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Sharaf al-Dīn al-Ṭūsī on the Number of Positive Roots of Cubic Equations

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In the second part of his *Algebra*, Sharaf al-Dīn al-Ṭūsī (12th-century) correctly determines the number of positive roots of cubic equations in terms of the coefficients. R. Rashed has recently published an edition of the *Algebra* [al-Ṭūsī 1985], and he has discussed al-Ṭūsī's work in connection with 17th century and more recent mathematical methods (see also [Rashed 1974]). In this paper we summarize and analyze the work of al-Ṭūsī using ancient and medieval mathematical methods. We show that al-Ṭūsī probably found his results by means of manipulations of squares and rectangles on the basis of Book II of Euclid's *Elements*. We also discuss al-Ṭūsī's geometrical proof of an algorithm for the numerical approximation of the smallest positive root of $x^3 + c = ax^2$. We argue that al-Ṭūsī discovered some of the fundamental ideas in his *Algebra* when he was searching for geometrical proofs of such algorithms.

Dans la seconde partie de son Algèbre, Sharaf al-Dīn al-Ṭūsī (XII^e siècle), a correctement déterminé le nombre de racines d'une équation du troisième degré en fonction de ses coefficients. R. Rashed a récemment publié une edition de cette Algèbre [al-Ṭūsī 1985] et a étudié l'ouvrage d'al-Ṭūsī en se servant des méthodes mathématiques du XVII^e siècle et de méthodes encore plus récentes (voir aussi [Rashed 1974]). Dans cet article, nous résumons et analysons l'ouvrage d'al-Ṭūsī en utilisant les méthodes mathématiques connues dans l'Antiquité et au Moyen-Age. Nous montrons qu'al-Ṭūsī a probablement trouvé les résultats auxquels il est parvenu par des opérations effectuées sur des carrés et des rectangles, opérations basées sur le Livre II des *Élements* d'Euclide. Nous étudions également la démonstration géométrique d'un algorithme utilisé par al-Ṭūsī pour calculer par approximation la valeur numérique de la plus petite racine positive de l'équation $x^3 + c = ax^2$. Nous essayons de montrer qu'al-Ṭūsī a trouvé certaines des idées fondamentales de son *Algèbre* alors qu'il tentait de trouver des démonstrations géométriques à de tels algorithmes.

شرف الدين الطوسي وعدد الجذور الإيجابية لمعادلات الدرجة الثالثة

ملخص ٠ في الجزء الثاني من كتابه في الجبر أتى شرف الدين الطوسي (القرن الثاني عشر الميلادى) بتحديد صحيح لعدد الجذور الإيجابية لمعادلات الدرجة الثالثة تبعًا لمعاملات المعادلة ـ٠ وقد نشر ر٠ راشد أخيرًا نص كتاب شرف الدين في الجبر [الطوسي ١٩٨٥] مع شرح لهذا الكتاب استعمل فيه أساليب رياضية ترجع إلى القرن السابع عشر وأساليب رياضية أحدث عهدًا (أنظر أيضًا [راشد ١٩٧٤] ٠ وفي هذا المقال

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نختصر ونحلل كتاب شرف الدين الطوسي ولا نستعمل في ذلك إلا الأساليب الرياضية التي كانت معروفة في العصور القديمة وفي العصور الوسطى •
ونبين هنا أنه من الراجح أن الطوسي قد وجد النتائج التي توصل إليها عن طريق استعماله مربعات ومستطيلات عالجها وفقًا لما جاء في الكتاب الثاني من كتاب أقليدس في الأصول • كما نتناول مناقشة البرهان الهندسي الذى أتى به الطوسي لإقامة الدليل على صحة الطريقة الحسابية التي سلكها في الحساب التقريبي لأصغر جذر إيجابي لمعادلة س^{٣} + ج = ا س^{٢} •
ونبين أن الطوسي اكتشف بعضا من آرائه الأساسية في الجبر أثناء محاولاته لإيجاد براهين هندسية لمثل هذه الطرق الحسابية •

AMS 1980 subject classifications: 01A30.
Key Words: history of algebra, cubic equations, Islamic mathematics, Sharaf al-Dīn al-Ṭūsī, geometrical algebra, numerical approximation.

