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A direct interval extension of TOPSIS method

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ABSTRACT

The *TOPSIS* method is a technique for order preference by similarity to ideal solution. This technique currently is one of the most popular methods for Multiple Criteria Decision Making (*MCDM*). The *TOPSIS* method was primary developed for dealing with only real-valued data. In many cases, it is hard to present precisely the exact ratings of alternatives with respect to local criteria and as a result these ratings are considered as intervals. There are some papers devoted to the interval extensions of *TOPSIS* method, but these extensions are based on different heuristic approaches to definition of positive and negative ideal solutions. These ideal solutions are presented by real values or intervals, which are not attainable in a decision matrix. Since this is in contradiction with basics of classical *TOPSIS* method, in this paper we propose a new direct approach to interval extension of *TOPSIS* method which is free of heuristic assumptions and limitations of known methods. Using numerical examples we show that "direct interval extension of *TOPSIS* method" may provide the final ranking of alternatives which is substantially different from the results obtained using known methods.

1. Introduction

The technique for order performance by similarity to ideal solution (*TOPSIS*) (Lai, Liu, & Hwang, 1994) is one of known classical *MCDM* method. It was first developed by Hwang and Yoon (1981) for solving *MCDM* problems.

The basic principle of the *TOPSIS* method is that the chosen alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. There exist a large amount of literature involving *TOPSIS* theory and applications.

Is was shown by Garca-Cascales and Lamata (2012) and Wang and Luo (2009) that "One of the problems attributable to *TOPSIS* is that it can cause the phenomenon known as rank reversal. In this phenomenon the alternative's order of preference changes when an alternative is added to or removed from the decision problem. In some cases this may lead to what is called total rank reversal, where the order of preferences is totally inverted, that is to say, that the alternative considered the best, with the inclusion or removal of an alternative from the process, then becomes the worst. Such a phenomenon in many cases may not be acceptable". Wang and Luo (2009) showed that rank reversal phenomenon occurs not only in the *TOPSIS* method, but in many other decision making approaches such as Analytic Hierarchy Process (*AHP*), the Borda–Kendall (*BK*) method for aggregating ordinal preferences, the simple additive weighting (*SAW*) method, and the cross-efficiency evaluation method in data envelopment analysis (*DEA*). Therefore, we can say that this problem is typical for known method of *MCDM*. In Garca-Cascales and Lamata (2012), the authors proposed a new method for the solution of this problem in the framework of *TOPSIS* method. Nevertheless it was pointed out in Garca-Cascales and Lamata (2012) that "the two methods "the classical" and "the new" do not have to give the same order. This is especially so in the case of evaluating alternatives which are very close". In other words, "the classical" and "the new" methods may provide different results based on the same decision matrix. Hence, it is not obvious that "the new" method performs better than classical one if there is no need to add or remove an alternative from the decision problem.

Therefore, hereinafter we shall consider only classical *TOPSIS* method and its interval extension.

In classical *MCDM* methods, the ratings and weights of criteria are known precisely. A survey of these methods is presented in Hwang and Yoon (1981). In the classical *TOPSIS* method, the ratings of alternatives and the weights of criteria are presented by real values.

Nevertheless, sometimes it is difficult to determine precisely the real values of ratings of alternatives with respect to local criteria, and as a result, these ratings are presented by intervals.

Jahanshahlo, Hosseinzade, and Izadikhah (2006) and Jahanshahloo, Hosseinzadeh Lotfi, and Davoodi (2009) extended the concept of *TOPSIS* method to develop a methodology for solving *MCDM* problem with interval data. The main limitation of this approach is that the ideal solutions are presented by real values, not by

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intervals. The similar approach to determining ideal solutions is used in Yue (2011) and Sayadi, Heydari, and Shahanaghi (2009) this approach is used in context of the so-called *VIKOR* method which is based on the measure of "closeness" to the ideal solutions, too.

In this paper, we show that these extensions may lead to the wrong results especially in the case of intersection of some intervals representing the ratings of alternatives.

In Chen (2011), Jahanshahloo et al. (2009), Jahanshahloo, Khodabakhshi, Hosseinzadeh Lotfi, and Moazami Goudarzi (2011), Tsaur (2011), Ye and Li (2009) and Yue (2011), the different definitions of interval positive and negative ideal solutions are proposed. They will be analysed in the next section, but their common limitation is that they are based on the heuristic assumptions (usually without any analysis and clear justification) and provide interval ideal solutions that are not always attainable in the interval-valued decision matrix.

