x and y are called indiscernible. The equivalence class of R and $\varnothing$ is called the primary set in approximation space (U,R).

We describe the definition of information systems (Huynh & Nakamori, 2005; Gong *et al.*, 2008; Sun *et al.*, 2008) as follows.

In general, a triple $(U,A,\{V_a\}_{a \in A})$ is called an information system, where U is the set of object, A is the set of attribute and V_a is the domain of attribute a, understood as a mapping $a : U \rightarrow V_a$. It can be easily seen that each subset of attribute set A induces an equivalence relation:

$$ind(A) = \{(x,y) \in U \times U | a(x) = a(y), a \in A\}.$$

Then, $ind(A)$ is called an indiscernibility relation and denotes $ind(A) = \cap_{a \in A} ind(a)$ (where $ind(a)$ means $ind(\{a\})$).

If $(x,y) \in ind(A)$, we then say that the objects x and y are indiscernible with respect to A. In other words, we cannot distinguish x from y in terms of the attributes in A and vice versa.

For any $X \subseteq U$, we define the two subsets of approximation space (U,R) as follows:

$$\underline{R}X = \{x \in U | [x]_R \subseteq X\}, \quad \bar{R}X = \{x \in U | [x]_R \cap X \neq \varnothing\}$$

We call $\underline{R}X$ and $\bar{R}X$ the lower and upper approximations of X with respect to (U,R).

If $\underline{R}X = \bar{R}X$, then X is called the definable set. Otherwise, X is called the rough set.

The lower approximation $\underline{R}X$ is the union of all elementary sets that are the subset of X, and the upper approximation $\bar{R}X$ is the union of all elementary sets that have a non-empty intersection with X.

The lower (upper) approximation $\underline{R}X$ ($\bar{R}X$) is interpreted as the collection of those elements of U that definitely (possibly) belong to X.(Huynh & Nakamori, 2005; Pawlak & Skowron, 2007; Gong *et al.*, 2008; Sun *et al.*, 2008).

In what follows, we give the definition of interval value in [0,1] and its operators.

We assume throughout that I is a unit closed interval, that is, $I = [0,1]$.

Let $[I] = \{[a,b] | a \leq b, a, b \in I\}$. For any $a \in I$, define $\bar{a} = [a, a]$, then $\bar{a} \in [I]$.

Definition 2.1.
Gorzafczary, 1988; Turksen, 1986; Gong *et al.*, 2008; Sun *et al.*, 2008; Zhang, 2010
For any $a_i \in I, i \in J$, we define

$$\vee_{i \in J} a_i = \sup\{a_i | i \in J\}, \wedge_{i \in J} a_i = \inf\{a_i | i \in J\},$$
$$\vee_{i \in J}[a_i, b_i] = [\vee_{i \in J} a_i, \vee_{i \in J} b_i],$$
$$\wedge_{i \in J}[a_i, b_i] = [\wedge_{i \in J} a_i, \wedge_{i \in J} b_i]$$

where $\vee$ means maximum and $\wedge$ means minimum.

For any interval-valued $[a_i, b_i] \in [I], i = 1, 2$, we define the following operations.

$[a_1, b_1] = [a_2, b_2]$ if and only if $a_1 = a_2, b_1 = b_2$,
$[a_1, b_1] \leq [a_2, b_2]$ if and only if $a_1 \leq a_2, b_1 \leq b_2$,
$[a_1, b_1] < [a_2, b_2]$ if and only if $[a_1, b_1] \leq [a_2, b_2]$ but $[a_1, b_1] \neq [a_2, b_2]$.

Remark 2.1.
It can be easily known that the following relation does not hold for any $[a_i, b_i] \in [I], i = 1, 2$.

$$[a_1, b_1] \nleq [a_2, b_2] \Leftrightarrow [a_1, b_1] > [a_2, b_2],$$
$$[a_1, b_1] \nless [a_2, b_2] \Leftrightarrow [a_1, b_1] \geq [a_2, b_2].$$

In general, the aforementioned two relations hold when $[a_i, b_i] \in [I], i = 1, 2$ is real-valued. That is, $a_i = b_i$ for any $a_i, b_i \in I, i = 1, 2$.

3. Dominance-based rough set model over interval-valued information systems

In this section, we discuss the basic rough set theory by defining an interval-valued dominance relation for interval-valued information systems.

3.1. Rough set model on interval-valued information systems

The interval-valued information system was first defined by Sun and Gong (Sun *et al.*, 2008; Gong *et al.*, 2008).

Here, we present the definition of interval-valued information systems as follows.

Let $U = \{x_1, x_2, \ldots, x_k\}, A = \{a_1, a_2, \ldots, a_m\}$. $F = \{f_i | i \leq m\}$ is a family of mapping set between U and A, where f_i is an interval-valued mapping from U to A. That is, $f_i: U \rightarrow [I] (i \leq m)$, and $[I]$ is the domain of attribute a_i.

Then, we call triple (U,A,F) as an interval-valued information system(Gong *et al.*, 2008; Sun *et al.*, 2008).

In general, every attribute set has determined an indiscernibility relation or equivalence relation in real-valued information systems. So, every attribute set defines an equivalence relation over interval-valued information systems as well. However, the equivalence relation could not be satisfied in practice, such as the multi-object comprehensive evaluation decision-making, the supplier selection decision-making with uncertainty and group decision-making in emergency events based on interval-valued information systems. So, there should be given a more suitable binary relation for interval-valued information systems. Therefore, we define an interval-valued dominance relation for interval-valued information systems in this section. It is also a generalization of the dominance relation in real-valued information systems (Zhang *et al.*, 2003a, 2003b; Zhang *et al.*, 2006; Xu & Zhang, 2008; Xu *et al.*, 2010).

Definition 3.1.
Let (U,A,F) be interval-valued information systems. For any $B \subseteq A$, we define

$$R_B^{\leq} = \{(x_i, x_j) | f_k(x_i) \leq f_k(x_j), \forall a_k \in B, x_i, x_j \in U\}$$

Then, $R_B^{\leq}$ is called an interval-valued dominance relation over interval-valued information systems.

Furthermore, we call triple (U,A,F) as interval-valued information systems based on interval-valued dominance relation.

For the sake of clarity, denote $(U, R_A^{\leq}, F) = (U, A, F)$, and we call $(U, R_A^{\leq}, F)$ as interval-valued dominance information systems.

Remark 3.1.
If the value of attribute $a \in A$ is real-valued, $(U, R_A^{\leq}, F)$ will be real-valued information systems. Moreover, the interval-valued dominance relation will be real-valued dominance relation (Zhang *et al.*, 2003a, 2003b; Zhang *et al.*, 2006; Xu & Zhang, 2008; Xu *et al.*, 2010).

Denote $[x]_B^{\leq} = \{y \in U | f_k(x) \leq f_k(y), a_k \in B\}$.

It is easy to see that the interval-valued dominance relation satisfies the following properties and that all of them are similar to real-valued information systems, ordered information systems, set-valued information systems and other information systems with dominance relation (Zhang *et al*., 2003a, 2003b; Wang & Zhang, 2006; Xu & Zhang, 2008; Xu *et al*., 2010).

Proposition 3.1.
Let $R_A^{\leq}$ be an interval-valued dominance relation of $\left(U, R_A^{\leq}, F\right)$. Then, the following conclusions hold.