1. INTRODUCTION

The recent edition of the *Algebra* of Sharaf al-Dīn al-Ṭūsī (12th century, not to be confused with Naṣīr al-Dīn), which was published by R. Rashed in [al-Ṭūsī 1985], is an important contribution to the history of Arabic mathematics. Until recently the mathematician and poet ʿUmar al-Khayyām (ca. 1048–1131) was supposed to have given the most advanced medieval treatment of cubic equations. Thanks to Rashed's publications [al-Ṭūsī 1985] and [Rashed 1974] we now know that al-Ṭūsī went considerably further.

The publication [al-Ṭūsī 1985] contains an edition of the Arabic text with a literal French translation, a transcription of al-Ṭūsī's reasoning in modern notation, and a discussion of most of the text in terms of modern algebra and analysis. Rashed conveniently divided the very long text of the *Algebra* into two parts, consisting of 116 and 127 pages of Arabic text, and printed in two volumes of [al-Ṭūsī 1985]; these volumes will henceforth be denoted as [T1] and [T2]. We will be concerned with the second part of the *Algebra,* on cubic equations that do not for all positive choices of the coefficients have a positive root. This second part consists mainly of a sequence of very long proofs in Euclidean style. The proofs are correct, but as Rashed points out, they do not necessarily reflect the way in which al-Ṭūsī found his results. In the introduction in [T1, xviii–xxxi], Rashed relates al-Ṭūsī's discussion of the cubic equation $f(x) = c$ to a method of P. de Fermat (1601–1665) for the determination of maxima and minima of a cubic curve $y = f(x)$. According to Rashed, the concept of the derivative of a function or of a polynomial is also implicit in al-Ṭūsī's work (see also [Rashed 1974, 272–273, 290] = [Rashed 1984, 175–176, 193], and for some further consequences [Rashed 1984, 312], reprinted from [Rashed 1978]).

As far as is known, cubic curves were never drawn by medieval mathematicians, and the method of Fermat and the derivative are not mentioned explicitly in any known medieval Arabic text. Thus the question arises of whether al-Ṭūsī's methods and motivation can also be explained in terms of standard ancient and medieval mathematics. In this paper I propose such an alternative explanation.

Section 2 of this paper is a concise analysis of the second part [T2] of al-Ṭūsī's *Algebra,* by means of methods and concepts attested to elsewhere in the Greek and Islamic tradition. Section 3 is about al-Ṭūsī's motivation. The appendices contain notes to the Arabic text and the French translation in [T2], for the reader who wishes to compare this paper with the original text. I conclude the present section with a brief summary in modern notation of the results that al-Ṭūsī proves in the *Algebra.*

The *Algebra* is a detailed treatment of linear, quadratic, and cubic equations in one unknown. Because the mathematicians in the Islamic tradition only recognized positive coefficients and roots, they had to distinguish 18 different types of cubic equations. Al-Khayyām had already shown that the five types without a constant term can be reduced to quadratic equations, and for each of the remaining 13 types he had given a geometrical construction of a root by means of two intersecting conic sections, or by means of one conic section intersecting a circle [al-Khayyām 1981].

Eight of these thirteen types have for all (positive) choices of the coefficients a (positive) root. In the first part of the *Algebra* [T1], al-Ṭūsī renders al-Khayyām's geometrical constructions for these eight types, and he describes a numerical procedure (essentially the Ruffini-Horner scheme, see [Luckey 1948]) for approximating the root.