Since this is in contradiction with basics of classical *TOPSIS* method, in this paper, we propose a new direct approach to interval extension of *TOPSIS* method which is free heuristic assumptions and limitations of known methods. This approach makes it possible to obtain the positive and negative ideal solutions in the interval form such that (opposite to the known methods) these interval-valued solutions are always attainable on the initial interval-valued decision matrix. Since this approach is based on the interval comparison, a new simple, but well-justified method for interval comparison is developed and presented in the special section.

It is worth noting that the most general approach to the solution of *MCDM* problems in the fuzzy setting is the presentation of all fuzzy values by corresponding sets of α-cuts. There are no restrictions on the shape of membership functions of fuzzy values in this approach and the fuzzy *TOPSIS* method is reduced to the solution of *MCDM* problems using interval extended *TOPSIS* method on the corresponding α-cuts (Wang & Elhag, 2006).

Therefore, the development of a reliable interval extension of *TOPSIS* method may be considered as a first step in the solution of *MCDM* problems using the fuzzy *TOPSIS* method.

The rest of the paper is set out as follows. In Section 2, we present the basics of *TOPSIS* method and analyse its known interval extensions. Section 3 presents the direct interval extension of *TOPSIS* method and the method for interval comparison which is needed to develop this extension. The results obtained using the method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) are compared with those obtained by the developed new method. Section 4 concludes with some remarks.

2. The basics of *TOPSIS* method and known approaches to its interval extension

The classical *TOPSIS* method is based on the idea that the best alternative should have the shortest distance from the positive ideal solution and the farthest distance from the negative ideal solution. It is assumed that if each local criterion is monotonically increasing or decreasing, then it is easy to define an ideal solution.

The positive ideal solution is composed of all the best achievable values of local criteria, while the negative ideal solution is composed of all the worst achievable values of local criteria.

Suppose a *MCDM* problem is based on m alternatives $A_1, A_2, \ldots, A_m$ and n criteria $C_1, C_2, \ldots, C_n$. Each alternative is evaluated with respect to the n criteria. All the ratings are assigned to alternatives and presented in the decision matrix $D[x_{ij}]_{m\times n}$, where x_{ij} is the rating of alternative A_i with respect to the criterion C_j. Let $W = (w_1, w_2, \ldots, w_n)$ be the vector of local criteria weights satisfying $\sum_{j=1}^{n} w_j = 1$.

The *TOPSIS* method consists of the following steps:

1. Normalize the decision matrix:

$$r_{ij} = \frac{x_{ij}}{\sqrt{\sum_{k=1}^{m} x_{kj}^2}}, \quad i = 1,\ldots,m;\ j = 1,\ldots,n. \tag{1}$$

Multiply the columns of normalized decision matrix by the associated weights:

$$v_{ij} = w_j \times r_{ij}, \quad i = 1,\ldots,m;\ j = 1,\ldots,n. \tag{2}$$

2. Determine the positive ideal and negative ideal solutions, respectively, as follows:

$$\mathrm{A}^+ = \{v_1^+, v_2^+, \ldots, v_n^+\} = \{(\max_i v_{ij} | j \in K_b)(\min_i v_{ij} | j \in K_c)\}, \tag{3}$$

$$\mathrm{A}^- = \{v_1^-, v_2^-, \ldots, v_n^-\} = \{(\min_i v_{ij} | j \in K_b)(\max_i v_{ij} | j \in K_c)\}, \tag{4}$$

where K_b is the set of benefit criteria and K_c is the set of cost criteria.

3. Obtain the distances of the existing alternatives from the positive ideal and negative ideal solutions: two Euclidean distances for each alternatives are, respectively, calculated as follows:

$$S_i^+ = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2}, \quad i = 1,\ldots,m,$$
$$S_i^- = \sqrt{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2}, \quad i = 1,\ldots,m. \tag{5}$$

4. Calculate the relative closeness to the ideal alternatives:

$$RC_i = \frac{S_i^-}{S_i^+ + S_i^-}, \quad i = 1,2,\ldots,m,\ 0 \leqslant RC_i \leqslant 1. \tag{6}$$

5. Rank the alternatives according to the relative closeness to the ideal alternatives: the bigger is the RC_i, the better is the alternative A_i.

In Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009), an interval extension of classical *TOPSIS* method was proposed. This approach may be described as follows.