1. $R_A^{\leq}$ is reflexive and transitive but not symmetric. Therefore, it is not an equivalence relation.
2. If $B_1 \subseteq B_2 \subseteq A$, then $R_A^{\leq} \subseteq R_{B_2}^{\leq} \subseteq R_{B_1}^{\leq}$.
3. If $B_1 \subseteq B_2 \subseteq A$, then $[x]_A^{\leq} \subseteq [x]_{B_2}^{\leq} \subseteq [x]_{B_1}^{\leq}$.
4. If $y \in [x]_A^{\leq}$, then $[y]_A^{\leq} \subseteq [x]_A^{\leq}$ and $[x]_A^{\leq} = \cup\left\{[y]_A^{\leq} \middle| y \in [x]_A^{\leq}\right\}$.
5. $[y]_A^{\leq} = [x]_A^{\leq}$ if and only if $f(x,a) = f(y,a), a \in A$.
6. $C^{\leq} = \left\{[x]_A^{\leq} | x \in U\right\}$ is a covering of universe U.

In what follows, we give the rough set model for interval-valued information systems.

Let $\left(U, R_A^{\leq}, F\right)$ be interval-valued dominance information systems. For any $X \subseteq U$, we define the lower and upper approximations of X as follows:

$$\underline{R}_A^{\leq}(X) = \left\{x \in U \middle| [x]_A^{\leq} \subseteq X\right\}, \overline{R}_A^{\leq}(X) = \left\{x \in U \middle| [x]_A^{\leq} \cap X \neq \varnothing\right\}$$

If $\underline{R}_A^{\leq}(X) = \overline{R}_A^{\leq}(X)$, then X is called a definable set with interval-valued dominance relation $R_A^{\leq}$. Otherwise, X is called a rough set in $\left(U, R_A^{\leq}, F\right)$.

Like the rough set model over the existing generalized information systems (Zhang *et al*., 2003a, 2003b; Wang & Zhang, 2006; Xu & Zhang, 2008; Xu *et al*., 2010), the following properties are clear.

Theorem 3.1.
Let $\left(U, R_A^{\leq}, F\right)$ be an interval-valued dominance information system. Then, the lower and upper approximations satisfy the following properties:

1. $\underline{R}_A^{\leq}(U) = U, \overline{R}_A^{\leq}(\varnothing) = \varnothing$,
2. $\underline{R}_A^{\leq}(X) = \simeq\overline{R}_A^{\leq}(\simeq X), \overline{R}_A^{\leq}(X) = \simeq\underline{R}_A^{\leq}(\simeq X)$,
3. $\underline{R}_A^{\leq}(X \cap Y) = \underline{R}_A^{\leq}(X) \cap \underline{R}_A^{\leq}(Y), \overline{R}_A^{\leq}(X \cup Y) = \overline{R}_A^{\leq}(X) \cup \overline{R}_A^{\leq}(Y)$,
4. $\underline{R}_A^{\leq}(X \cup Y) \supseteq \underline{R}_A^{\leq}(X) \cup \underline{R}_A^{\leq}(Y), \overline{R}_A^{\leq}(X \cap Y) \subseteq \overline{R}_A^{\leq}(X) \cap \overline{R}_A^{\leq}(Y)$,
5. $\underline{R}_A^{\leq}(X) \subseteq X \subseteq \overline{R}_A^{\leq}(X)$,
6. $\underline{R}_A^{\leq}(X) \subseteq \underline{R}_A^{\leq}\left(\underline{R}_A^{\leq}(X)\right), \overline{R}_A^{\leq}\left(\overline{R}_A^{\leq}(X)\right) \subseteq \overline{R}_A^{\leq}(X)$,

where $\simeq X$ stands for the complement of X.

Proof
It can be easily proven by the definitions earlier.

Example 3.1.
Table 1 gives an example of interval-valued information systems.

It is easy to obtain the following dominance classes by Definition 3.1:

$$[x_1]_A^{\leq} = \{x_1, x_2, x_5\} \quad [x_2]_A^{\leq} = \{x_2, x_5\}$$
$$[x_3]_A^{\leq} = \{x_2, x_3, x_4, x_5\} \quad [x_4]_A^{\leq} = \{x_4\}$$
$$[x_5]_A^{\leq} = \{x_5\}$$

Setting $X = \{x_2, x_3, x_5\}$.

Table 1: *An interval-valued information systems*

U	a_1	a_2	a_3
x_1	[0.1, 0.5]	[0.2, 0.6]	[0.1, 0.6]
x_2	[0.3, 0.8]	[0.2, 0.6]	[0.2, 0.7]
x_3	[0.1, 0.5]	[0.1, 0.6]	[0.2, 0.6]
x_4	[0.2, 0.9]	[0.1, 0.7]	[0.3, 0.8]
x_5	[0.3, 1]	[0.3, 0.9]	[0.2, 0.8]

Then, come the following relations:

$$\underline{R}_A^{\leq}(X) = \{x_2, x_5\}, \overline{R}_A^{\leq}(X) = \{x_1, x_2, x_3, x_5\}$$

Clearly, $\underline{R}_A^{\leq}(X) \subseteq X \subseteq \overline{R}_A^{\leq}(X)$.

Remark 3.1.
By Example 3.1, the following relations could not hold.

1. $\underline{R}_A^{\leq}(X) = \cup\left\{[x]_A^{\leq} | [x]_A^{\leq} \subseteq X, x \in U\right\} = \left\{x \in U | [x]_A^{\leq} \subseteq X\right\}$.
2. $\overline{R}_A^{\leq}(X) = \cup\left\{[x]_A^{\leq} | [x]_A^{\leq} \cap X \neq \varnothing, x \in U\right\} = \left\{x \in U | [x]_A^{\leq} \cap X \neq \varnothing\right\}$.

Similar to the definition by Huynh and Nakamori (Huynh & Nakamori, 2005; Gong *et al*., 2008; Zhang, 2010), we give the following definitions.

Definition 3.2.
Let $\left(U, R_A^{\leq}, F\right)$ be interval-valued dominance information systems. For any $X, Y \subseteq U$,
X and Y are called lower rough equal if $\underline{R}_A^{\leq}(X) = \underline{R}_A^{\leq}(Y)$. Denote as $X \eqsim Y$.
X and Y are called upper rough equal if $\overline{R}_A^{\leq}(X) = \overline{R}_A^{\leq}(Y)$. Denote as $X \simeq Y$.
X and Y are called rough equal if they are both lower and upper equal. Denote as $X \approx Y$.

It is easy to know that the operators $\eqsim$, $\simeq$ and $\approx$ are equivalence relations. Then, the following conclusions are satisfied.

Theorem 3.2.
Let $\left(U, R_A^{\leq}, F\right)$ be interval-valued dominance information systems. For any subset $X, Y, X', Y' \subseteq U$, the following properties hold:

1. $X \eqsim Y \Leftrightarrow (X \cap Y) \eqsim X$ and $(X \cap Y) \eqsim Y$,
2. $X \simeq Y \Leftrightarrow (X \cup Y) \simeq X$ and $(X \cup Y) \simeq Y$,
3. If $X \eqsim X'$ and $Y \eqsim Y'$, then $(X \cap Y) \eqsim (X' \cap Y')$,
4. If $X \simeq X'$ and $Y \simeq Y'$, then $(X \cup Y) \simeq (X' \cup Y')$,
5. If $X \eqsim Y$, then $X \cap (\sim Y) \eqsim \varnothing$,
6. If $X \simeq Y$, then $X \cup (\sim Y) \simeq U$.

