

GRASP with Path Relinking for Commercial Districting

Hugo Jair Escalante

Computer Science Department

Instituto Nacional de Astrofísica, Óptica y Electrónica (INAOE)

Tonantzintla, Puebla, Mexico

hugojair@inaoep.mx

Roger Z. Ríos-Mercado¹

Graduate Program in Systems Engineering

Universidad Autónoma de Nuevo León (UANL)

San Nicolás de los Garza, NL, Mexico

roger.rios@uanl.edu.mx

March 2014

¹Corresponding author

Abstract

The problem of grouping basic units into larger geographic territories subject to dispersion, connectivity, and balance requirements is addressed. The problem is motivated by a real-world application from the bottled beverage distribution industry. For addressing dispersion minimization, a more robust measure based on the diameter of the formed territories is used. For solving this particular territory design problem, a greedy randomized adaptive search procedure (GRASP) that incorporates a novel construction procedure where territories are formed simultaneously in two main stages using different criteria is proposed. The GRASP is further enhanced with two variants of forward-backward path relinking, namely static and dynamic. The proposed algorithm and its components have been extensively evaluated over a wide set of data instances. Experimental results reveal that the construction mechanism produces feasible solutions of acceptable quality, which are improved by an effective local search procedure. In addition, empirical evidence indicate that the two path relinking strategies have a significant impact on solution quality when incorporated within the GRASP framework. The ideas and components of the developed method can be further extended to other districting problems under balancing and connectivity constraints.

Keywords: Service industry; Districting; Metaheuristics; GRASP; Path relinking.

1 Introduction

The territory design problem (TDP) may be viewed as the problem of grouping small geographic basic units (BUs) into larger geographic clusters, called territories, in a way that the territories are acceptable (or optimal) according to relevant planning criteria. Territory design or districting has a broad range of applications such as political districting, sales territory design, school districting, power districting, and public services, to name a few. The reader can find in the works of Kalcsics, Nickel, and Schröder [17] and Duque, Ramos, and Suriñach [11] state of the art surveys on models, algorithms, and applications to districting problems.

The problem addressed in this paper is a commercial territory design problem (CTDP) motivated by a real-world application from the bottled beverage distribution industry. The problem, introduced by Ríos-Mercado and Fernández [28], considers finding a design of p territories with minimum dispersion subject to planning requirements such as exclusive BU-to-territory assignment, territory connectivity, and territory balancing with respect to three BU attributes: number of customers, product demand, and workload.

An important criterion in territory design problems is compactness. Typically this is achieved by minimizing a dispersion function. In commercial territory design, several models based on dispersion functions from the well-known p -center and p -median location problems have been studied in the past. These are center-based dispersion functions, that is, the dispersion is measured with respect to a centroid of a territory. However, there are other non-center-based measures of dispersion that can be used. Center-based functions rely heavily on the location of the centers; if the centers are “badly” located, the resulting design may cause a serious deterioration in objective function. In addition, in location problems, the centers represent a physical entity or facility that provides some service; however, in CTDPs the centers are artificially located as no facility is actually placed there, it is just a reference for the dispersion measure. These limitations motivate the study on other ways of measuring dispersion. For instance, a measure such as the diameter, which measures the longest distance between any two basic units in a territory, is a more robust function since it does not depend on a center location, providing more flexibility. Even from the algorithmic perspective, heuristic methods for tackling TDPs under center-based dispersion functions need to constantly

update and recompute as centers keep moving along every time the territory suffers a change. This time-consuming task can be avoided if other measures such as the diameter are used.

In this work, we focus our study in a commercial territory design problem that seeks to minimize territory dispersion based on a diameter dispersion measure. To the best of our knowledge, this type of problem has not been addressed before in the territory design literature. Since the aim is to target large instances, we present a Greedy Randomized Adaptive Search Procedure (GRASP) with Path Relinking for this NP-hard CTDP. The algorithm is denoted as GPR_CTDP. In our proposed GRASP we develop a procedure that builds exactly p territories at once simultaneously, that is, we start with p node seeds and start associating nodes to the seeds until all of them are assigned. By growing the territories simultaneously rather than one at a time one expects that the violation of the balancing constraints be considerably lower. In addition, we develop two path relinking (PR) strategies, one dynamic and one static, motivated by the work of Resende et al. [21], who successfully applied it to the max-min diversity problem. In our work, these PR strategies rely on finding a “path” between two different territory designs. To this end, an associated assignment subproblem for finding the best match between territory centers is solved. The solution to this problem provides a very nice way of generating the trajectory between two given designs. This idea is novel in any districting of territory design application to the best of our knowledge.

To assess its efficiency, the proposed GPR_CTDP with many of its components and strategies, has been extensively evaluated over a wide set of data instances. We have found, for instance, that building territories simultaneously results in feasible solutions of acceptable quality. The two PR variants implemented in GPR_CTDP allowed us to obtain better solutions than those obtained when using straight local search; although, the static strategy resulted more helpful. The main algorithmic ideas incorporated in the developed algorithm can be extended so as to handle other districting problems with similar structure.

The paper is organized as follows. In Section 2 we describe the problem in detail and present a combinatorial optimization model. Section 3 gives an overview of relevant previous related work. Section 4 describes in detail the components of the proposed heuristic, and Section 5 presents the empirical evaluation of the method. We end the paper in Section 6, with some conclusions and

final remarks.

2 Problem Description

Let $G = (V, E)$ denote a graph where V is the set of city blocks or basic units (BUs), and E is the set of edges representing adjacency between blocks, that is, $(i, j) \in E$ if and only if BUs i and j are adjacent blocks. Let d_{ij} denote the Euclidean distance between BUs i and j , with $i, j \in V$. For each BU $i \in V$ there are three associated parameters. Let w_i^a be the value of activity a at node i , where $a = 1$ (number of customers), $a = 2$ (product demand), and $a = 3$ (workload). The number of territories is given by the parameter p . A p -partition of V is denoted by $X = (X_1, \dots, X_p)$, where $X_k \subset V$ is called a territory of V . Let $w^a(X_k) = \sum_{i \in X_k} w_i^a$ denote the size of territory X_k with respect to activity $a \in A = \{1, 2, 3\}$ and $k \in K = \{1, \dots, p\}$. The balancing planning requirements are modeled by introducing a user-specified tolerance parameter τ^a that measure the allowable relative deviation from the target average size μ^a , given by $\mu^a = w^a(V)/p$, for each activity $a \in A$. Another planning requirement is that all of the nodes assigned to each territory are connected by a path contained totally within the territory. In other words, each of the territories X_k must induce a connected subgraph of G . Finally, we seek to maximize territory compactness or, equivalently, minimize territory dispersion, where dispersion is given by the largest diameter over all territories, that is $\max_{k=1, \dots, p} \max_{i, j \in X_k} \{d_{ij}\}$.

Let Π be the collection of all p -partitions of V . The combinatorial optimization model is given as follows.

Model (CTDP)

$$\min_{X \in \Pi} f(X) = \max_{k \in K} \max_{i, j \in X_k} \{d_{ij}\} \quad (1)$$

$$\text{subject to } \frac{w^a(X_k)}{\mu^a} \in [1 - \tau^a, 1 + \tau^a] \quad k \in K, a \in A \quad (2)$$

$$G_k = G(V_k, E(V_k)) \text{ is connected} \quad k \in K \quad (3)$$

Objective (1) measures territory dispersion. Constraints (2) represent the territory balance with

respect to each activity measure as it establishes that the size of each territory must lie within a range (measured by tolerance parameter τ^a) around its average size. Constraints (3) guarantee the connectivity of the territories, where G_k is the graph induced in G by the set of nodes X_k . Note that there is an exponential number of such constraints.

The model can be viewed as partitioning G (the contiguity graph representing the BUs) into p connected components (contiguous districts) under the additional side constraints on balancing product demand, number of customers, and workload of each territory, and minimizing a dispersion measure of the BUs in a territory. The basic contiguity graph model for the representation of a territory divided into elementary units has been adopted in political districting [26].

3 Related Work

Territory design or districting has a broad range of applications such as political districting [4, 15, 3, 18, 27, 20], sales territory design [10, 32], school districting [6], power districting [1, 8], and public services [2, 7, 19], to name a few. The reader can find in the works of Kalcsics, Nickel, and Schröder [17] and Duque, Ramos, and Suriñach [11] state of the art surveys on models, algorithms, and applications to districting problems. Zoltners and Sinha [33] present a survey focusing on sales districting and Ricca et al. [26] present a survey on political districting.