Let $[x_{ij}] = [x_{ij}^L, x_{ij}^U]$ be an interval value of jth criterion for ith alternative (x_{ij}^L and x_{ij}^U are the lower and upper bounds of interval, respectively), $W = (w_1, w_2, \ldots, w_n)$ be the weight vector satisfying $\sum_{j=1}^{n} w_j = 1$. Then $D[[x_{ij}^L, x_{ij}^U]]_{m\times n}$ is the interval-valued decision matrix. The method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) consists of the following steps:

1. Normalizing the decision matrix using the following expressions:

$$r_{ij}^L = \frac{x_{ij}^L}{\left(\sum_{k=1}^{m}\left(\left(x_{kj}^L\right)^2 + \left(x_{kj}^U\right)^2\right)\right)^{\frac{1}{2}}}, \quad i = 1,\ldots,m; j = 1,\ldots,n, \tag{7}$$

$$r_{ij}^U = \frac{x_{ij}^U}{\left(\sum_{k=1}^{m}\left(\left(x_{kj}^L\right)^2 + \left(x_{kj}^U\right)^2\right)\right)^{\frac{1}{2}}}, \quad i = 1,\ldots,m; j = 1,\ldots,n. \tag{8}$$

2. Taking into account the importance of criteria, the weighted normalized interval-valued decision matrix is obtained using the following expressions:

$$v_{ij}^L = w_j \times r_{ij}^L, \quad v_{ij}^U = w_j \times r_{ij}^U, \quad i = 1,\ldots,m;\ j = 1,\ldots,n.$$

3. The positive and negative ideal solutions are obtained as follows:

$$A^{+} = \{v_1^{+}, v_2^{+}, \ldots, v_n^{+}\} = \left\{ \left(\max_i v_{ij}^{U} | j \in K_b \right), \left(\min_i v_{ij}^{L} | j \in K_c \right) \right\}, \quad (9)$$

$$A^{-} = \{v_1^{-}, v_2^{-}, \ldots, v_n^{-}\} = \left\{ \left(\min_i v_{ij}^{L} | j \in K_b \right), \left(\max_i v_{ij}^{U} | j \in K_c \right) \right\}. \quad (10)$$

4. The separation of each alternative from the positive ideal solution is calculated using the n-dimensional Euclidean distance:

$$S_i^{+} = \left\{ \sum_{j \in K_b} \left(v_{ij}^{L} - v_j^{+} \right)^2 + \sum_{j \in K_c} \left(v_{ij}^{U} - v_j^{+} \right)^2 \right\}^{\frac{1}{2}}, \quad i = 1, \ldots, m. \quad (11)$$

Similarly, the separation from the negative ideal solution is calculated as follows:

$$S_i^{-} = \left\{ \sum_{j \in K_b} \left(v_{ij}^{U} - v_j^{-} \right)^2 + \sum_{j \in K_c} \left(v_{ij}^{L} - v_j^{-} \right)^2 \right\}^{\frac{1}{2}}, \quad i = 1, \ldots, m. \quad (12)$$

5. Calculate the relative closeness to the ideal alternatives:

$$RC_i = \frac{S_i^{-}}{S_i^{+} + S_i^{-}}, \quad i = 1, 2, \ldots, m, \; 0 \leqslant RC_i \leqslant 1. \quad (13)$$

6. Rank the alternatives according to the relative closeness to the ideal alternatives: the bigger is the RC_i, the better is the alternative A_i.

In Jahanshahloo et al. (2009) and Jahanshahloo et al. (2011), the interval extension of *TOPSIS* method based on interval-valued ideal solution was proposed. This method is based on the assumption that the ideal and negative-ideal solutions are changed for each alternative. Nevertheless, analysing this approach we can see that obtained interval ideal solutions are not always attainable in the interval decision matrix.

In Ye and Li (2009), an interval *TOPSIS* method is used in the framework of group multi-attribute decision model. For the decision maker k, the positive and negative interval-valued solutions are presented in Ye and Li (2009) (in our notation) as follows:

$$A^{+k} = \langle (v_1^{+kU}, v_2^{+kU}, \ldots, v_n^{+kU}) \rangle = \left\{ \left[\max_i v_{ij}^{kL}, \max_i v_{ij}^{kU} \right] \right\},$$

$$A^{-k} = \langle (v_1^{+kU}, v_2^{+kU}, \ldots, v_n^{+kU}) \rangle = \left\{ \left[\min_i v_{ij}^{kL}, \min_i v_{ij}^{kU} \right] \right\}.$$

There is no any justification of this approach in Ye and Li (2009) and only what we can say about it is that in the case of only benefit criteria the upper bound of A^{+k} may be obtained from (9) and the lower bound of A^{-k} may be obtained from (10). It is also important that interval-valued ideal solutions obtained using the method proposed by Ye and Li (2009) are not always attainable in the interval-valued decision matrix. The similar problem may be found in Chen (2011).