Proof
It is easy to prove by the aforementioned definition.

In general, the formula will not be satisfied when $\simeq$ is replaced by $\eqsim$ in Theorem 3.2 and vice versa.

Remark 3.2.
Let $\left(U, R_A^{\lesssim}, F\right)$ be interval-valued dominance information systems. For any X, $Y \subseteq U$, the following relations do not hold according to Remark 3.1 and Theorem 3.1.

1. $\underline{R}_A^{\lesssim}(X) = \cap\{Y \subseteq U | Y \approx X\}$,
2. $\overline{R}_A^{\lesssim}(X) = \cup\{Y \subseteq U | Y \simeq X\}$.

However, these two equations hold for real-valued information systems. This is the difference between the interval-valued dominance and real-valued information systems (Zhang *et al.*, 2003a, 2003b).

Definition 3.3.
Let $\left(U, R_A^{\lesssim}, F\right)$ be interval-valued dominance information systems. For any $X \subseteq U$, the precision of X in $\left(U, R_A^{\lesssim}, F\right)$ is defined as follows:

$$\alpha_A(X) = \frac{\left|\underline{R}_A^{\lesssim}(X)\right|}{\left|\overline{R}_A^{\lesssim}(X)\right|}$$

Moreover, $\rho_A(X) = 1 - \alpha_A(X)$ is called the roughness of X in $\left(U, R_A^{\lesssim}, F\right)$.

The ratio $\gamma_A(X) = \frac{\left|\underline{R}_A^{\lesssim}(X)\right|}{|U|}$ defines the quality of approximation of X by $R_A^{\lesssim}$ in $\left(U, R_A^{\lesssim}, F\right)$.

Clearly, there is $0 \leq \alpha_A(X) \leq 1$ and $\alpha_A(X) = 1$ if and only if $\underline{R}_A^{\lesssim}(X) = X = \overline{R}_A^{\lesssim}(X)$.

3.2. Dominance-based attribute reduction for interval-valued information systems

In this section, we study attribute reduction for interval-valued dominance information systems.

In general, objects are described by different attributes. However, it is not necessary to know all attributes for the classification of information systems. That is, some attributes are unnecessary and do not affect the result of classification when removed from the attribute set. Meanwhile, some attributes are indispensable to the result of classification when removed from the attribute set. Furthermore, some attributes are relatively necessary for the classification and may determine the result by associating with other attributes. The attribute reduction presents a minimum attribute subset completely describing the classification as the original attribute set for information systems.

First of all, we define the reduction for the interval-valued dominance information systems.

Let $\left(U, R_A^{\lesssim}, F\right)$ be interval-valued dominance information systems. If $B \subseteq A$ satisfies the following relations,

1. $R_B^{\lesssim} = R_A^{\lesssim}$;
2. For any $b \in B$, $R_{B-\{b\}}^{\lesssim} \neq R_A^{\lesssim}$.

then, B is called a reduction of $\left(U, R_A^{\lesssim}, F\right)$.

Clearly, it is easy to know that there is not only one reduction of interval-valued dominance information systems.

In what follows, we give the concepts of discernibility attribute set and discernibility matrix.

Definition 3.4.
Let $\left(U, R_A^{\lesssim}, F\right)$ be interval-valued dominance information systems.

$$D_P(x, y) = \{a_i \in A | f_{a_i}(x) \nleq f_{a_i}(y), x, y \in U\}$$

is called a discernibility attribute set of an interval-valued dominance information system.

Remark 3.3.
According to Remark 2.1, we know that the discernibility attribute set will be defined as

$$D_P(x, y) = \{a_i \in A | f_{a_i}(x) > f_{a_i}(y), x, y \in U\}$$

when $\left(U, R_A^{\lesssim}, F\right)$ is the real-valued information system.

Remark 3.4.
The relation of $D_P(x,y) \cap D_P(y,x) = \varnothing, x, y \in U$ could not hold in $\left(U, R_A^{\lesssim}, F\right)$, but it is correct for real-valued information systems.

This is the difference between the interval-valued dominance and real-valued information systems. The following Example 3.2 shows this difference directly.

Example 3.2. Continued from Example 3.1
By Table 1, we have, respectively, the values of elements x_1 and x_4 with respect to attribute A as follows:

$$f_{a_1}(x_1) = [0.1, 0.5], f_{a_1}(x_4) = [0.2, 0.9],$$
$$f_{a_2}(x_1) = [0.2, 0.6], f_{a_2}(x_4) = [0.1, 0.7],$$
$$f_{a_3}(x_1) = [0.1, 0.6], f_{a_3}(x_4) = [0.3, 0.8]$$

By Definition 3.4, we obtain the discernibility attribute set of x_1 and x_4 as follows:

$$D_P(x_1, x_4) = \{a_i \in A | f_{a_i}(x_1) \nleq f_{a_i}(x_4)\} = \{a_2\},$$
$$D_P(x_4, x_1) = \{a_i \in A | f_{a_i}(x_4) \nleq f_{a_i}(x_1)\} = \{a_1, a_2, a_3\}$$

Then, $D_P(x_1,x_4) \cap D_P(x_4,x_1) = \{a_2\} \neq \varnothing$.

In what follows, we discuss the attribute reduction for interval-valued dominance information systems.

Denote

$$D_P = \{D_P(x, y) | x, y \in U\}$$

Then, D_P is called the discernibility matrix of interval-valued dominance information systems $\left(U, R_A^{\lesssim}, F\right)$.

Denote

$$D_{P_0} = \{D_P(x, y) | D_P(x, y) \neq \varnothing\}$$

By the aforementioned definitions, the following conclusions are clear.

Theorem 3.4.
Let $\left(U, R_A^{\lesssim}, F\right)$ be interval-valued dominance information systems. For any $B \subseteq A$, the following propositions are equivalent.

1. $R_B^{\lesssim} = R_A^{\lesssim}$.
2. $B \cap B' \neq \varnothing$, for any $B' \in D_{P_0}$.
3. If $B \cap B' = \varnothing$, for any $B' \subseteq A$, then $B' \notin D_{P_0}$.

Proof
Firstly, we prove that 1 is equivalent to 2.

$1 \Rightarrow 2$: For any $x, y \in U$, there is $[x]_A^{\leqslant} = [y]_B^{\leqslant}$ according to $R_B^{\leqslant} = R_A^{\leqslant}$ .So, if $y \notin [x]_B^{\leqslant}$, it can be easily obtained that $f_{a_i}(x) \nleqslant f_{a_i}(y)$ for any $a_i \in B, x \in U$ by the definition of $[x]_B^{\leqslant}$. That is, $a_i \in D_P(x,y)$.

It proves $B' \cap B \neq \varnothing$ for any $B' \in D_{P_0}, x, y \in U$.