Here we discuss the related work on commercial territory design. Ríos-Mercado and Fernández [28] introduced the commercial TDP by incorporating a territory compactness criterion and a fixed number of territories p . They seek to maximize this compactness criterion subject to planning requirements such as exclusive BU-to-territory assignment, territory connectivity, and territory balancing with respect to three BU attributes: number of customers, product demand, and workload. In their work, the authors consider as a minimization function a dispersion function based on the objective function of the well-known p -Center Problem. After establishing the NP-completeness of the problem, the authors propose a Reactive GRASP for obtaining high-quality solutions to this problem. The core of their GRASP is a three-phase iterative procedure composed by a construction phase, an adjustment phase, and a local search phase. In the construction phase a solution with q territories, where q is usually larger than p , satisfying the connectivity constraints is built. Then

an adjustment phase based on a pairwise merging mechanism is applied to obtain a solution with p territories. Afterwards, a local search phase attempting both to eliminate the infeasibility with respect to the balancing requirements and to improve the dispersion objective function is applied. One interesting observation is that the construction and adjustment phases produce solutions with very high degree of infeasibility. This is very nicely repaired by the local search, at a very high computational cost though. The reason for this is that attempting to merge two territories into one in the adjustment phase may result in a high violation of the upper bound of the balancing constraints.

Aguilar-Salazar et al. [30] present an exact optimization framework for tackling relatively small instances of several CDTP models. They studied two linear models that differ in the way they measure dispersion, one model uses a dispersion function based on the objective of the p -Median Problem (MPTDP) and the other is based on the p -Center Problem (CPTDP). They can successfully solve instances of up to 100 BUs for the CPTDP and up to 150 BUs for the MPTDP. This concludes that p -center-based dispersion measures yield more difficult models as they have weaker LP relaxations than the median-based models.

Salazar-Acosta and Ríos-Mercado [29] present a heuristic based on GRASP and adaptive memory programming for a CTDTP that considers the minimization of a p -Center Problem function subject to additional budget routing constraints.

One of the most popular methods for addressing districting problems is the location-allocation technique [17]. However, this technique is not applicable to our problem mainly because the nature of the dispersion objective function is different. As it has been show, the location-allocation method seems to work well when a p -Median Problem-based objective function is used.

As stated previously, CTDTPs with diameter-based dispersion functions have not been studied in the past. One of the reasons is that center-based functions yield well-structured mixed-integer programming problems which in turn can lead to relatively good optimization algorithms. However, this structure is somewhat lost when addressing a problem from the heuristic perspective. In fact, using non-center-based functions such as the diameter may be more convenient since no time consuming center updating operations are needed.

4 Proposed Heuristic

This section introduces the proposed GRASP heuristic with path relinking for the commercial territory design problem (GPR_CTDTP). GRASP is a well known meta-heuristic based on greedy search and random construction mechanisms [14] that has been successfully used for many combinatorial optimization problems, including CTDTP [28]. We propose a GRASP improved with path relinking (PR). The heuristic comprises a new construction procedure and a very effective PR mechanism. The construction procedure intelligently handles a strategy for building territories simultaneously, while the PR formulation allows us to obtain better solutions than those obtained when using straight local search, see Section 5. The rest of this section describes in detail the components of the GPR_CTDTP approach, which receives as input an instance of the CTDTP and a set of parameters as described below.

4.1 GRASP

A GRASP is an iterative process in which each major iteration consists of two phases: construction and local search [14]. The construction phase attempts to build a feasible solution and the local search phase attempts to improve it. This process is repeated for a fixed number of iterations and the best overall solution is returned as the result. GRASP incorporates greedy search and randomization mechanisms that allow it to obtain high quality solutions to combinatorial problems in acceptable times. Despite the simplicity of this multi-start heuristic it has proved to be very effective in a wide range of problems and applications [22]. Previous work on GRASP for the CTDTP is presented in Section 3. In this paper we propose procedure GPR_CTDTP, which is in essence a GRASP augmented with PR mechanisms, accordingly, in this section we describe the particular construction and local search procedures of the GRASP and the next subsection presents the PR strategies.

Construction phase

At a given iteration, the construction phase consists of building p territories, X_1, \dots, X_p , simultaneously in such a way that connectivity is always satisfied while infeasibility in terms of dispersion

and balance is allowed to some extent. Each territory X_k is formed by a subset of BUs or nodes such that $\cup_{k=1,\dots,p} X_k = V$ and $X_k \cap X_l = \emptyset$, for all $k \neq l$. Under the proposed procedure each territory X_k is associated to a center, $c(k)$. This is not a requirement of the problem but a feature of the proposed formulation that was adopted for convenience when measuring dispersion of territories.

Procedure 1 grasp_construction(δ, L, α)

Input: δ : fraction of nodes assigned by the distance criteria;
 L : interval for updating centers;
 α : RCL quality parameter;
Output: X : A p -partition of V ;
 $(c(1), \dots, c(p)) \leftarrow \mathbf{max_disp}(p)$; {Compute p initial centers}
 $i \leftarrow 0$; $\bar{V} \leftarrow V$;
while ($n - |\bar{V}| \leq \delta n$) **do**
 for all ($k \in \{1, \dots, p\}$) **do**
 $N_q(X_k) \leftarrow q$ nearest (unassigned) neighbors of X_k ;
 $X_k \leftarrow X_k \cup N_q(X_k)$; $\bar{V} \leftarrow \bar{V} \setminus N_q(X_k)$;
 end for
 $i \leftarrow i + 1$;
 if ($i \bmod L = 0$) **then**
 $c(k) \leftarrow \min(\max d_{v,w}), \forall v, w \in X_k, k = 1, \dots, p$; {Update centers}
 end if
end while
 $open(k) \leftarrow \text{TRUE}, k = 1, \dots, p$;
while ($|\bar{V}| > 0$ and $\exists k$ such that $open(k) == \text{TRUE}$) **do**
 for all ($k = 1, \dots, p$) **do**
 if ($open(k) = \text{TRUE}$) **then**
 Compute $\phi_k(v)$ in Eq. (4), $\forall v \in N(X_k)$;
 $\Phi_{\min} \leftarrow \min\{\phi_k(v)\}$; $\Phi_{\max} \leftarrow \max\{\phi_k(v)\}$;
 RCL $\leftarrow \{h \in N(X_k) : \phi_k(h) \leq \Phi_{\min} + \alpha(\Phi_{\max} - \Phi_{\min})\}$;
 Choose $v \in \text{RCL}$ randomly; $X_k \leftarrow X_k \cup \{v\}$; $\bar{V} \leftarrow \bar{V} \setminus \{v\}$;
 if ($N(X_k) = \emptyset$ or $w^a(X_k) > (1 + \tau^a)$ for any a) **then**
 $open(k) \leftarrow \text{FALSE}$; {Close this territory}
 end if
 end if
 end for
end while
if ($|\bar{V}| > 0$) **then**
 for all ($v \in \bar{V}$) **do**
 $X_v \leftarrow$ Nearest territory to node v ;
 $X_v \leftarrow X_v \cup \{v\}$; $\bar{V} \leftarrow \bar{V} \setminus \{v\}$;
 end for
end if
return $X = \{X_1, \dots, X_p\}$;

Procedure 1 presents the construction phase of the proposed GPR-CTDP. \bar{V} denotes the set of nodes that have not been assigned to any territory and $n = |V|$ the number of BUs. The process starts by selecting p seeds or centers, $\{c(1), \dots, c(p)\}$, which are the first nodes assigned to each territory; that is, $c(k) \in X_k, k \in \{1, \dots, p\}$. Territories are then built iteratively in two main stages followed by a postprocessing stage. In the first stage q BUs are iteratively assigned to each territory

X_k . For each territory X_k , we iteratively assign the q (unassigned) nearest neighboring nodes of that territory, $v \in N_q(X_k)$. The BUs in $N_q(X_k)$ that are assigned to X_k must be connected by an edge to a BU already assigned to X_k . The latter process is iterated until a fraction δ of the total of BUs have been assigned to one of the p territories (i.e. $\lfloor \delta n \rfloor$ BUs have been assigned), where the centers $c(1), \dots, c(p)$ are updated every L iterations. One should note that the notion of centers is only used for this very-first phase of the construction procedure and it is not used elsewhere.