A more complicated approach to the definition of interval-valued ideal solutions was proposed by Tsaur (2011), where the author wrote "Theoretically, a for pivot value $\widehat{v}_j^{+} = [\widehat{v}_j^{+L}, \widehat{v}_j^{+U}]$ for criterion j in the positive ideal solution, we know that both of $\widehat{v}_j^{+L}$ and $\widehat{v}_j^{+U}$ might be obtained from different alternatives.". Therefore, the ideal solutions obtained using the method proposed by Tsaur (2011) are not always attainable in the interval-valued decision matrix.

In Yue (2011), the group decision making problem was solved using the modified interval extension of *TOPSIS* method. In this approach, the positive and negative ideal solutions were presented by interval-valued matrices. For example, the negative ideal solution was presented as follows $A^{-} = ([v_{ij}^{-L}, v_{ij}^{-U}])_{m \times n}$, where $v_{ij}^{-L} = \min_k v_{ij}^{kL}, v_{ij}^{-U} = \max_k v_{ij}^{kU}$ (k is a number of decision maker).

It is easy to see that obtained A^{-} is not always attainable in the interval decision matrices provided by decision makers.

Summarising, we can say that the common limitation of known approaches to interval extension of *TOPSIS* method is that they (based on the different assumptions) provide interval-valued ideal solutions which are not always attainable in corresponding interval-valued decision matrices. This is in contradiction with basics of classical *TOPSIS* method and is a consequence of heuristic assumptions which are not usually justified enough.

In the most of analysed approaches, the upper bound of positive interval-valued solution is calculated as in the expression (9) and the lower bound of negative interval solution is calculated as in (10).

Hence, we can say that the approach developed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) providing real-valued ideal solutions attainable in the interval-valued decision matrix seems to be more justified than the other analysed here approaches providing interval-valued ideal solutions, which are not always attainable in the interval-valued decision matrix. Therefore, to compare a direct interval extension of *TOPSIS* method we propose in this paper with other known approaches, it seems to be enough to compare the results obtained by our method with those obtained using the method developed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) (see expressions (7)–(13)).

3. A new approach to the interval extension of *TOPSIS* method

3.1. The problem formulation

Therefore, a more correct and straightforward approach to calculation of ideal solutions is representing them in the interval form using the expressions:

$$A^{+} = \{[v_1^{+L}, v_1^{+U}], [v_2^{+L}, v_2^{+U}], \ldots, [v_n^{+L}, v_n^{+U}]\} = \left\{ \left(\max_i \left[v_{ij}^{L}, v_{ij}^{U} \right] | j \in K_b \right), \left(\min_i \left[v_{ij}^{L}, v_{ij}^{U} \right] | j \in K_c \right) \right\}, \quad (14)$$

$$A^{-} = \{[v_1^{-L}, v_1^{-U}], [v_2^{-L}, v_2^{-U}], \ldots, [v_n^{-L}, v_n^{-U}]\} = \left\{ \left(\min_i \left[v_{ij}^{L}, v_{ij}^{U} \right] | j \in K_b \right), \left(\max_i \left[v_{ij}^{L}, v_{ij}^{U} \right] | j \in K_c \right) \right\}. \quad (15)$$

As there are no any type reductions (representation of intervals by real values) and additional assumptions concerned with expressions (14) and (15), we call our approach Direct Interval Extension of *TOPSIS* method.

It is easy to see that expressions (14) and (15) provide the positive and negative interval-valued ideal solutions which are always attainable in the corresponding interval-valued decision matrix.

To perform the difference of proposed approach from known ones it is enough to compare it with the method developed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) (see explanation at the end of previous section).

Table 1
Decision matrix.

	C_1	C_2
A_1	[5,7]	[22,32]
A_2	[0,10]	[25,27]

Suppose we deal with the interval-valued decision matrix presented in Table 1, where $[x_{11}] = [5,7]$, $[x_{12}] = [22,32]$, $[x_{21}] = [0,10]$ and $[x_{22}] = [25,27]$ represent the ratings of alternatives A_1 and A_2 with respect to the benefit criteria C_1 and C_2.

Since we deal with the only benefit criteria C_1 and C_2, then expression (9) in our case is reduced to $A^+ = \{v_1^+, v_2^+\} = \max_i(x_{ij}^U)$. Using this expression, from the first column of Table 1 we get $v_1^+ = 10$ and from the second column $v_1^+ = 32$. Therefore $A^+ = \{v_1^+, v_2^+\} = \{10, 32\}$. Similarly, from (10) we get $A^- = \{v_1^-, v_2^-\} = \min_i(x_{ij}^L) = \{0, 22\}$.