$2 \Rightarrow 1$: Clearly, there exists $a_i \in B$ that satisfies $a_i \in B'$ according to $B \cap B' \neq \varnothing$ for any $B' \in D_{P_0}$ .This shows $f_{a_i}(x) \nleqslant f_{a_i}(y), x, y \in U$. So, $[x]_B^{\leqslant} \cap [y]_B^{\leqslant} = \varnothing$. Furthermore, there are $[x]_A^{\leqslant} \subseteq [x]_B^{\leqslant}$ and $[y]_A^{\leqslant} \subseteq [y]_B^{\leqslant}$ by the relation $B \subseteq A$. So, there are $[x]_A^{\leqslant} = [x]_B^{\leqslant}$ and $[y]_A^{\leqslant} = [y]_B^{\leqslant}$ for any $x, y \in U$. Then, $R_B^{\leqslant} = R_A^{\leqslant}$.

Therefore, 1 is equivalent to 2.

The equivalence of 2 and 3 has a similar proof.

By the aforementioned discussion, the following theorem is clear.

Theorem 3.5.
Let $(U, R_A^{\leqslant}, F)$ be interval-valued dominance information systems. For any $B \subseteq A$, then B is a reduction of $(U, R_A^{\leqslant}, F)$ if and only if the following conditions are satisfied:

1. $B \cap B' \neq \varnothing$, for any $B' \in D_{P_0}$.
2. For any $b \in B$, there exists $B' \in D_{P_0}$ that satisfies $(B - \{b\}) \cap B' = \varnothing$.

Proof
It is easy to prove by the definition of reduction and Theorem 3.4.

Example 3.3. Continued from Example 3.1
Table 2 gives the discernibility matrix of interval-valued dominance information systems in Table 1.

By the aforementioned definition, we can obtain the discernible attribute set as follows:

$$D_{P_0} = \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, A\}$$

Let $B = \{a_2, a_3\}$.

By Table 1, it is easy to obtain the interval-valued dominance classes as follows:

$$[x_1]_B^{\leqslant} = \{x_1, x_2, x_5\} \qquad [x_2]_B^{\leqslant} = \{x_2, x_5\}$$
$$[x_3]_B^{\leqslant} = \{x_2, x_3, x_4, x_5\} \quad [x_4]_B^{\leqslant} = \{x_4\}$$
$$[x_5]_B^{\leqslant} = \{x_5\}$$

The conclusion (1) of Theorem 3.4 $R_B^{\leqslant} = R_A^{\leqslant}$ can be easily verified. Meanwhile, it is easy to verify that $B \cap B' \neq \varnothing$ holds for any $B' \in D_{P_0}$.

Table 2: *Discernibility matrix D_P*

U	x_1	x_2	x_3	x_4	x_5
x_1	$\varnothing$	$\varnothing$	$\{a_2\}$	$\{a_2\}$	$\varnothing$
x_2	$\{a_1, a_3\}$	$\varnothing$	A	$\{a_1, a_2\}$	$\varnothing$
x_3	$\{a_3\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_4	A	A	A	$\varnothing$	$\{a_3\}$
x_5	A	A	A	$\{a_1, a_2\}$	$\varnothing$

Let $B' = \{a_1\}$ and $B' = \varnothing$, respectively.

It also can be easily seen that the conclusion (3) of Theorem 3.4 holds.

Thus, the validity of Theorem 3.4 is tested. Furthermore, Theorem 3.5 also can be easily illuminated as the way to Theorem 3.4.

Therefore, $B = \{a_2, a_3\}$ is a reduction of interval-valued dominance information systems $(U, R_A^{\leqslant}, F)$.

4. Dominance-based knowledge reduction for interval-valued decision information systems

In this section, we establish some conclusions of knowledge reduction for interval-valued decision information systems by the rough set theory based on interval-valued dominance relation.

Actually, another similar problem in knowledge reduction is feature selection, which has been a hot problem investigated by many researchers from the artificial intelligence and machine learning community (Kohavi & John, 1997; Praczyk, *et al*., 1999; Dash & Liu, 1997, 2003; Xiong & Funk, 2006, 2010; Vale *et al*., 2010). Feature selection is an effective technique in dealing with dimensionality reduction. Many valuable algorithms and results have been established for feature selection problem by these scholars. Moreover, rough set theory also has been used as an effective approach to feature selection problems in recent years (Kuncheva, 1992; Thangavel & Pethalakshmi, 2009; Tsenga & Huang, 2007; Maria & Maite, 2011). In this paper, we discuss the knowledge reduction or feature selection problem for interval-valued decision information systems by using dominance-based rough set theory over interval-valued information systems in detail.

Decision information systems are special information systems with condition and decision attributes simultaneously. The decision information systems study the interrelationship between condition and decision attributes, and then acquire the decision rule from information systems.

Let $(U, R_A^{\leqslant}, F, D, g_D)$ be interval-valued dominance decision information systems, where $(U, R_A^{\leqslant}, F)$ is an interval-valued dominance information system, and D is a decision attribute set and satisfies $A \cap D = \varnothing$.

$F = \{f_{a_i} | U \to [I], a_i \in A\}$ is a family of mapping between U and A;

$g_D : U \to V_D$ is a discrete integer-valued mapping from universe U to decision attribute set D, where $V_D = \{1,2,\ldots,k\}$ is the domain of decision attribute.

So, an equivalence relation R_D is defined by the integer-valued mapping g_D. That is, $R_D = \{(x,y) | g_D(x) = g_D(y), x, y \in U\}$ is an equivalence relation over universe U.

Denote

$$U/R_D = \{[x]_D | x \in U\}$$

where $[x]_D = \{y | (x,y) \in R_D, x, y \in U\}$.

In what follows, we present the concept of consistent and inconsistent interval-valued dominance decision information systems.

Definition 4.1.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be interval-valued dominance decision information systems. If $R_A^{\leq} \subseteq R_D$, that is, $[x]_A^{\leq} \subseteq [x]_D$. Then $\left(U, R_A^{\leq}, F, D, g_D\right)$ is called a consistent interval-valued dominance decision information system.

Otherwise, $\left(U, R_A^{\leq}, F, D, g_D\right)$ is called an inconsistent interval-valued dominance decision information system.

Remark 4.1.
The binary relation based on condition and decision attributes is different over consistent and inconsistent interval-valued dominance information systems, that is, a dominance relation on condition attribute and an equivalence relation on decision attribute. However, the binary relation defined on condition and decision attributes is identical for real-valued decision information systems. That is, they define two equivalence relations or two dominance relations on condition and decision attributes (Zhang *et al.*, 2003a, 2003b; Zhang *et al.*, 2006; Xu & Zhang, 2008; Xu *et al.*, 2010).

In what follows, we discuss the methods of knowledge reduction for consistent and inconsistent interval-valued dominance decision information systems, respectively.

4.1. Consistent interval-valued dominance decision information systems

We first introduce the concept of inclusion degree.

Definition 4.2. Zhang *et al.*, 2003a, 2003b
Let $(X,\leq)$ be a partial order set. For any $x, y \in X$, there exists a number $D(y/x)$ that satisfies the following conditions:

1. $0 \leq D(y/x) \leq 1$.
2. $x \leq y \Rightarrow D(y/x) = 1$.
3. $x \leq y \leq z \Rightarrow D(x/z) \leq D(x/y)$.

Then, $D(\cdot)$ is called the inclusion degree over X.

Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be interval-valued dominance decision information systems for any $B \subseteq A$.