Figure 1 shows the BUs assigned after stage one of the construction phase for an instance of the CTDP considered for experimentation. From this stage the p territories have been simultaneously built by using a neighborhood criteria completely ignoring the balance constraints. The rationale behind this is that nodes that belong to the same territory must be close to each other, hence a portion of nodes can be assigned with a closeness criterion. The remaining nodes will lie at boundaries among territories, therefore, balance and dispersion information is taken into account for assigning those nodes.

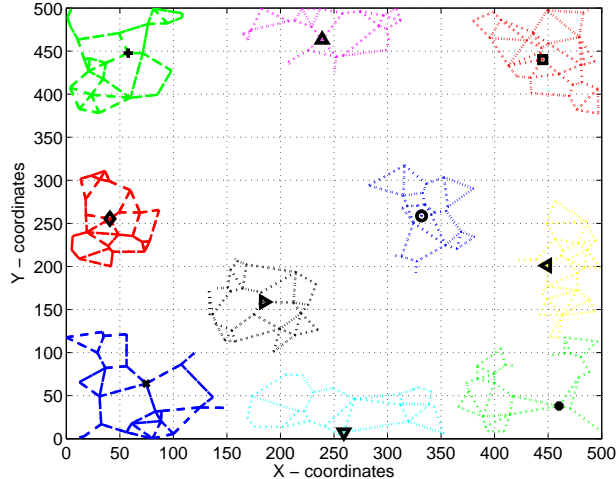


Figure 1: First stage of the proposed construction procedure for an instance of the CTDP.

An important aspect of stage one is that of selecting seed centers. Clearly, randomness must be considered for this process as we want to generate fairly different centers at each iteration of the GPR-CTDP approach. To this end, we view the problem of choosing an appropriate set of p initial seeds as a p -Dispersion Problem [12], which is a combinatorial optimization problem that places p points in the plane as far away of each other as possible by using an appropriate measure

for maximizing dispersion. In our procedure, we used an approach that selects centers randomly with a maximum dispersion criteria. The particular strategy starts with a randomly selected node as the center for the first territory and the rest of centers are obtained by using a greedy heuristic for the p -dispersion problem [12].

The second stage of the construction phase consists of assigning the remaining $n - \lfloor \delta n \rfloor$ nodes that were not assigned in stage one. For this stage BUs are assigned to territories using a greedy randomized adaptive procedure that takes into account both balance and dispersion constraints. For each territory X_k , the cost of assigning every neighboring node $v \in N(X_k)$ to X_k is evaluated according to Equation (4). Then a restricted candidate list (RCL) is formed, from which a single BU is randomly selected and assigned to the current territory X_k . This RCL is restricted by a quality parameter α , that is, RCL is formed by those BUs whose greedy function evaluation falls within α percent from the best evaluation. Equation (4) determines the cost incurred when assigning node v to a territory X_k . This cost is determined by a linear combination of the weights assigned to nodes in territory $X_k \cup \{v\}$, as determined by the term $G_k(v)$, and the dispersion of those nodes, as estimated by the term $F_k(v)$, with $G_k(v)$ and $F_k(v)$ defined in Equations (5) and (6), respectively

$$\phi_k(v) = \lambda F_k(v) + (1 - \lambda)G_k(v), \quad (4)$$

$$G_k(v) = \sum_{a \in A} g_k^a(v), \quad (5)$$

$$F_k(v) = \left(\frac{1}{d_{\max}} \right) f(X_k \cup \{v\}) = \left(\frac{1}{d_{\max}} \right) \max \left\{ f(X_k), \max_{i \in X_k, v} \{d_{iv}\} \right\}, \quad (6)$$

where $f(X_k) = \max_{k \in K} \max_{i, j \in X_k} \{d_{ij}\}$ is the dispersion measure (as dictated by the objective function) and $g_k^a(v) = \frac{1}{\mu^a} \max\{w^a(X_k \cup \{v\}) - (1 + \tau^a)\mu^a, 0\}$ accounts for the sum of relative infeasibilities for the balancing constraints. Here $d_{\max} = \max_{i, j \in V} \{d_{ij}\}$, the maximum distance between any pair of nodes, is used for normalizing the objective function. One should note that $g_k^a(v)$ represents the infeasibility with respect to the upper bound of the balance constraint for activity a . Both factors dispersion and balancing are weighted by a parameter λ in expression (4). The process is repeated for every territory k . If a territory exceeds the expected average weight for a territory it is considered *closed* (i.e., $open(j) = false$) and no further node can be assigned

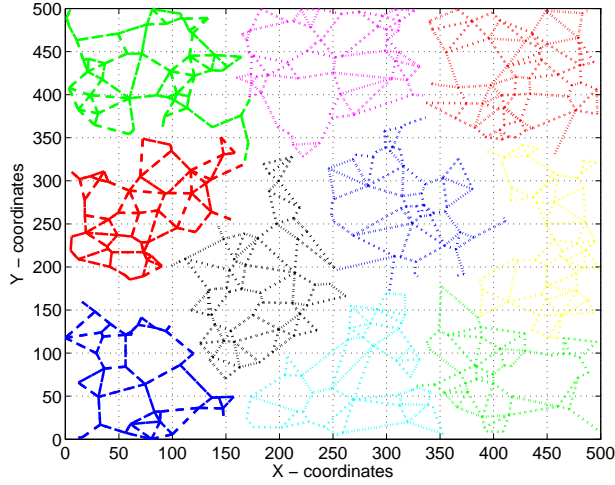


Figure 2: Second and third stages of the proposed construction procedure for an instance of the CTDP.

to it. The latter process iterates until either every node has been assigned to a territory or every territory is considered closed. Since stage two of this construction phase does not guarantee that all nodes will be assigned to a territory, a third stage is applied in which each unassigned node gets assigned to its nearest territory. Figure 2 shows the distribution of BUs for an instance of the CTDP after stages two and three of the construction procedure.

Local search

After a solution is build a postprocessing phase consisting of local search is performed. The goal in this phase is to improve the objective function value and recovering feasibility (if violated) in the constructed solution, X . In this local search, a merit function that weights both the infeasibility with respect to balancing constraints and the objective function value is used. This function is indeed similar to the greedy function used in the construction phase with the exception that now the sum of relative infeasibilities take into consideration lower and upper bound violation of the balancing constraints. Specifically, the merit function for a given territory design $X = \{X_1, \dots, X_p\}$ is given by

$$\psi(X) = \lambda F(X) + (1 - \lambda)G(X) \quad (7)$$

where

$$F(X) = \left(\frac{1}{d_{\max}} \right) \max_{k \in K} \max_{i, j \in X_k} \{d_{ij}\} \quad (8)$$

and

$$G(X) = \sum_{k=1}^p \sum_{a \in A} g^a(X_k), \quad (9)$$

with $g^a(X_k) = \frac{1}{\mu^a} \max\{w^a(X_k) - (1 + \tau^a)\mu^a, (1 - \tau^a)\mu^a - w^a(X_k), 0\}$ being the sum of the relative infeasibilities of the balancing constraints. The quality of solutions is then determined by Expression (7), we now describe the mechanism for exploring solutions around the constructed territory design.

Let $t(i)$ denote the territory node i belongs to, $i = 1, \dots, n$. A move $move(i, j)$ is defined as moving a node i from its current territory to a territory $t(j)$, where $t(j) \neq t(i)$. Only moves $move(i, j)$ where $(i, j) \in E$ and $t(i) \neq t(j)$ are allowed. Thus, $move(i, j)$ transforms a solution $X = (X_1, \dots, X_{t(i)}, \dots, X_{t(j)}, \dots, X_p)$ into $X^T = (X_1, \dots, X_{t(i)} \setminus \{i\}, \dots, X_{t(j)} \cup \{i\}, \dots, X_p)$. If connectivity must be kept, only moves where $X_{t(i)} \setminus \{i\}$ remains connected are allowed. Note that in general $move(i, j)$ is asymmetric.