The second part of the *Algebra* [T2] is entirely devoted to the five remaining types of cubic equations, namely

$$x^3 + c = ax^2 \tag{1}$$

$$x^3 + c = bx \tag{2}$$

$$x^3 + ax^2 + c = bx \tag{3}$$

$$x^3 + bx + c = ax^2 \tag{4}$$

$$x^3 + c = ax^2 + bx \tag{5}$$

with $a, b, c > 0$.

Al-Khayyām pointed out that the number of roots of these equations depends on the number of intersections of the two conic sections used in the construction. He does not give the precise relation between the number of intersections and the coefficients of the cubic equation (cf. [al-Khayyām 1981, 71]). For a given choice of the coefficients one could of course draw the conic sections on a piece of paper and determine the number of intersections empirically. Al-Khayyām does not mention this procedure, perhaps because it cannot be completely accurate. However, al-Ṭūsī succeeded in determining the exact relationship between the number of roots and the coefficients of the equation. Neither al-Khayyām nor al-Ṭūsī was able to determine the roots themselves in terms of the coefficients; there is no evidence whatsoever that the algebraic solution of the cubic equation was known before the Italian Renaissance.

Al-Ṭūsī treats the five equations in the order (1) [T2, 1–18]; (2) [T2, 19–34]; (3) [T2, 34–48]; (4) [T2, 49–70]; and (5), case $a = \sqrt{b}$ [T2, 70–76], case $a > \sqrt{b}$ [T2, 76–104], case $a < \sqrt{b}$ [T2, 104–127]. For each of the types (1)–(4), and for each of the three cases of (5), the treatment is structured as follows (for detailed references to the text, see note [1]). For sake of brevity I write the equations (1)–(5) as $f(x) = c$.

A. First al-Ṭūsī defines a quantity m in a way that depends on the type of equation: (1) $m = (\frac{2}{3})a$, (2) $m = \sqrt{(b/3)}$, (3) $m^2 + (\frac{2}{3})am = b/3$, (4) $m^2 + (b/3) = (\frac{2}{3})am$ (here m is the largest of the two positive roots), and (5) $m^2 = (\frac{2}{3})am + b/3$. (In all five cases we have $f'(m) = 0$, but in my opinion al-Ṭūsī did not know the concept of a derivative.) He then proves $f(x) < f(m)$ for all (positive) $x \neq m$. Thus if $c > f(m)$, $f(x) = c$ has no root and if $c = f(m)$ there is exactly one root $x = m$.

B. He then supposes $c < f(m)$, and he considers the equation

$$y^3 + py^2 = d, \tag{6}$$

with $d = f(m) - c$ for all types and p depending on the type of equation, as follows: (1) $p = a$, (2) $p = 3m$, (3) $p = 3m + a$, (4), (5) $p = 3m - a$ with m defined as above; it can be shown that $p > 0$ always. The (unique positive) root y_1 of (6) had already been constructed geometrically in [T1, 56–57] by means of a parabola and a hyperbola, and an algorithm for the computation of y_1 had been described in [T1, 58–66]. Al-Ṭūsī proves that $x_1 = m + y_1$ is a root of $f(x) = c$. Thus the existence of at least one root $x_1 > m$ is guaranteed (by the geometrical construction of y_1), and in part F it will turn out that there is no other root $x > m$. The root x_1 can be computed from m and y_1.

C. For types (4) and (5) al-Ṭūsī provides an upper bound of x_1 in terms of a and b.

D1. For type (1) only, al-Ṭūsī geometrically constructs a segment of length q such that

$$q^2 + q(a - x_1) = x_1(a - x_1), \tag{7}$$

where $x_1 > m$ is the unique positive root of (1) constructed in B. He shows that $x_2 = a - x_1 + q$ is another root of (1) with $x_2 < m$. He also proves that if $z_2 = m - x_2$, then $z = z_2$ is a root of

$$z^3 + d = pz^2, \tag{8}$$

with $d = f(m) - c = (4/27)a^3 - c$, $p = a$ as above.