Denote

$$U/A_B^{\leq} = \left\{[x]_B^{\leq} | x \in U\right\}$$
$$U/R_D = \{D_1, D_2 \ldots, D_r\}$$

where $D_j = [x_j]_D, j = 1, 2, \ldots, r$.

Obviously, $U/A_B^{\leq}$ is a covering, but U/R_D is a partition of universe U.

Denote

$$D\left(D_j/[x]_B^{\leq}\right) = \frac{\left|D_j \cap [x]_B^{\leq}\right|}{\left|[x]_B^{\leq}\right|}, D_j \in U/R_D$$

where $|\cdot|$ stands for the cardinality of the set.

Then, $D\left(D_j/[x]_B^{\leq}\right)$ is an inclusion degree of P(U) (where P(U) stands for the power set of U) according to Definition 4.2.

On the basis of the inclusion degree $D\left(D_j/[x]_B^{\leq}\right)$, the following conclusion is clear.

Theorem 4.1.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be consistent interval-valued dominance decision information systems. For any $B \subseteq A$, the following propositions are equivalent.

1. $D\left([x]_D/[x]_B^{\leq}\right) = 1$, for any $x \in U$.
2. $R_B^{\leq} \subseteq R_D$.
3. $\frac{1}{|U|} \sum_{X \in U/R_D} \left|\underline{R}_B^{\leq}(X)\right| = 1$.

Proof
Because $D\left([x]_D/[x]_B^{\leq}\right) = 1 \Leftrightarrow [x]_B^{\leq} \subseteq [x]_D$, this proves that 1 is equivalent to 2.

According to Theorem 3.1, we have $\underline{R}_B^{\leq}(X) \subseteq X$.

Therefore, $\frac{1}{|U|} \sum_{X \in U/R_D} \left|\underline{R}_B^{\leq}(X)\right| = 1$ means that $\underline{R}_B^{\leq}(X) = X$.

That is, $R_B^{\leq} \subseteq R_D$.

So, we prove that 3 is equivalent to 2.

Lemma 4.1. Zhang *et al.*, 2003a, 2003b
Let $\mathcal{P}$ be a partition of universe U, $(\mathcal{P}, \leq)$ be a partial order set and D be an inclusion degree over $(P(U),\subseteq)$, for any two partitions $\mathcal{A}$ and $\mathcal{B}$ of universe U, where $\mathcal{A} = \{E_1, , E_2, \ldots, , E_k\}, \mathcal{B} = \{F_1, , F_2, \ldots, , F_l\}$. Denote

$$\mathcal{D}(\mathcal{B}/\mathcal{A}) = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{l} D\left(F_j/E_i\right)$$

Then, $\mathcal{B}$ is an inclusion degree over $(\mathcal{P}, \leq)$.

Proof
It is easy to obtain that $0 \leq \mathcal{D}(\mathcal{B}/\mathcal{A}) \leq 1$ by $0 \leq D(F_j/E_i) \leq 1$. Meanwhile, the relation $\mathcal{A} \leq \mathcal{B}$ means that there exists $F_j \in \mathcal{B}$ and satisfies the relation $E_i \subseteq F_j$ for any $E_i \in \mathcal{A}$. This is equivalent to $\vee_{j=1}^{l} D\left(F_j/E_i\right) = 1$.

Hence, it is also equivalent to $\mathcal{D}(\mathcal{B}/\mathcal{A}) = 1$.

Given $\mathcal{C} = \{G_1, G_2 \ldots, G_r\}$ and $\mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$.

Then, for any $E_i \in \mathcal{A}$, there exists $F_j \in \mathcal{B}$, and $G_p \in \mathcal{C}$ satisfies the relation $E_i \subseteq F_j \subseteq G_p$. Because $\mathcal{D}$ is the inclusion degree over $(\mathcal{P}(U), \subseteq)$, so there is $D(E_i/G_p) \leq D(E_i/F_j)$.

Therefore, it proves the relation $\mathcal{D}(\mathcal{A}/\mathcal{C}) \leq \mathcal{D}(\mathcal{A}/\mathcal{B})$.

Lemma 4.2.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be consistent interval-valued dominance decision information systems. For any $B \subseteq A$, denote $C_B\left(R_B^{\leq}\right)$ as the covering of universe U that is generated by the interval-valued dominance relation $R_B^{\leq}$. Then,

$$D\left((U/R_D)/C_B\left(R_B^{\leq}\right)\right) = \min_{x \in U} D\left([x]_D/[x]_B^{\leq}\right)$$

and

$$D\left((U/R_D)/C_B\left(R_B^{\leq}\right)\right) = \frac{1}{|U|} \sum_{X \in U/R_D} \left|\underline{R}_B^{\leq}(X)\right|$$

are the inclusion degree of universe U.

Proof
It can be easily proved by Theorem 4.1 and Lemma 4.1.

In the following, we establish the definition of reduction and also provide the judgement theorem of knowledge reduction based on the inclusion degree for consistent interval-valued dominance decision information systems by Theorem 4.1 and Lemma 4.2.

Definition 4.3.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be consistent interval-valued dominance decision information systems. For any $B \subseteq A$, if the following equations are satisfied

1. $D\left((U/R_D)/C_B\left(R_B^{\leq}\right)\right) = 1.$
2. $D\left((U/R_D)/C_B\left(R_{B-\{b\}}^{\leq}\right)\right) < 1$ for any $b \in B$.

then B is called a reduction of (U,A,F,D,g_D).

Definition 4.4. Zhang *et al.*, 2006; Xu & Zhang, 2008; Xu *et al.*, 2010; Li *et al.*, 2010
Let $\alpha = (a_1,a_2,\ldots,a_n), \beta = (b_1,b_2,\ldots,b_n)$ be two n dimension vectors. If $a_i = b_i (i = 1,2,\ldots,n)$, then α is called equal to β, denoted as $\alpha = \beta$. If $a_i \leq b_i (i = 1,2,\ldots,n)$, then α is called less than β, denoted as $\alpha \leq \beta$.

Otherwise, if there exists $i_0 (i_0 \in \{1,2,\ldots,n\})$, and $a_{i_0} > b_{i_0}$ is satisfied, then α is defined as no greater than β, denoted as $\alpha \nleq \beta$.

Remark 4.2.
If $a_i, b_i \in [I]$, then α and β are called n-dimension interval-valued vectors. Therefore, similar to Definition 4.4, the definition of the relations $\alpha = \beta$, $\alpha \leq \beta$ and $\alpha \nleq \beta$ is clear for an n-dimension interval-valued vector.

Remark 4.3.
It can be easily seen that $\alpha \nleq \beta$ is not equivalent to $\alpha > \beta$ because both α and β are interval-valued vectors. So is $\alpha < \beta$ and $\alpha \neq \beta$.

In what follows, we present the description of knowledge reduction for consistent interval-valued dominance decision information systems.

Definition 4.5.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be consistent interval-valued dominance decision information systems. Define

$$D_g(x,y) = \begin{cases} \{a_k \in A | f_{a_k}(x) \nleq f_{a_k}(y)\}, & g_D(x) \neq g_D(y), \\ A, & g_D(x) = g_D(y), \end{cases}$$

where $f_{a_k}(x)$ stands for the value of object $x \in U$ about attribute a_k.