Procedure 2 local_search(X)

Input: X : A solution to the CTDP;

Output: X : Improved solution to the CTDP;

$nmoves \leftarrow 0$; $local_optima \leftarrow \text{FALSE}$;

$k \leftarrow 1$; {starting territory}

while ($nmoves \leq limit_evals$ AND $\neg local_optima$) **do**

$improvement \leftarrow \text{FALSE}$;

while ($|N(X_k)| > 0$ and $\neg improvement$) **do**

$move(i, j) \leftarrow$ Choose valid move from $N(X_k)$;

$N(X_k) \leftarrow N(X_k) \setminus \{(i, j)\}$;

Evaluate $\psi(X^T)$ using Expression (7);

if ($\psi(X^T) < \psi(X)$) **then**

$X \leftarrow X^T$; {perform move}

$nmoves \leftarrow nmoves + 1$;

$improvement \leftarrow \text{TRUE}$;

$kend \leftarrow k$;

$k \leftarrow (k + 1) \bmod p$;

end if

end while

if ($\neg improvement$) **then**

$k \leftarrow (k + 1) \bmod p$;

end if

if ($k = kend$) **then**

$local_optima \leftarrow \text{TRUE}$;

end if

end while

return X

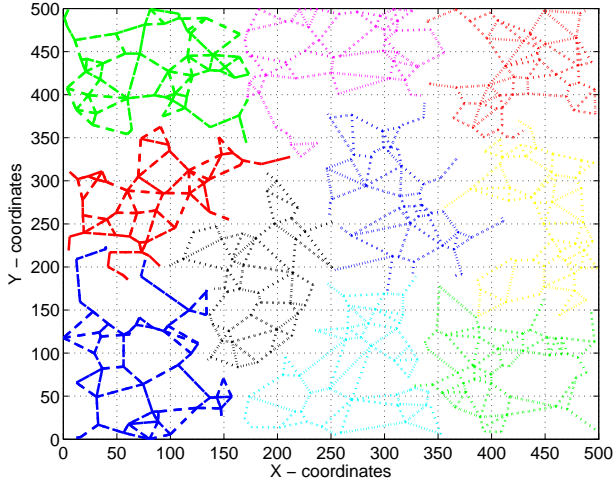


Figure 3: Solution found after applying the local search procedure for an instance of the CTDp.

The basic idea of the local search is to start the search with a given territory, say territory k , and then consider first the moves emanating from territory k , that is, if we let $N(X_k)$ denote the feasible moves $move(i, j)$ with $t(i) = k$ evaluate first all the moves in $N(X_k)$, and take the best that improves the current solution, if any. If none found, proceed with territory $(k + 1) \bmod p$. As soon as a better move is found, perform the move, and restart the search from this new solution X^T but setting $k + 1$ as the starting territory, where k was the last territory examined, that is, in a new move the starting territory is $k + 1$ and the final territory to be examined is k . By using this cyclic strategy for starting territory we avoid performing many unnecessary move evaluations. A move is performed using a different territory each time until no improvements can be found. In practice an additional stopping criterion: the maximum number of allowed evaluations of the fitness function (*limit_evals*), is added to avoid performing an extensive search for long periods of time. Therefore, the postprocessing step stops when either a local optima is found or the number of moves exceeds *limit_evals*. The postprocessing phase is described in Procedure 2. Figure 3 shows a solution obtained after applying the local search procedure.

4.2 Path relinking

Path Relinking (PR) was originally proposed by Glover and colleagues as a way of incorporating intensification and diversification strategies in tabu search [16]. PR consists of exploring the path

of intermediate solutions between two selected solutions called starting (X^S) and target (X^T) with the hypothesis that some of the intermediate solutions can be either better than X^S and X^T (intensification) or comparable but different enough from X^S and X^T (diversification). Intermediate solutions are generated by performing moves from the starting solution in such a way that these moves introduce attributes that are present in the target solution. Successful applications of PR in the context of Tabu and Scatter Search are reported in Resende et al. [23].

Despite the fact that PR was originally proposed for Tabu and Scatter search, it has been successfully used with GRASP as well [22, 21]. In the context of GRASP, PR can be considered as a way of introducing memory into the search process. To the best of our knowledge PR has not been used in the context of territory design, although it has been recently applied to the related problem of capacitated clustering by Deng and Bard [9]. Whereas both problems are related, the proposed formulations differ significantly. For example, Deng and Bard did not consider centers in their PR approach and they proposed a single PR variant (at a cluster-level basis). Deng and Bard report experiments with less than 90 nodes and 5 clusters, while in Section 5 we report instances of up to 500 nodes and 10 territories.

Different PR variants have been proposed so far each having benefits and limitations in terms of efficiency and efficacy. In this work we consider two variants of forward-backward PR, namely static and dynamic, that have proven very effective in related problems [21]. For excellent surveys on applications of GRASP with PR we refer the reader to the work of Resende and Ribeiro [22].

The so called, forward-backward PR strategies explore the paths between X^S and X^T in two different ways (i.e., from X^S to X^T and viceversa) [22]. The main benefit of these strategies is that more and different solutions can be generated, although it has been found that there is little gain over one-way strategies [25]. This can be due to the greediness of usual PR methods, which evaluate every possible solution that can be generated by making a move from a initial solution and choose the move that results in the best intermediate solution [25, 21]. Thus, these methods explore a large number of solutions and, therefore, forward-backward PR does not help to improve the quality of final solutions. In this work we select moves in such a way that a single move is evaluated for generating intermediate solutions. This form of PR is more efficient at the expense of

sacrificing the benefit of greedy strategies. Nevertheless, we believe that in the considered setting the use of a forward-backward PR strategy is advantageous.

Besides the direction of the search, there are other aspects that make PR strategies different [22, 21]. For example, greedy randomized PR methods form a RCL with candidate moves and select a move randomly as in GRASP [13]. Truncated PR techniques explore partially the trajectory between X^S and X^T . Evolutionary PR consists of evolving a reference set of solutions in a similar way as the reference set is evolved in scatter search [24]. In this work we developed static and dynamic PR strategies that resulted very effective for the CTDP. Both strategies have been successfully used in other combinatorial optimization problems [21]. The rest of this section describes the PR strategies incorporated in GPR_CTDP.

Recall each solution of the CTDP is an assignment of every node $i \in V$ to one of p territories X_1, \dots, X_p . Let $t(X, i) \in \{1, \dots, p\}$ denote the index of the territory to which node i is assigned according to solution X . Given two particular solutions X^S and X^T , PR aims at generating intermediate solutions or p -partitions in the path starting at X^S and finishing at X^T . In GPR_CTDP intermediate solutions are created by changing $t(X^S, i)$, the territory to which node i is assigned in solution X^S into the corresponding territory $t(X^T, i)$. Because both X^S and X^T solutions are created independently, and the territory ordering may be arbitrary, it is not clear what territory in X^S corresponds to what territory in X^T . Hence, a correspondence between territories must be obtained before starting the search process. The problem of finding the best match between territories can be set as an Assignment Problem (AP) by considering the territory centers only. Let $C(X)$ the set of p node centers corresponding to solution X . Then a complete bipartite graph is formed with sets $C(X^S)$ and $C(X^T)$, where the cost between node $i \in C(X^S)$ and $j \in C(X^T)$ is given by d_{ij} . The AP can be solved in polynomial time. We use one of the most recent implementations of the Hungarian algorithm [5]. A solution to the AP represents a minimum cost assignment between territory centers, and therefore a match between territories. Let M be the solution to AP given by $M = \{(i_1, j_1), \dots, (i_p, j_p)\}$. The idea of the PR is then to “transform” each territory $X_{t(i_k)}$ to territory $X_{t(j_k)}$ for each $(i_k, j_k) \in M$. The rationale for this matching stems from the fact that it is expected that relatively close territories (from different designs) will have many BUs in common.

This scheme is illustrated in Figure 4. One should note that the notion of centers is adopted at this stage for convenience, as centers allows us to establish a correspondence between territories in an efficient way.

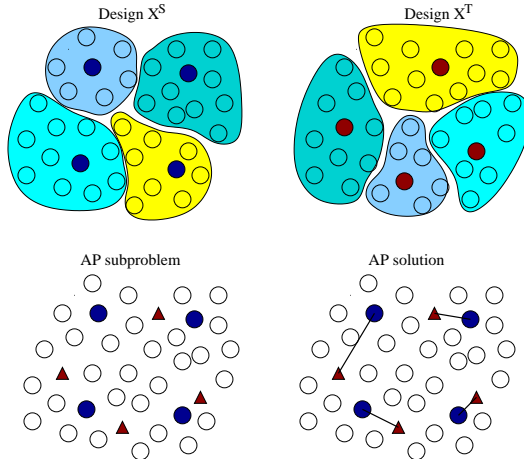


Figure 4: Illustration of how to set up a search trajectory from two given designs (top) by solving an associated Assignment Problem (bottom).