On the other hand, using any method for interval comparison (see Sevastjanov (2007), Wang & Kerre (2001) and subSection 3.2) we obtain that $[x_{21}] < [x_{11}]$, $[x_{22}] < [x_{12}]$ and for the positive and negative interval ideal solutions we get: $[A^+] = \{[v]_1^+, [v]_2^+\} = \{[5,7], [22,32]\}$ and $[A^-] = \{[v]_1^-, [v]_2^-\} = \{[0,10], [25,27]\}$.

It is easy to see that $\{v_1^+, v_2^+\} = \{10, 32\}$ is not included in $\{[v]_1^+, [v]_2^+\} = \{[5,7], [22,32]\}$ and $\{v_1^-, v_2^-\} = \{0, 22\}$ is not included in $\{[v]_1^-, [v]_2^-\} = \{[0,10], [25,27]\}$.

Thus, we can say that in the cases when some intervals in the decision matrix intersect, the approach proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) may lead to wrong results.

It is seen that our method provides the interval-valued ideal solutions which are strongly attainable in the considered decision matrix (see Table 1.). Since the other known methods analysed in previous section may produce interval-valued ideal solutions which are not attainable in the considered decision matrix (see Table 1.), they may produce the wrong results too.

We use here the words "wrong results" to emphasize that only our approach based on the expressions (14), (15) guarantees that obtained interval-valued ideal solutions will be always attainable in the considered interval-valued decision matrix and this is in compliance with basics of classical *TOPSIS* method.

As in (14) and (15) the minimal and maximal intervals must be chosen, the main difficulty in the implementation of the above method is the problem of interval comparison.

3.2. The methods for interval comparison

The problem of interval comparison is of perennial interest, because of its direct relevance in practical modeling and optimization of real-world processes.

To compare intervals, usually the quantitative indices are used (see reviews in Sevastjanov (2007) and Wang & Kerre (2001)). Wang, Yang, and Xu (2005) proposed a simple heuristic method which provides the degree of possibility that an interval is greater/lesser than another one.

For intervals $B = [b^L, b^U]$, $A = [a^L, a^U]$, the possibilities of $B \geqslant A$ and $A \geqslant B$ are defined in Wang et al. (2005) Wang, Yang, and Xu (2005) as follows:

$$P(B \geqslant A) = \frac{\max\{0, b^U - a^L\} - \max\{0, b^L - a^U\}}{a^U - a^L + b^U - b^L}, \tag{16}$$

$$P(A \geqslant B) = \frac{\max\{0, a^U - b^L\} - \max\{0, a^L - b^U\}}{a^U - a^L + b^U - b^L}. \tag{17}$$

The similar expressions were proposed earlier by Facchinetti, Ricci, and Muzzioli (1998) and by Xu and Da (2002). Xu and Chen (2008) showed that the expressions proposed in Facchinetti et al. (1998), Wang et al. (2005)
and Xu and Da (2002) are equivalent ones.

A separate group of methods is based on the so-called probabilistic approach to the interval comparison (see review in Sevastjanov (2007)). The idea to use the probability interpretation of interval is not a novel one. Nevertheless, only in Sevastjanov (2007) the complete consistent set of interval and fuzzy interval relations involving separated equality and inequality relations developed in the framework of probability approach is presented. Nevertheless, the results of interval comparison obtained using expressions (16) and (17) generally are similar to those obtained with the use of probabilistic approach to the interval comparison.

The main limitations of described above methods is that they provide an extent to which an interval is greater/lesser than another one if they have a common area (the intersection and inclusion cases should be considered separately (Sevastjanov (2007))). If there are no intersections of compared intervals, the extent to which an interval is greater/lesser than another one is equal to 0 or 1 regardless of the distance between intervals. For example, Let $A = [1,2]$, $B = [3,4]$ and $C = [100,200]$. Then using described above approaches we obtain: $P(C > A) = P(B > A) = 1$, $P(A > B) = 0$.

Thus, we can say that in the case of overlapping intervals the above methods provide the possibility (or probability) that an interval is greater/lesser than another one and this possibility (or probability) can be treated as the strength of inequality or (in some sense) as the distance between compared intervals.

On the other hand, the above methods can not provide the measure of intervals inequality (distance) when they have no a common area.

Of course, the Hamming distance

$$d_H = \frac{1}{2}(|a^L - b^L| + |a^U - b^U|). \tag{18}$$

or Euclidean distance

$$d_E = \frac{1}{2}((a^L - b^L)^2 + (a^U - b^U)^2)^{\frac{1}{2}} \tag{19}$$

can be used as the distance between intervals, but these distances give no information about which interval is greater/lesser.