Then, $D_g(x,y)$ is called the dominated discernibility attribute set of objects x and y with respect to consistent interval-valued dominance decision information systems.

Moreover, $D_g = (D_g(x,y) \,|\, x,y \in U)$ is called the dominated discernibility matrix of consistent interval-valued dominance decision information systems.

Here, the term for the aforementioned definition comes from Zhang *et al.* (2006), but the conditions in the definition are defined in a different manner.

Theorem 4.2.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be consistent interval-valued dominance decision information system. For any $B \subseteq A$,

1. $\forall x, y \in U, R_B^{\leq} \subseteq R_D \Leftrightarrow B \cap D_g(x,y) \neq \varnothing$,
2. $R_B^{\leq} \subseteq R_D \Leftrightarrow \forall B' \subseteq A$, if $B \cap B' = \varnothing$, then $B' \notin D_g$.

Proof
1. If $R_B^{\leq} \subseteq R_D$, then there is $D_g(x,y) = A$ when $g_D(x) = g_D(y)$. So, $B \cap D_g(x,y) \neq \varnothing$.

Meanwhile, there is $y \notin [x]_D$ when $g_D(x) \neq g_D(y)$. Then, $[x]_B^{\leq} \subseteq [x]_D$ is satisfied by $R_B^{\leq} \subseteq R_D$. So, $y \notin [x]_B^{\leq}$ and $a_k \in B$ satisfy $f_{a_k}(x) \nleq f_{a_k}(y)$. That is, $a_k \in D_g(x,y)$.

This proves $B \cap D_g(x,y) \neq \varnothing$.

Conversely, there is $g_D(x) \neq g_D(y)$ when $y \notin [x]_D$ for any $x, y \in U$. Then, by $B \cap D_g(x,y) \neq \varnothing$, we know there exists $a_k \in B$ satisfying $f_{a_k}(x) \nleq f_{a_k}(y)$. So, $y \notin [x]_B^{\leq}$. That is, $[x]_B^{\leq} \subseteq [x]_D$.

This proves $R_B^{\leq} \subseteq R_D$.

2. By the conclusion of 1, $R_B^{\leq} \subseteq R_D \Leftrightarrow$ for any $B' \in D_g$, and there is $B \cap B' \neq \varnothing$.

By Theorem 4.2, the following conclusion is clear.

For $B \subseteq A$, denote

$$\gamma(B) = \Pi_{x,y \in U} \chi_B\left(D_g(x,y)\right)$$

where

$$\chi_B\left(D_g(x,y)\right) = \begin{cases} 1, & B \cap D_g(x,y) \neq \varnothing, \\ 0, & B \cap D_g(x,y) = \varnothing \end{cases}$$

Theorem 4.3.
Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be consistent interval-valued dominance decision information systems. $D_g = \{D_g(x,y) \,|\, x,y \in U\}$ is a dominated discernibility matrix. For any $B \subseteq A$, there is

$$\gamma(B) = 1 \Leftrightarrow R_B^{\leq} \subseteq R_D$$

Proof
It is easy to prove by the aforementioned definition.

Theorems 4.2 and 4.3 give the conditions of sufficiency and necessity of knowledge reduction for consistent interval-valued dominance decision information systems.

Next, we discuss the knowledge reduction for inconsistent interval-valued dominance decision information systems.

4.2. Inconsistent interval-valued dominance decision information systems

In general, we also can define the distribution reduction, generalized distribution reduction and maximum distribution reduction with corresponding judgement theorems of knowledge reduction similar to real-valued inconsistent decision, inconsistent ordered and set-valued decision and others (Wang & Zhang, 2006; Xu & Zhang, 2008; Xu *et al.*, 2010; Li *et al.*, 2010; Zhang, 2010).

In this section, we discuss another two reductions for inconsistent interval-valued dominance decision information systems.

We begin with some basic definitions as follows.

Let $\left(U, R_A^{\leq}, F, D, g_D\right)$ be inconsistent interval-valued dominance decision information systems.

For any $B \subseteq A$, denote

$$\eta_B = \frac{1}{|U|}\sum_{j=1}^{r}|\bar{R}_B^{\leq}(D_j)|$$

$$\delta_B(x) = \{D_j | D_j \cap [x]_B^{\leq} \neq \varnothing, x \in U\}$$

where $D_j \in U/R_D = \{[x]_D \mid g_D(x) = g_D(y), x, y \in U\}$.

It is easy to know that the following two propositions are clear by $\delta_B(x)$ and η_B.

Proposition 4.1.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A$,

1. $\delta_A(x) \subseteq \delta_B(x)$ for any $x \in U$.
2. If $[x]_B^{\leq} \subseteq [y]_B^{\leq}$, then $\delta_B(x) \subseteq \delta_B(y)x$, for any $y \in U$.

Proposition 4.2.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A$, the following conditions hold.

1. $1 \leq \eta_B \leq |U/R_D|$.
2. For any $D_j \in U/R_D$,

$$\eta_B = 1 \Leftrightarrow \bar{R}_B^{\leq}(D_j) = D_j,$$
$$\eta_B = |U/R_D| \Leftrightarrow \bar{R}_B^{\leq}(D_j) = U, D_j \in U/R_D$$

3. If $\bar{R}_A^{\leq}(D_j) \subseteq \bar{R}_B^{\leq}(D_j)$, then $\eta_A \leq \eta_B$.

On the basis of $\delta_B(x)$ and η_B, we give two reductions for inconsistent interval-valued dominance decision information system as follows.

Definition 4.6.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A$,

1. If $\delta_A(x) = \delta_B(x)$ for any $x \in U$, then B is called a dominated allocation consistent set. Furthermore, if $B - \{a\}$ for any $a \in B$ is not a dominated allocation consistent set, then B is called a dominated allocation reduction;
2. If $\eta_B = \eta_A$ for any $x \in U$, then B is called an approximation-dominated consistent set. Furthermore, if $B - \{a\}$ for any $a \in B$ is not an approximation-dominated consistent set, then B is called an approximation-dominated reduction.

Theorem 4.4.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A$, then B is a dominated allocation consistent set if and only if B is an approximation-dominated consistent set.

Proof
Let B be a dominated allocation consistent set. That is, $\delta_B(x) = \delta_A(x)$ for any $x \in U$. Then,

$$\begin{aligned} x \in \bar{R}_B^{\leq}(D_j) &\Leftrightarrow [x]_B^{\leq} \cap D_j \neq \varnothing \\ &\Leftrightarrow D_j \in \delta_B(x) \Leftrightarrow D_j \in \delta_A(x) \\ &\Leftrightarrow [x]_A^{\leq} \cap D_j \neq \varnothing \Leftrightarrow x \in \bar{R}_A^{\leq}(D_j) \end{aligned}$$

This proves $\bar{R}_B^{\leq}(D_j) = \bar{R}_A^{\leq}(D_j)$.

Therefore, $\eta_B = \eta_A$, and B is the approximation-dominated consistent set.

Conversely, if B is an approximation-dominated consistent set, then $\eta_B = \eta_A$.

That is, $\sum_{j=1}^{r}|\bar{R}_B^{\leq}(D_j)| = \sum_{j=1}^{r}|\bar{R}_A^{\leq}(D_j)|$.