Once that correspondence between territories has been established it is possible to perform moves from one solution X^S to another X^T . As a consequence, in order to arrive at solution X^T starting from X^S , every node in X^S such that $t(X^S, i) \neq t(X^T, i)$ must be moved to its associated territory in X^T . We define a PR move, $move_{PR}(X^S, X^T, i)$, as a function that moves or reassigns a node i from territory $t(X^S, i)$ to territory $t(X^T, i)$. The move is valid as long as $t(X^S, i) \neq t(X^T, i)$ and the resulting p -partition remains connected, that is, if and only if $X_{t(X^T, i) \cup \{i\}}$ is connected and $X_{t(X^S, i) \setminus \{i\}}$ remains connected. One should note that moves are always made between boundary nodes as it is not possible to exchange a non-boundary node from one territory to another territory in a single move because loss of connectivity.

Intermediate solutions between X^S and X^T are generated by making moves from X^S to X^T and updating the solution X^S accordingly. Clearly, the order in which nodes i are selected may give rise to different trajectories between X^S and X^T . In this work we chose nodes i in lexicographical order, we also tried a random node selection approach although no difference in performance was obtained. After an intermediate solution is created it is evaluated using Formula (8). The generation-evaluating process is repeated for every node with $t(X^S, i) \neq t(X^T, i)$ and the process

stops when $t(X^S, i) = t(X^T, i)$ for all $i \in V$. Thus, the PR procedure receives as input a pair of solutions X^S and X^T , generates and evaluates all of the intermediate solutions from X^S to X^T and the best intermediate solution X^R is returned as output. In the following we denote with $PR(X^S, X^T)$ the application of PR starting at solution X^S and finishing at solution X^T .

Procedures 3 and 4 present the static and dynamic variants of PR implemented in GPR_CTDTP, respectively. Both static and dynamic variants maintain a set of b elite solutions $B = \{B_1, \dots, B_b\}$. B is initialized by running the construction and local search procedures for b times. Solutions in B are always kept sorted in ascending order of their objective function value estimated with Equation (8).

Procedure 3 grasp_pr_static(i_{\max})

Input: i_{\max} : number of global iterations;
Output: X^{best} : A p -partition of V ;

```

for all (  $i \in \{1, \dots, b\}$  ) do
   $X \leftarrow$  grasp_construction();
   $B_i \leftarrow$  local_search(  $X$  );
end for
Sort  $B$  from best to worst;
for all (  $iter = 1, \dots, i_{\max}$  ) do
   $X^S \leftarrow$  grasp_construction();
   $X^S \leftarrow$  local_search(  $X^S$  );
  if ( (  $\psi(X^S) < \psi(B_1)$  ) or (  $\psi(X^S) < \psi(B_b)$  and  $d_{\mu}^{sol}(X^S, B) > \theta$  ) ) then
     $E_j \leftarrow$  closest solution to  $X^S$  in  $B$  with  $\psi(X^S) < \psi(B_j)$ ;
     $E_j \leftarrow X^S$ ;
    Update  $B$ ;
  end if
end for
 $X^{best} \leftarrow B_1$ ;
for all (  $i \in \{1, \dots, b-1\}$  ) do
  for all (  $j \in \{i+1, \dots, b\}$  ) do
    Apply  $PR(B_i, B_j)$  and  $PR(B_j, B_i)$  and let  $X^S \leftarrow$  best solution found;
     $X^S \leftarrow$  local_search(  $X^S$  );
    if (  $\psi(X^S) < \psi(X^{best})$  ) then
       $X^{best} \leftarrow X^S$ ;
    end if
  end for
end for
return  $X^{best}$ ;
```

4.2.1 Static GPR_CTDTP

In the static variant, PR is performed at the end of i_{\max} iterations of a typical GRASP. In each iteration of the GRASP a solution is constructed and improved with local search, X^S . This solution is compared with the solutions in B . If X^S is better than the best solution in B (i.e., B_1) or if X^S is

better than the worst solution in B (i.e., B_b) and is at a distance larger than a given threshold θ from solutions in B , then the most similar solution to X^S in B is replaced by X^S . Solutions in B are then sorted from best to worst. After i_{\max} iterations the static PR starts. Every path between solutions in B is evaluated and the best solution is returned. The distance between X^S and solutions in B is estimated as $d_{\mu}^{sol}(X^S, B) = \frac{1}{b} \sum_{i=1}^b g(X^S, B_i)$, where $g(X^S, B_i)$ is the fraction of nodes in X^S and B_i that are assigned to different territories; that is, $d_{\mu}^{sol}(X^S, B)$ is the average number of nodes assigned to different territories in X^S and B_i . Alternative measures of similarity/distance between territory designs have been described before, see for example the work by Tavares Pereira and Rui Figueira [31]. However, such measures do not take advantage of the information we have available when solving the AP. That is, those measures do not know the correspondence between territories beforehand. Besides, distance measures described in [31] are defined in terms of a single attribute and it is not clear how to extend the similarity measure to incorporate information of more than one attribute (e.g., the three activities considered in this work). For that reasons we adopted a simple, yet very informative, measure for computing the distance between territory designs. The pseudocode of the static variant of PR is shown in Procedure 3. $\theta \in [0, 1]$ is a scalar that is set empirically.

4.2.2 Dynamic GPR_CTDP

The dynamic PR variant differs from the static one in that in each iteration of the GRASP the solution X^S is compared to a randomly selected solution from B , say B' . The intermediate solutions between X^S and B' are evaluated, and the best solution found in the path is denoted X^R . Then if X^R is better than B_1 or if X^R is better than B_b and it is at a distance of at most θ from the solutions in B , then the closest solution in B to X^R is replaced with X^R . Then solutions in B are sorted from best to worst. After i_{\max} iterations the best solution, namely B_1 , is returned. The pseudocode is shown in Procedure 4.

A number of parameters are associated with GPR_CTDP in both variants, namely δ the fraction of nodes assigned with a distance criterion, k the number of neighbors that are considered for building a territory, λ the tradeoff parameter of the objective function, α the GRASP quality

Procedure 4 `grasp_pr_dynamic(i_{\max})`

Input: i_{\max} : number of global iterations;**Output:** X^{best} ; A p -partition of V ;**for all** ($i = \{1, \dots, b\}$) **do** $X^S \leftarrow$ `grasp_construction()`; $B_i \leftarrow$ `local_search(X^S)`;**end for**Sort B in ascending order;**for all** ($iter = 1, \dots, i_{\max}$) **do** $X^S \leftarrow$ `grasp_construction()`; $X^S \leftarrow$ `local_search(X^S)`;Randomly select B' from B ;Apply $PR(X^S, B')$ and $PR(B', X^S)$ and let $X^R \leftarrow$ best solution found;**if** (($\psi(X^R) < \psi(B_1)$) or ($\psi(X^R) < \psi(B_b)$ and $d_{\mu}^{sol}(X^R, B) > \theta$)) **then** $B_j \leftarrow$ closest solution to X^R in B with $\psi(X^R) < \psi(B_j)$; $B_j \leftarrow X^R$;Update B ;**end if****end for****return** $X^{best} \leftarrow B_1$;

parameter for the RCL, *limit_evals* the maximum number of evaluations for the local search, b the number of solutions in the elite set B and θ the distance threshold in PR. In this work we have fixed all of these parameters based on preliminary experimentation. The next section reports experimental results with the proposed GPR_CTDTP.

5 Computational experiments

This section reports experimental results obtained with GPR_CTDTP. The proposed method was implemented in Matlab^R. The code and data sets are publicly available for research purposes from the authors upon request. All of the experiments were run in a 64-bit workstation with a Corei7 processor at 3.4GHz and 8 GB in RAM.

5.1 Experimental setting

For the experiments we used the data base from [28]. These are randomly generated instances based on real-world data. Data sets DS and DT are considered for experimentation. The former generate the BU weights from a uniform distribution and the latter uses a triangular distribution. Data set DT more closely resembles real-world instances. These data sets are fully described in [28]. For each of DS and DT data sets there are 20 different instances of size $n = 500$ and $p = 10$.

For all of the instances in both DS and DT data sets we use a tolerance level $\tau^a = 0.05$, $a \in A$. Recall that τ^a measures the allowable relative deviation from the target average size μ^a for activity a . Hence, a value of $\tau^a = 0.05$ implies that instances are tightly constrained in all activities and therefore the problem is more difficult to solve than instances that use a larger value of τ^a . In previous work [28], experiments have been reported with other values for $\tau^a \in [0.05, 0.30]$. Here we focus on the most difficult instances.