It can be seen that they can not be used directly for interval comparison especially when an interval is included into another one.

Therefore, here we propose to use directly the operation of interval subtraction (Moore, 1966) instead of Hamming and Euclidean distances. This method makes it possible to calculate the possibility (or probability) that an interval is greater/lesser that another one when they have a common area and when they do not intersect.

So for intervals $A = [a^L, a^U]$ and $B = [b^L, b^U]$, the result of subtraction is the interval $C = A - B = [c^L, c^U]$; $c^L = a^L - b^U$, $c^U = a^U - b^L$. It is easy to see that in the case of overlapping intervals A and B, we always obtain a negative left bound of interval C and a positive right bound.

Therefore, to get a measure of distance between intervals which additionally indicate which interval is greater/lesser, we propose here to use the following value:

$$\Delta_{A-B} = \frac{1}{2}((a^L - b^U) + (a^U - b^L)). \tag{20}$$

It is easy to prove that for intervals with common center, Δ_{A-B} is always equal to 0. Really, expression (20) may be rewritten as follows:

$$\Delta_{A-B} = \left(\frac{1}{2}(a^L + a^U) - \frac{1}{2}(b^U + b^L)\right). \tag{21}$$

We can see that expression (21) represents the distance between the centers of compared intervals A and B. This is not a surprising result as Wang et al. (2005) noted that most of the proposed methods for interval comparison are "totally based on the midpoints of interval numbers". It easy to see that the result of subtraction of intervals with common centers is an interval centered around 0. In the framework of interval analysis, such interval is treated as the interval 0.

More strictly, if a is a real value, then 0 can be defined as $a - a$. Similarly, if A is an interval, then interval zero may be defined as an interval $A - A = [a^L - a^U, a^U - a^L]$ which is centered around 0. Therefore, the value of Δ_{A-B} equal to 0 for A and B having a common center may be treated as a real-valued representation of interval zero.

The similar situation we have in statistics. Let A and B be samples of measurements with corresponding uniform probability distributions such that they have a common mean ($mean_A = mean_B$), but different variances ($\sigma_A > \sigma_B$). Then using statistical methods it is impossible to prove that the sample B is greater than the sample A or that the sample A is greater than the sample B.

Taking into account the above consideration we can say that the interval comparison based on the assumption that intervals having a common center are equal ones seems to be justified and reasonable.

Obviously, the comparison of intervals based on comparison of their centers seems to be too simple. Nevertheless, as it is shown above, this approach is based directly on conventional operation of interval subtraction. Therefore it is not a heuristic one. Moreover this method coincides better with common sense than more complicated known approaches. Let us consider two intervals $A = [3,5]$ and $B = [1,4]$. Since $a^U > b^U$ and $a^L > b^L$, then according to the Moore (1966) and common sense we have $A > B$. Since A and B are not identical and have no a common center there is no chance for A and B to be equal ones. Finally, according to common sense in this case the possibility of $A < B$ should be equal to 0. There is no chance for B to be greater than A as a whole, although some point belonging to B in the common area of A and B may be greater than the points of A in this area.

Nevertheless, in our case from (16) and (17) we get $P(B \geqslant A) = \frac{1}{5}$ and $P(A \geqslant B) = \frac{4}{5}$. Thus, we can see that the known approaches may provide counterintuitive results.

In Table 1, we present the values of $P(A \geqslant B)$, $P(B \geqslant A)$ (see expressions (16), (17)), the Hamming d_H and Euclidean d_H distances (see expressions (18), (19)) between A_i and B, and Δ_{A_i-B} for intervals $A_1 = [4,7]$, $A_2 = [5,8]$, $A_3 = [8,11]$, $A_4 = [13,16]$, $A_5 = [18,21]$, $A_6 = [21,24]$, $A_7 = [22,25]$ and $B = [7,22]$ placed as it is shown in Fig. 1. The numbers in the first row in Table 1 correspond to the numbers of intervals A_i, i = 1 to 7. (see Table 2).