Meanwhile, $\bar{R}_A^{\leq}(D_j) \subseteq \bar{R}_B^{\leq}(D_j)$ holds for any $D_j \in U/R_D$. Therefore, $\bar{R}_A^{\leq}(D_j) = \bar{R}_B^{\leq}(D_j)$. So,

$$\begin{aligned} D_j \in \delta_B(x) &\Leftrightarrow [x]_B^{\leq} \cap D_j \neq \varnothing \\ &\Leftrightarrow x \in \bar{R}_B^{\leq}(D_j) \Leftrightarrow x \in \bar{R}_A^{\leq}(D_j) \\ &\Leftrightarrow [x]_A^{\leq} \cap D_j \neq \varnothing \Leftrightarrow D_j \in \delta_A(x) \end{aligned}$$

This proves $\delta_B(x) \subseteq \delta_A(x)$.

Furthermore, we know that $\delta_A(x) \subseteq \delta_B(x)$ by Proposition 4.2. Thus, we prove that $\delta_A(x) = \delta_B(x)$.

This proves that B is a dominated allocation consistent set.

Example 4.1.
Table 3 gives an inconsistent interval-valued dominance decision information system.

It is easy to obtain the interval-valued dominance classes and equivalence classes by condition attribute A and decision attribute D, respectively, as follows:

$$\begin{aligned} &[x_1]_A^{\leq} = \{x_1, x_2, x_5\} && [x_2]_A^{\leq} = \{x_2, x_5\} \\ &[x_3]_A^{\leq} = \{x_2, x_3, x_4, x_5\} && [x_4]_A^{\leq} = \{x_4\} \\ &[x_5]_A^{\leq} = \{x_5\} \end{aligned}$$

and

$$\begin{aligned} &[x_1]_D = [x_5]_D = \{x_1, x_5\} \\ &[x_2]_D = [x_4]_D = \{x_2, x_4\} \\ &[x_3]_D = \{x_3\} \end{aligned}$$

It can be easily seen that $\cup_{x \in U}[x]_A^{\leq}$ is a covering, and $\cup_{x \in U}[x]_D$ is a partition of universe U. Let

$$D_1 = [x_1]_D = [x_5]_D \quad D_2 = [x_2]_D = [x_4]_D$$
$$D_3 = [x_3]_D$$

Table 3: *Inconsistent interval-valued dominance decision information system*

U	a_1	a_2	a_3	D
x_1	[0.1, 0.5]	[0.2, 0.6]	[0.1, 0.6]	3
x_2	[0.3, 0.8]	[0.2, 0.6]	[0.2, 0.7]	2
x_3	[0.1, 0.5]	[0.1, 0.6]	[0.2, 0.6]	1
x_4	[0.2, 0.9]	[0.1,0.7]	[0.3, 0.8]	2
x_5	[0.3, 1]	[0.3,0.9]	[0.2, 0.8]	3

We calculate the value of $\delta_A(x)(x \in U)$ as follows:

$$\delta_A(x_1) = \{D_1, D_2\}, \delta_A(x_2) = \{D_1, D_2\},$$
$$\delta_A(x_3) = \{D_1, D_2, D_3\}, \delta_A(x_4) = \{D_2\},$$
$$\delta_A(x_5) = \{D_1\}$$

Taking $B = \{a_2, a_3\} \subseteq A = \{a_1, a_2, a_3\}$. for any $x \in U$, we can easily obtain that $[x]_A^{\leq} = [x]_B^{\leq}$.

Then, $\delta_B(x) = \delta_A(x)$ holds.

So, B is a dominated allocation consistent set.

In what follows, we give the two descriptive theorems of knowledge reduction for inconsistent interval-valued dominance decision information systems.

Theorem 4.5.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A, x, y \in U$, B is a dominated allocation consistent set if and only if $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$ when $\delta_A(x) \nsubseteq \delta_A(y)$.

Proof
1. Suppose that $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$ does not hold when $\delta_A(x) \nsubseteq \delta_A(y)$ for any $x, y \in U$. That is, $[x]_B^{\leq} \subseteq [y]_B^{\leq}$, then we know $\delta_B(x) \subseteq \delta_B(y)$ by Proposition 4.1.

Moreover, B is a dominated allocation consistent set. So, $\delta_A(x) = \delta_B(x)$ and $\delta_A(y) = \delta_B(y)$. Therefore, there is $\delta_A(x) \subseteq \delta_A(y)$. This is a contradiction.

Conversely, we only need to prove that $\delta_B(x) = \delta_A(x)$ for any $x \in U$. It is easy to know that $\delta_A(x) \subseteq \delta_B(x)$ holds. Then, we only prove $\delta_B(x) \subseteq \delta_A(x)$.

In what follows, we prove this conclusion by two different cases.

- If $\delta_B(x) = \varnothing$, the conclusion holds.
- If $\delta_B(x) \neq \varnothing$, $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$ when $\delta_A(x) \nsubseteq \delta_A(y)$.

That is, if $[x]_B^{\leq} \subseteq [y]_B^{\leq}$, then $\delta_A(x) \subseteq \delta_A(y)$ holds for any $x, y \in U$.

Because $\delta_B(x) \neq \varnothing$ and $D_j \cap [x]_B^{\leq} \neq \varnothing$ hold for any $D_j \in U/R_D$, let $y \in D_j \cap [x]_B^{\leq}$. Then, there are $y \in [x]_B^{\leq}$ and $y \in D_j$. So, $[y]_B^{\leq} \subseteq [x]_B^{\leq}$.

This proves that $\delta_A(y) \subseteq \delta_A(x)$ holds.

It is easy to know that $y \in [y]_A^{\leq}$, that is, $y \in D_j \cap [y]_A^{\leq}$. So, $D_j \cap [x]_A^{\leq} \subseteq D_j \cap [y]_A^{\leq}$.

This proves that $\delta_B(x) \subseteq \delta_A(y)$.

The aforementioned two cases prove that $\delta_B(x) \subseteq \delta_A(x)$ hold.

So, the conclusion is proved.

The following is a corollary.

Corollary 4.1.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A$, the following two conclusions are equivalent.

1. B is an approximation-dominated consistent set.
2. $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$ when $\delta_A(x) \nsubseteq \delta_A(y), x, y \in U$.

On the basis of dominated consistent attribute sets, we can obtain the reduction for inconsistent interval-valued dominance decision information systems.

In what follows, we give another descriptive theorem of knowledge reduction for inconsistent interval-valued dominance decision information systems.

First of all, we present the following definition.

Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. Denote

$$D_\delta^* = \{(x, y) | \delta_A(x) \subseteq \delta_A(y)\}$$

Definition 4.7.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. Define

$$D_\delta(x, y) = \begin{cases} \{a_k \in A | f_{a_k}(x) \nleq f_{a_k}(y)\}, & (x, y) \in D_\delta^*, \\ A, & (x, y) \notin D_\delta^* \end{cases}$$

Then, $D_\delta(x,y)$ is called the dominated allocation discernibility attribute set of x and y.

Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. Denote

$$D_\delta = (D_\delta(x, y) | x, y \in U)$$

Then, D_δ is called the dominated allocation discernibility matrix for inconsistent interval-valued dominance decision information systems.

On the basis of the aforementioned definitions, we have the following descriptive theorem.