Throughout the evaluation, the GRASP is run with $i_{\max} = 500$. Based on preliminary experimentation for fine-tuning the algorithmic parameters for GPR_CTDP, we will use the values reported in Table 1. Showing the fine-tuning of these parameters is out of the scope of this paper.

Table 1: Summary of values used for the algorithmic parameters of GPR_CTDP.

Parameter	Value	Description
δ	0.5	Fraction of nodes assigned with a distance criterion.
k	3	Number of neighbors that are considered for growing a territory.
λ	0.7	Weight parameter in the meritfunction.
α	0.3	RCL quality parameter.
<i>limit_evals</i>	1,000	The maximum number of fitness function evaluations in the local search.
b	20	The number of solutions in the elite set E .
θ	0.6	The distance threshold in PR.
i_{\max}	500	Number of global iterations for GPR_CTDP.

In the following sections we report the obtained experimental results. We have divided experimental results in three sections that aim at assessing different aspects of the GPR_CTDP.

5.2 Assessing the construction and local search procedures within a GRASP framework

This section describes results of experiments designed to evaluate the effect of the proposed construction and local search procedures. To this end we apply the new construction phase within a GRASP framework, that is, no PR phase is applied in this experiment. First, we apply the GRASP with construction phase only and then we apply the complete GRASP with both construction and local search phases. For each of these, we tested the two different data sets. Figures 5 and 6 show the performance of the construction and local search procedures for DT and DS data sets, respectively. In each figure we plot the values of the objective, $F(S)$, and infeasibility, $G(S)$, for each instance and for each mechanism. As expected, from these figures we can see that local search

improves significantly the construction procedure, in terms of both infeasibility and dispersion. For both data sets, local search (triangle marker) obtains feasible solutions (i.e., $G(S) = 0$) for most of the instances starting from the highly infeasible solutions generated by the construction mechanism (diamond marker). Besides, there are considerable improvements in terms of $F(S)$ for all of the instances in the DT data set, see Figure 5. Lower improvements in terms of dispersion are observed for the DS data set, see Figure 6; although local search always obtained returned better solutions. It is expected that the PR mechanisms further improve the dispersion of solutions obtained with plain local search.

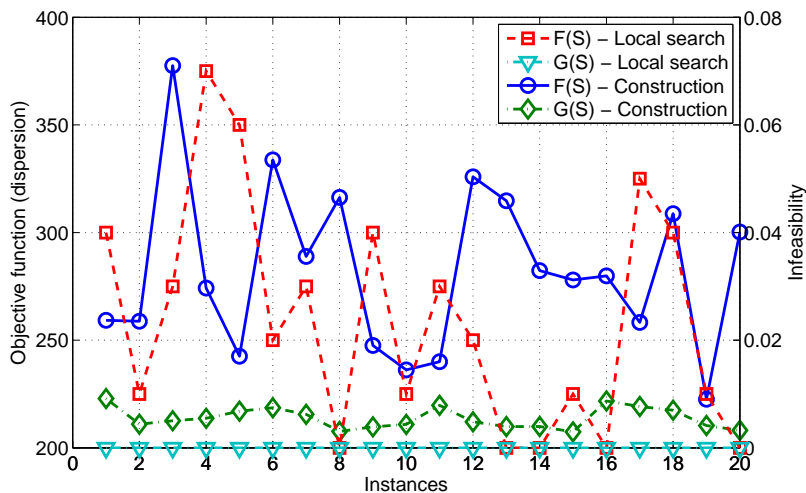


Figure 5: Performance of the construction and local search mechanisms for instances in the DT data set. We show the values of $F(S)$ (left y -axis) and $G(S)$ (right y -axis).

Table 2 summarizes the performance of the construction and local search procedures across all instances of both DT and DS data sets. For the dispersion term $F(S)$, we show the relative deviation between the solution obtained with each procedure and the best known solution for each instance $RDB = (F(S) - F(S_{best})/F(S_{best}))$. The column labeled “local search” indicates that both construction and local search phases are applied. From this table we can see that the average of the sum of relative infeasibilities is maintained low in the construction procedure for both data sets. This result shows that the proposed procedure is able to obtain acceptable solutions in terms of the degree of satisfaction of the balance constraints despite the fact part of the construction procedure is based on a purely distance-based criterion.

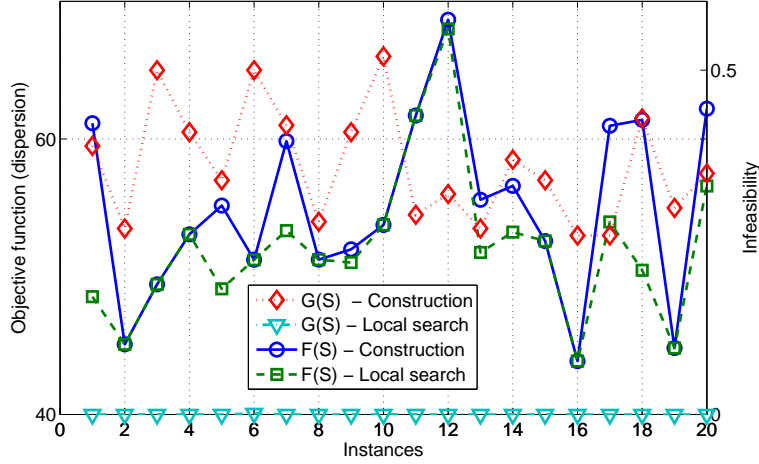


Figure 6: Performance of the construction and local search mechanisms for instances in the DS data set. We show the values of $F(S)$ (left y -axis) and $G(S)$ (right y -axis).

Table 2: Evaluation of the construction and local search procedures of GPR_CTDP.

Data set		DT		DS	
Measure / Mechanism		Construction	Local search	Construction	Local search
RDB	Best	5.81%	0.00%	0.00%	0.00%
	Average	34.12%	1.51%	20.91%	14.51%
	Worst	81.45%	6.04%	58.28	56.76%
$G(S)$	Best	$0.00E + 00$	$0.00E + 00$	$2.60E - 01$	$0.00E + 00$
	Average	$2.37E - 02$	$0.00E + 00$	$3.61E - 01$	$3.01E - 04$
	Worst	$7.06E - 02$	$0.00E + 00$	$5.24E - 01$	$3.55E - 03$

After applying local search to the constructed solutions, the dispersion measure $F(S)$ is improved as it shows a reduction in the relative deviation with respect to the best dispersion value. In the case of the DT data set the solutions obtained with local search are very close to the best ones in terms of dispersion (average deviation of 1.51%), while for DU there is much more room for improvement (average deviation of 14.51%). For the DT data set the objective function is improved in average by 32.61%, while for the DS data set the improvement is of 6.4%. These are rather important differences that evidence the effectiveness of the proposed local search mechanism. It is very important to emphasize that dispersion is improved by also considerably reducing $G(S)$.

5.3 GRASP vs. GPR_CTDP

This section reports experimental results on the improvements of the PR strategies over the straight GRASP implementation described in Section 4.1. Table 3 shows the performance of GPR_CTDP

under both static (GPR-ST column) and dynamic (GPR-DY column) PR strategies for DT and DS data sets. In the table, we compare the performance of GPR_CTDP when using PR and when only GRASP without PR is adopted. We show the relative deviation between the best solution obtained with each method and the best known solution for each instance.

Table 3: Evaluation of GPR_CTDP with static and dynamic PR.

Data set		DT			DS		
Measure		GRASP	GPR-ST	GPR-DY	GRASP	GPR-ST	GPR-DY
RDB	Best	0%	0%	0%	0%	0%	0%
	Average	1.51%	0.51%	1.27%	14.51%	0.76%	13.92%
	Worst	6.04%	3.09%	3.91%	56.76%	11.44%	56.76%
$G(S)$	Best	$0.00E + 00$	$0.00E + 00$	$0.00E + 00$	$0.00E + 00$	$0.00E + 00$	$0.00E + 00$
	Average	$0.00E + 00$	$0.00E + 00$	$0.00E + 00$	$3.01E - 04$	$2.53E - 04$	$2.84E - 04$
	Worst	$0.00E + 00$	$0.00E + 00$	$0.00E + 00$	$3.55E - 03$	$5.07E - 04$	$3.55E - 03$

As we can see, for the DT data set, the improvements obtained with PR over local search are small yet non-negligible. We believe this result can be due to the fact that we are approaching to the global optimum for this data set and since the local search procedure provides very competitive solutions by itself the improvements due to PR are rather small. However, it is important to emphasize that all of the solutions found with GRASP and GPR_CTDP are feasible for this data set. For this data set the static PR strategy outperformed the dynamic one by less than 1% in terms of the objective function. For the DS data set the improvements due to PR are larger. GPR_CTDP with static PR outperforms the results of local search by an average of $\approx 13\%$ in terms of the objective function, whereas the dynamic strategy outperforms local search by less than 1%. The static variant of PR achieve important improvements in terms of the dispersion objective ($F(S)$), while also reducing the infeasibility term.