We can see that the values of Δ_{A_i-B} are negative when $A_i \leqslant B$ and become positive for $A_i \geqslant B$. These estimates coincide (at least qualitatively) with $P(A_i \geqslant B)$ and $P(B \geqslant A_i)$. So we can say that the sign of Δ_{A_i-B} indicates which interval is greater/lesser and the values of $abs(\Delta_{A_i-B})$ may be treated as the distances between intervals since these values are close the to the values of d_E and d_H in both cases:

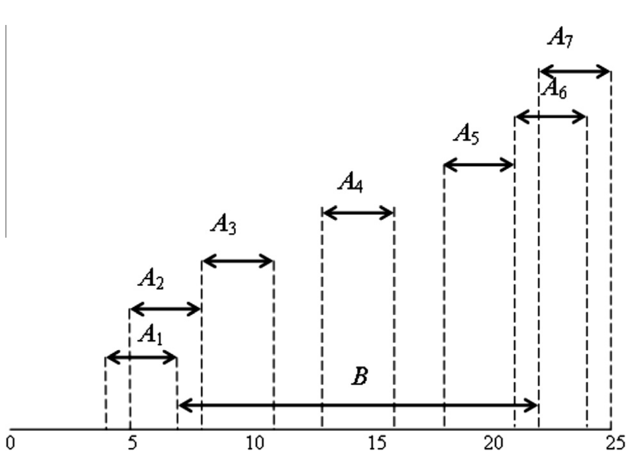

Fig. 1. Compared intervals.

Table 2
Results of interval comparison.

Method	1	2	3	4	5	6	7
$P(A \geqslant B)$	0	0.06	0.22	0.5	0.78	1	1
$P(A \leqslant B)$	1	0.94	0.78	0.5	0.22	0	0
d_E	10.82	10	7.81	6	7.81	10	10.82
d_H	9	8	6	6	6	8	9
Δ_{A-B}	−9	−8	−5	0	5	8	9

when intervals have a common area and when the there is no such an area.

3.3. The comparison of the direct interval extension of TOPSIS method with the known method

Using Δ_{A-B}, it is easy to obtain from (14), (15) the ideal interval solutions

$$A^+ = \{[v_1^{+L}, v_1^{+U}], [v_2^{+L}, v_2^{+U}], \ldots, [v_n^{+L}, v_n^{+U}]\},$$
$$A^- = \{[v_1^{-L}, v_1^{-U}], [v_2^{-L}, v_2^{-U}], \ldots, [v_n^{-L}, v_n^{-U}]\}.$$

As in the framework of our approach the distance between intervals A and B is presented by the value of Δ_{A-B}, there is no need to use Hamming or Euclidean distances for calculation of S_i^+ and S_i^-.

Since Δ_{A-B} is the subtraction of the midpoints of A and B, the values of S_i^+ and S_i^- may be calculated as follows:

$$S_i^+ = \frac{1}{2}\sum_{j \in K_B}((v_j^{+L} + v_j^{+U}) - (v_{ij}^L + v_{ij}^U)) + \frac{1}{2}\sum_{j \in K_C}((v_{ij}^L + v_{ij}^U) - (v_j^{+L} + v_j^{+U})). \tag{22}$$

$$S_i^- = \frac{1}{2}\sum_{j \in K_B}((v_{ij}^L + v_{ij}^U) - (v_j^{-L} + v_j^{-U})) + \frac{1}{2}\sum_{j \in K_C}((v_j^{-L} + v_j^{-U}) - (v_{ij}^L + v_{ij}^U)). \tag{23}$$

Finally, using expression (13) we obtain the relative closeness RC_i to the ideal alternative.

Let us consider some illustrative examples.

Example 1. Suppose we deal with three alternatives A_i, i = 1 to 3 and four local criteria C_j, j = 1 to 4 presented by intervals in Table 3, where C_1 and C_2 are benefit criteria, C_3 and C_4 are cost criteria.

Suppose $W = (0.25, 0.25, 0.25, 0.25)$. To stress the advantages of our method, in this example many intervals representing the values of ratings intersect.

Then using the known method for interval extension of *TOPSIS* method Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) (expressions (7)–(13)) we obtain $R_1 = 0.5311$, $R_2 = 0.6378$, $R_3 = 0.3290$ and therefore $R_2 > R_1 > R_3$, whereas with the use of our method (expressions 7, 8, 22, 23 and 13) we get $R_1 = 0.7688$, $R_2 = 0.7528$, $R_3 = 0.0717$ and therefore $R_1 > R_2 > R_3$.

We can see that there is a considerable difference between the final ranking obtained by the known method and using our method based on the direct extension of *TOPSIS* method. This can be explained by the fact that the method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) has some limitations

Table 3
Decision matrix.

	C_1	C_2	C_3	C_4
A_1	[6,22]	[10,15]	[16,21]	[18,20]
A_2	[15,18]	[8,11]	[20,30]	[19,28]
A_3	[9,13]	12,17]	[42,48]	[40,49]

Table 4
Decision matrix.