Theorem 4.6.
Let $(U, R_A^{\leq}, F, D, g_D)$ be inconsistent interval-valued dominance decision information systems. For any $B \subseteq A$, B is a dominated allocation consistent set if and only if $B \cap D_\delta(x,y) \neq \varnothing$ for any $(x, y) \in D_\delta^*$.

Proof
1. For any $x, y \in U$, $\delta_A(x) \nsubseteq \delta_A(y)$ holds when $(x,y) \in U$. Furthermore, we have $[x]_B^{\leq} \subseteq [y]_B^{\leq}$ because B is a dominated allocation consistent set.

Then, there are three cases for $[x]_B^{\leq}$ and $[y]_B^{\leq}$.

(a) $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$,
(b) $[x]_B^{\leq} \cap [y]_B^{\leq} = \varnothing$,
(c) $[x]_B^{\leq} \cap [y]_B^{\leq} \subseteq [x]_B^{\leq}$ and $[x]_B^{\leq} \cap [y]_B^{\leq} \subseteq [y]_B^{\leq}$.

In what follows, we prove $B \cap D_\delta(x,y) \neq \varnothing$ for the aforementioned three cases.

(a) If $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$, there exists $z \in [y]_B^{\leq}$ that satisfies $z \notin [x]_B^{\leq}$. Then, there exists $a_k \in B$ that satisfies $f_{a_k}(x) \nleq f_{a_k}(z)$. So, $f_{a_k}(y) \leq f_{a_k}(z)$ according to $z \in [y]_B^{\leq}$. Therefore, $f_{a_k}(y) \leq f_{a_k}(x)$.

This proves $a_k \in D_\delta(x,y)$.

That is, $B \cap D_\delta(x,y) \neq \varnothing$.

(b) If $[x]_B^{\leq} \cap [y]_B^{\leq} = \varnothing$, there exists $a_k \in B$ that satisfies $f_{a_k}(x) \nleq f_{a_k}(y)$. That is, $B \cap D_\delta(x,y) \neq \varnothing$.

Otherwise, there is $f_{a_k}(x) \leq f_{a_k}(y)$ for any $a_k \in B$. So, it proves that $y \in [x]_B^{\leq}$.

This is contradicted by $[x]_B^{\leq} \cap [x]_B^{\leq} = \varnothing$.

(c) If $[x]_B^{\leq} \cap [y]_B^{\leq} \subseteq [x]_B^{\leq}$ and $[x]_B^{\leq} \cap [y]_B^{\leq} \subseteq [y]_B^{\leq}$, then it can be proved in the same way as (*a*).

Considering the aforementioned discussion, $B \cap D_\delta$ $(x, y) \neq \varnothing$ when B is a dominated allocation consistent set for $(x, y) \in D_\delta^*$.

Conversely, there is $B \cap D_\delta(x,y) \neq \varnothing$ for $(x, y) \in D_\delta^*$. Then, there exists $a_k \in B$ satisfying $a_k \in D_\delta(x,y)$.

So, we have $f_{a_k}(x) \nleq f_{a_k}(y)$. That is, $y \notin [x]_B^{\leq}$.

Furthermore, there is $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$ because $y \in [y]_B^{\leq}$. Meanwhile, there is $\delta_A(x) \nsubseteq \delta_A(y)$ for any $(x, y) \in D_\delta^*$. Hence, we have $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$ when $\delta_A(x) \nsubseteq \delta_A(y)$.

Thus, B is a dominated allocation consistent set.

Example 4.2. Continued from Example 4.1

Table 4 gives the discernibility matrix for inconsistent interval-valued dominance decision information systems.

From Table 4, it easy to know that $B = \{a_2, a_3\}$ is a dominated allocation reduction of an inconsistent interval-valued dominance decision information system $\left(U, R_A^{\leq}, F, D, g_D\right)$.

Moreover, the following results can be easily calculated according to Example 4.1:

$$[x_1]_B^{\leq} = \{x_1, x_2, x_5\}, \quad [x_2]_B^{\leq} = \{x_2, x_5\},$$
$$[x_3]_B^{\leq} = \{x_2, x_3, x_4, x_5\}, \quad [x_4]_B^{\leq} = \{x_4\},$$
$$[x_5]_B^{\leq} = \{x_5\}$$

and

$$\delta_A(x_1) = \{D_1, D_2\}, \delta_A(x_2) = \{D_1, D_2\},$$
$$\delta_A(x_3) = \{D_1, D_2, D_3\}, \delta_A(x_4) = \{D_2\},$$
$$\delta_A(x_5) = \{D_1\}$$

By these results, we can verify the condition:

If $\delta_A(x) \nsubseteq \delta_A(y)$, then $[x]_B^{\leq} \nsubseteq [y]_B^{\leq}$, for any $x, y \in U$.

This is the conclusion of Theorem 4.5.

Furthermore, it can be easily known that $D_\delta(x,y) = \{\{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, A\}$ by Definition 4.7.

Then we have $B \cap D_\delta(x,y) \neq \varnothing$, for any $x, y \in U$.

This is the conclusion of Theorem 4.6.

Table 4: *Dominated allocation discernibility matrix* D_δ

U	x_1	x_2	x_3	x_4	x_5
x_1	$\varnothing$	$\varnothing$	$\{a_2\}$	A	A
x_2	$\{a_1, a_3\}$	$\varnothing$	A	A	A
x_3	$\{a_3\}$	$\varnothing$	$\varnothing$	A	A
x_4	A	A	A	$\varnothing$	A
x_5	A	A	A	$\{a_1, a_2\}$	$\varnothing$

5. Conclusions and remarks

Knowledge reduction over information systems and the generalization of real-valued information systems are two main research directions both in theory and application of rough set theory. This paper studies a type of extended real-valued information systems, that is, interval-valued information systems. Moreover, we study the basic rough set theory over interval-valued information systems. We systematically discuss attribute reduction for interval-valued dominance information systems based on the proposed rough set model. To discuss knowledge reduction for interval-valued dominance decision information systems, some concepts concerning the reduction of both consistent and inconsistent interval-valued dominance decision information systems have been established by the definitions of an interval-valued dominance relation and an equivalence relation in $\left(U, R_A^{\leq}, F, D, g_D\right)$.Some descriptive theorems for knowledge reduction are given in detail for interval-valued dominance decision information systems. It is also an extension of real-valued decision information systems. A numerical example is applied to test the methods and conclusions of the interval-valued dominance decision information systems as well.

There are at least two aspects in the study of rough set theory: constructive and axiomatic approaches (Wu and Zhang 2004). The same is true for interval-valued information systems. In this paper, we define the rough lower (upper) approximation operator and discuss the basic properties by the constructive method. So further work should consider the axiomatic approaches to rough lower (upper) approximation operator and decision-making in the context of an interval-valued environment.

Acknowledgements

The authors thank the editor-in-chief Dr Jon Hall and the anonymous reviewers for valuable comments to improve the quality of this paper. The work was partly supported by the National Science Foundation of China (71161016 and 71071113), the Foundation for the Author of National Excellent Doctoral Dissertation of PR China (200782), the Shuguang Plan of Shanghai Education Development Foundation and Shanghai Education Committee (08SG21), the Shanghai Pujiang Program, and Shanghai Philosophical and Social Science Program (2010BZH003) and the Fundamental Research Funds for the Central Universities. The third author is supported by the National Science Foundation of China (71061013).

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