Finally, it is important to point out that even in the case when GRASP is allowed to run by itself for an amount of time equal to the total amount of time employed by GPR_CTDP, the results reported by the later are still better. This is due to the fact that the GRASP seems to converge within the first iterations, thus a better solution is hardly found by GRASP afterwards.

Figures 7 and 8 show the territories obtained with the construction, local search, and PR GPR_CTDP procedures for a particular instance of the DT data set. Figure 7 shows the solution from a run of the static PR GPR_CTDP and Figure 7 shows the corresponding solution for the

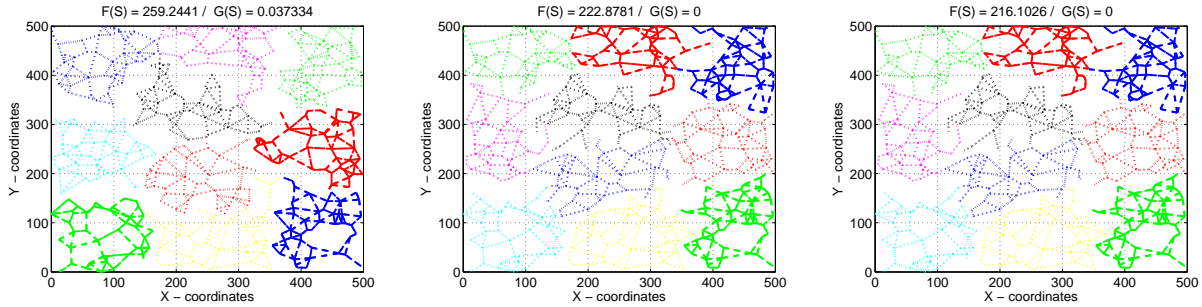


Figure 7: Solutions obtained by the construction, local search and static PR procedures for a particular instance of the DT data set.

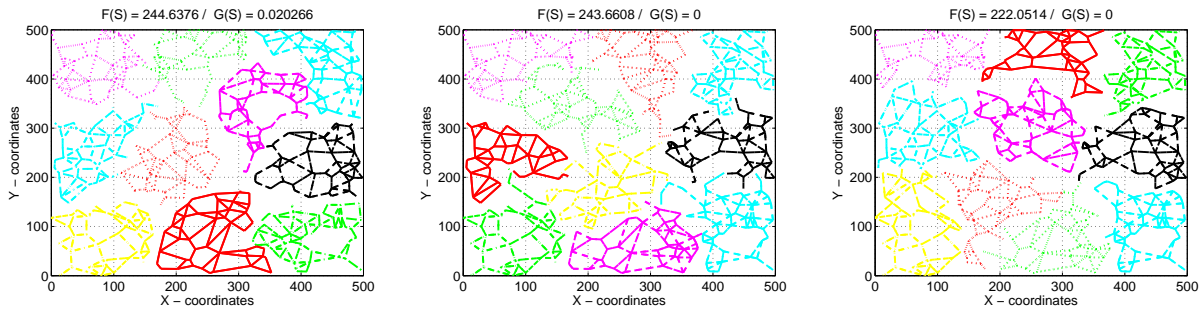


Figure 8: Solutions obtained by the construction, local search and dynamic PR procedures for a particular instance of the DT data set.

dynamic PR GPR_CTDP. These figures illustrate the advantages of GPR_CTDP over the construction and local search mechanisms. Territories generated after the construction procedure present infeasibilities. The local search process eliminates infeasibilities and reduces the dispersion objective. However, the dispersion is further minimized with both PR variants. Visually, it can be seen that territories generated with local search (center plots) are more disperse than those generated with GPR_CTDP (rightmost plots). For this particular instance, a better solution was obtained with the static version of PR, which agrees with results presented in this section.

5.4 Static vs. dynamic path relinking

This section elaborates on the difference in performance between the static and dynamic PR variants of GPR_CTDP. From Table 3 we can see that the improvements of static and dynamic GPR_CTDP over local search are of 1% and 0.24% for the DT data set and of 13.75% and 0.59% for the DS data set (in terms of the objective function). Thus, despite the fact both strategies resulted effective, the use of the static one is advantageous. We think this can be due to the fact that static GPR_CTDP

explores all of the paths between elite solutions at the end of the search process. Hence a global picture of the search process is considered during the execution of static GPR_CTDTP. Dynamic GPR_CTDTP on the other hand, explores the paths between every solution processed by local search and a random solution from the elite set. Since it is not guaranteed that PR is performed over two competitive solutions, it is less likely that an effective solution can be found after exploring the paths.

Figure 9 shows the relative deviation of the solutions found with each tested method and the best known solution for each instance for DT data set. This figure give us more insight into the performance of the different methods across the instances, it is rather clear that the static PR strategy obtained the best solutions for most of the instances (those instances for which the relative deviation is zero), followed by the dynamic PR approach.

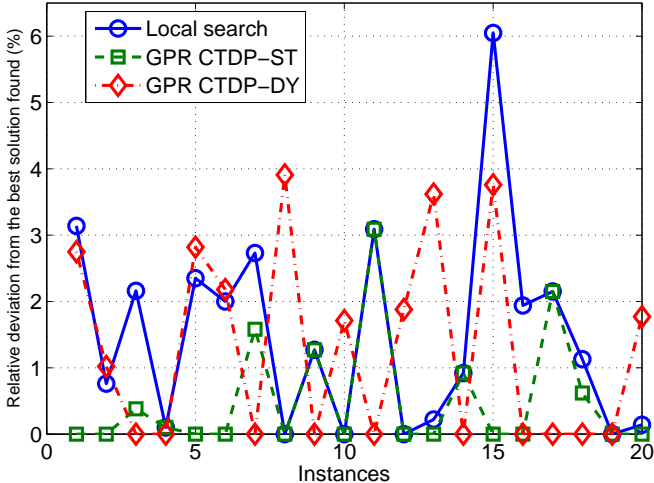


Figure 9: A comparison among the methods in terms of relative deviation from best objective function value for the DT data set on each individual instance.

Table 4 reports the processing time for each variant of GPR_CTDTP and for each data set. In general terms a new territory design can be obtained with either variant of GPR_CTDTP in a few hours. This processing times are acceptable, since from the practical standpoint, this decision is taken every 3-4 months.

One final comment, it was observed that, in the GPR_CTDTP method, around 80% of the time is spend in the GRASP and 20% doing the Path Relinking. Therefore, we have empirically observed

Table 4: CPU time (min) comparison for static and dynamic GPR_CTDTP.

	DT		DS	
	GPR-ST	GPR-DY	GPR-ST	GPR-DY
Best	124.84	119.93	179.74	179.37
Average	136.25	133.70	204.42	200.24
Worst	152.96	150.34	240.30	227.46

that this additional amount of effort pays off significantly.

6 Conclusions

We have introduced a new model in commercial territory design. The new model makes use of a diameter-based dispersion function instead of the traditional center-based functions.

We have described a GRASP with path relinking (GPR_CTDTP) for this CTDTP. The problem, motivated by a real-world application, consists of grouping commercial units into geographic territories subject to dispersion, connectivity and balance constraints. A novel construction procedure was developed and two variants of PR were explored in GPR_CTDTP, namely, static and dynamic PR. The components of GPR_CTDTP were evaluated and compared extensively in instances that are known to be very challenging from previous work.

Experimental results show that the proposed construction procedure is able to construct very competitive solutions, mainly in terms of the dispersion criterion. The local search of the GPR_CTDTP improves solutions in terms of both dispersion and balance requirements. Both versions of PR improve the performance of the application of the construction and local search mechanisms, confirming previous work on the combination of GRASP and PR. In particular we found that with the static PR variant better solutions can be obtained for the TDP. This can be due to the fact that the PR process is applied over elite instances, which increases the chances of finding a better solution. In general terms the processing time of both PR variants lies in reasonable ranges for the application.