	C_1	C_2	C_3	C_4
A_1	[6,22]	[10,15]	[13,19]	[40,48]
A_2	[3,4]	[17,21]	[20,30]	[22,28]
A_3	[25,28]	[8,10]	[42,48]	[18,20]

concerned with the presentation of intervals by real values in the calculation of ideal solutions and using the Euclidean distance when intervals intersect.

Example 2. In this example, there are no intersecting intervals in the columns of decision matrix (see Table 4). As in the previous example, C_1 and C_2 are benefit criteria, C_3 and C_4 are cost criteria, $W = (0.25, 0.25, 0.25, 0.25)$.

Then using the known method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009), in this example we get $R_1 = 0.4825$, $R_2 = 0.4984$, $R_3 = 0.5413$ and therefore $R_3 > R_2 > R_1$. With the use of our method we obtain $R_1 = 0.5671$, $R_2 = 0.4675$, $R_3 = 0.4488$ and therefore $R_1 > R_2 > R_3$. Hence, we can conclude that even in the case when there are no intersecting intervals in the columns of interval valued decision table, the known method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) and our methods may provide very different final rankings of alternatives. That may be explained by the fact that in the method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009), the real-valued ideal solutions are used, whereas in our method they are presented by intervals attainable in the interval-valued decision table.

Nevertheless, when there are no intersecting intervals in the columns of interval-valued decision table, the final ratings obtained by the method proposed in Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) may coincide with those obtained using our method.

Consider the illustrative examples.

Example 3. In this example we will use the decision table presented in Table 4, where C_1 and C_2 are benefit criteria, C_3 and C_4 are cost criteria, but $W = (0.5, 0.1, 0.25, 0.15)$.

Then using the method from Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) we obtain $R_1 = 0.4812$, $R_2 = 0.2232$, $R_3 = 0.5798$ and therefore $R_3 > R_1 > R_2$. Using our method we get $R_1 = 0.6653$, $R_2 = 0.2758$, $R_3 = 0.6363$ and therefore $R_3 > R_1 > R_2$.

Thus, in this case two considered methods provide coincided ratings.

Example 4. Let us consider the decision matrix presented in Table 5, which differs from Table 4 by only one element $[x_{31}]$ so that $[x_{31}]$ intersects with $[x_{11}]$. There are no other intersections in the columns of Table 5.

Using, as in previous case $W = (0.5, 0.1, 0.25, 0.15)$, and the method from Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009) we get $R_1 = 0.4812$, $R_2 = 0.25322$, $R_3 = 0.5798$ and therefore $R_3 > R_1 > R_2$ as in the previous example, whereas with the use of our method we obtain $R_1 = 0.6653$, $R_2 = 0.2758$, $R_3 = 0.6363$ and therefore $R_1 > R_3 > R_2$.

Table 5
Decision matrix.

	C_1	C_2	C_3	C_4
A_1	[6,22]	[10,15]	[13,19]	[40,48]
A_2	[3,4]	[17,21]	[20,30]	[22,28]
A_3	[12,28]	[8,10]	[42,48]	[18,20]

Thus, we can see that even the change of only one element in a decision matrix which leads to the appearance of intersecting intervals in the corresponding column, may lead to the significant changes in the results obtained by our method, whereas the known method (Jahanshahlo et al. (2006) and Jahanshahloo et al. (2009)) does not provide different final ratings of compared alternatives.

4. Conclusion

The critical analysis of known approaches to the interval extension of *TOPSIS* method is presented. It is shown that these extensions are based on different heuristic approaches to definition of positive and negative ideal solutions. These ideal solutions are presented by real values or intervals, which are not attainable in a decision matrix.

Since this is in contradiction with basics of classical *TOPSIS* method, a new approach to the solution of *MCDM* problems with the use of *TOPSIS* method in the interval setting is proposed. This method called "direct interval extension of *TOPSIS* method" is free of heuristic limitations of known methods concerned with the definition of positive and negative ideal solutions and using the Euclidean distance when intervals in a decision matrix intersect.

The main advantage of the proposed method is that (opposite to the known methods) it provides interval-valued positive and negative ideal solutions which in compliance with the basics of classical *TOPSIS* method are always attainable in the interval-valued decision matrix.

It is shown that the use of known methods may lead to the wrong results as well as the use of the Euclidean distance when intervals representing the values of local criteria intersect.

Using numerical examples, it is shown that the proposed "direct interval extension of *TOPSIS* method" may provide the final ranking of alternatives which is substantially different from the results obtained using the known methods especially when some interval-valued ratings in the columns of decision table intersect.

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