We have identified several future work directions in the context of GPR_CTDTP. In particular we would like to explore other variants of PR that are known to be very effective, for example, evolutionary PR. Further, we are interested in the development of an adaptive filtering step that

allows us to identify pairs of solutions that can be potentially improved by applying PR. This is in addition to the rules used for updating the set of elite solutions. We think that such a filtering strategy will have a very positive impact in the efficiency of GPR_CTD. Since we found evidence that maintaining a set of elite solutions can be beneficial for TDP, we would like to explore the use of other “population-based” metaheuristics such as scatter search.

It is important to note that the method developed in this work can also be extended and applied to other districting problems under balancing and connectivity constraints. The presence of the connectivity constraints make the path relinking process more challenging. For instance, path relinking has been applied in a different manner in related partitioning problems such as capacitated clustering [9]. In this particular work, we have successfully exploited the problem structure by solving an associated Assignment Problem whose solution will guide the relinking process in a more intelligent fashion. To the best of our knowledge this PR idea is novel and worthwhile for further exploration in other districting or clustering problems under connectivity constraints.

Acknowledgements: The first author was supported by PROMEP under grant 103.5/11/4330. The second author was supported by the Mexican National Council for Science and Technology (CONACYT) under grants CB-2005-01/48499-Y and CB-2011-01/166397, and by Universidad Autónoma de Nuevo León under its Scientific and Technological Research Support through grants UANL-PAICYT CE012-09, IT511-10, and CE728-11.

References

- [1] P. K. Bergey, C. T. Ragsdale, and M. Hoskote. A simulated annealing genetic algorithm for the electrical power districting problem. *Annals of Operations Research*, 121(1):33–55, 2003.
- [2] M. Blais, S. D. Lapierre, and G. Laporte. Solving a home-care districting problem in an urban setting. *Journal of the Operational Research Society*, 54(11):1141–1147, 2003.
- [3] B. Bozkaya, E. Erkut, and G. Laporte. A tabu search heuristic and adaptive memory procedure for political districting. *European Journal of Operational Research*, 144(1):12–26, 2003.

- [4] M. H. Browdy. Simulated annealing: An improved computer model for political redistricting. *Yale Law and Policy Review*, 8:163–179, 1990.
- [5] R. E. Burkard, M. Dell’Amico, and S. Martello. *Assignment Problems*. SIAM, Philadelphia, 2009.
- [6] F. Caro, T. Shirabe, M. Guignard, and A. Weintraub. School redistricting: Embedding GIS tools with integer programming. *Journal of the Operational Research Society*, 55(8):836–849, 2004.
- [7] S. J. D’Amico, S.-J. Wang, R. Batta, and C. M. Rump. A simulated annealing approach to police district design. *Computers & Operations Research*, 29(6):667–684, 2002.
- [8] L. S. de Assis, P. M. Franca, and F. L. Usberti. A redistricting problem applied to meter reading in power distribution networks. *Computers & Operations Research*, 41:65–75, 2014.
- [9] Y. Deng and J. F. Bard. A reactive GRASP with path relinking for capacitated clustering. *Journal of Heuristics*, 17(2):119–152, 2011.
- [10] A. Drexler and K. Haase. Fast approximation methods for sales force deployment. *Management Science*, 45(10):1307–1323, 1999.
- [11] J. C. Duque, R. Ramos, and J. Suriñach. Supervised regionalization methods: A survey. *International Regional Science Review*, 30(3):195–220, 2007.
- [12] E. Erkut, Y. Ülküsal, and O. Yeniçerioglu. A comparison of p -dispersion heuristics. *Computers & Operations Research*, 21(10):1103–1113, 1994.
- [13] H. J. Faria, S. Binato, M. G. C. Resende, and D. J. Falcao. Transmission network design by a greedy randomized adaptive path relinking approach. *IEEE Transactions on Power Systems*, 20(1):43–49, 2005.
- [14] T. A. Feo and M. G. C. Resende. Greedy randomized adaptive search procedures. *Journal of Global Optimization*, 6(2):109–133, 1995.

- [15] S. L. Forman and Y. Yue. Congressional districting using a TSP based genetic algorithm. In E. Cantú-Paz, J. A. Foster, K. Deb, L. David, and R. Rajkumar, editors, *Genetic and Evolutionary Computation – GECCO 2003*, volume 2723 of *Lecture Notes in Computer Science*, pages 2072–2083. Springer, Berlin, 2003.
- [16] F. Glover. Tabu search and adaptive memory programming – Advances, applications, and challenges. In R. S. Barr, R. V. Helgason, and J. L Kennington, editors, *Interfaces in Computer Science and Operations Research*, chapter 1, pages 1–75. Kluwer, Dordrecht, 1996.
- [17] J. Kalcsics, S. Nickel, and M. Schröder. Toward a unified territorial design approach: Applications, algorithms, and GIS integration. *TOP*, 13(1):1–56, 2005.
- [18] A. Mehrotra, E. L. Johnson, and G. L. Nemhauser. An optimization based heuristic for political districting. *Management Science*, 44(8):1100–1114, 1998.
- [19] L. Muyldermans, D. Cattrysse, D. Van Oudheusden, and T. Lotan. Districting for salt spreading operations. *European Journal of Operational Research*, 139(3):521–532, 2002.
- [20] F. Pukelsheim, F. Ricca, B. Simeone, A. Scozzari, and P. Serafini. Network flow methods for electoral systems. *Networks*, 59(1):73–88, 2012.
- [21] M. G. C. Resende, R. Martí, M. Gallego, and A. Duarte. GRASP and path relinking for the max-min diversity problem. *Computers & Operations Research*, 37(3):498–508, 2010.
- [22] M. G. C. Resende and C. C. Ribeiro. Greedy randomized adaptive search procedures: Advances, hybridizations, and applications. In M. Gendreau and J.-Y. Potvin, editors, *Handbook of Metaheuristics*, volume 146 of *International Series in Operations Research & Management Science*, chapter 10, pages 283–319. Springer, Berlin, 2010.
- [23] M. G. C. Resende, C. C. Ribeiro, F. Glover, and R. Martí. Scatter search and path-relinking: Fundamentals, advances, and applications. In M. Gendreau and J.-Y. Potvin, editors, *Handbook of Metaheuristics*, volume 146 of *International Series in Operations Research & Management Science*, chapter 4, pages 87–107. Springer, Berlin, 2010.

- [24] M. G. C. Resende and R. F. Werneck. A hybrid heuristic for the p -median problem. *Journal of Heuristics*, 10(1):59–88, 2004.
- [25] C. C. Ribeiro, E. Uchoa, and R. F. Werneck. A hybrid GRASP with perturbations for the Steiner problem in graphs. *INFORMS Journal on Computing*, 14(3):228–246, 2002.
- [26] F. Ricca, A. Scozzari, and B. Simeone. Political districting: From classical models to recent approaches. *Annals of Operations Research*, 204(1):271–299, 2013.
- [27] F. Ricca and B. Simeone. Local search algorithms for political districting. *European Journal of Operational Research*, 189(3):1409–1426, 2008.
- [28] R. Z. Ríos-Mercado and E. A. Fernández. A reactive GRASP for a commercial territory design problem with multiple balancing requirements. *Computers & Operations Research*, 36(3):755–776, 2009.
- [29] R. Z. Ríos-Mercado and J. C. Salazar-Acosta. A GRASP with strategic oscillation for a commercial territory design problem with a routing budget constraint. In I. Batyrshin and G. Sidorov, editors, *Advances in Soft Computing*, volume 7095 of *Lecture Notes in Artificial Intelligence*, pages 307–318. Springer, Heidelberg, Germany, 2011.
- [30] M. A. Salazar-Aguilar, R. Z. Ríos-Mercado, and M. Cabrera-Ríos. New models for commercial territory design. *Networks & Spatial Economics*, 11(3):487–507, 2011.
- [31] F. Tavares Pereira, J. Rui Figueira, V. Mousseau, and B. Roy. Comparing two territory partitions in districting problems: Indices and practical issues. *Socio-Economic Planning Sciences*, 43(1):72–88, 2009.
- [32] A. A. Zoltner and P. Sinha. Sales territory alignment: A review and model. *Management Science*, 29(11):1237–1256, 1983.
- [33] A. A. Zoltner and P. Sinha. Sales territory design: Thirty years of modeling and implementation. *Marketing Science*, 24(3):313–331, 2005.