He then explains an algorithm for the computation of x_2 from (1), assuming that $c \leq (\frac{2}{27})a^3$ (see below for more details). If $c > (\frac{2}{27})a^3$ we have $d < (\frac{2}{27})a^3$; in this case he first computes $z_2 = (\frac{2}{3})a - x_2$ by the same algorithm applied to (8).

D2. For types (2)–(5), al-Ṭūsī considers the auxiliary equation

$$z^3 + d = pz^2, \tag{9}$$

with p and d as in (6); this equation is of type (1). Let z_2 be the smallest (positive) root and $x_2 = m - z_2$. He then proves that x_2 is a root of $f(x) = c$. The root x_2 can be computed from z_2 and m.

E. For types (4) and (5), al-Ṭūsī discusses positive lower bounds for x_2 in terms of a and b if such bounds exist.

F. Al-Ṭūsī proves separately that if $x_1 > m$ is a root of $f(x) = c$, $y_1 = x_1 - m$ is a root of (6).

G. He proves similarly that if $x_2 < m$ is a root of $f(x) = c$, $z_2 = m - x_2$ is a root of (9).

H. He finishes the discussion of most types with a summary or a numerical example.

Thus al-Ṭūsī determines the number of solutions directly from the coefficients, and he shows that al-Khayyām's separate geometrical constructions for (1)–(5) are superfluous, because they can all be reduced to the geometrical construction for (6) in [T1, 56–57]. Therefore [T2] does not contain conic sections at all. Al-Ṭūsī does not mention the fact that the equation $x^3 + bx = ax^2 + c$ can have two or three positive roots for suitable positive coefficients a, b, c (compare [T1, 107–116]).

2. ANALYSIS OF THE SECOND PART OF AL-ṬŪSĪ'S *ALGEBRA*

In the *Algebra* al-Ṭūsī uses similar reasoning in many different situations, and his solutions of Eqs. (1)–(5) are to a large extent analogous. This makes it possible to render the essentials of the 127 pages of Arabic text in [T2] in a concise way. The purpose of the following presentation is to make al-Ṭūsī's ideas easily accessible to the reader, and to explain his ideas in the context of ancient and medieval mathematics. The presentation is very close in spirit to the text of the *Algebra,* although I do not follow the order of the arguments in the text, labeled A–H in the preceding section. I rather intend to give a plausible reconstruction of how al-Ṭūsī found his results. In ancient terminology one could say that al-Ṭūsī's *Algebra* is a synthesis and my reconstruction is a corresponding analysis. The text of the *Algebra* contains several indications of al-Ṭūsī's original line of thought (see the parts labeled F and G in Section 1, and also, for example, [T2, 36, 39, 57]), and my reconstruction is consistent with these indications.

For sake of brevity and clarity I use some modern notation in the transcription of ancient and medieval concepts. I indicate the algebraical "cube," "square," and "root" as x^3, x^2, and x (or y^3, y^2, y, z^3, z^2, z), and I transcribe equations such as "a cube plus a number equals squares plus roots" as $x^3 + c = ax^2 + bx$; here a and b stand for the "number of squares" and the "number of roots," respectively.

The second part of the *Algebra* contains very little of what we would call algebra, i.e., direct manipulation of algebraic equations (for an exception see [T2, 8–9]). Al-Ṭūsī immediately casts his equations in a geometrical form, and he works with the resulting geometrical expressions. Thus in the case of $x^3 + c = ax^2$

$+ bx$, he chooses on a straight line three segments $BE = x$, $BC = a$, and $BA = \sqrt{b}$ (the square root is necessary for reasons of homogeneity). Then c can be interpreted as "the excess of BC times the square of BE and the square of AB times BE over the cube of BE." I will transcribe this as $c = BC \cdot BE^2 + AB^2 \cdot BE - BE^3$. I denote the points in the geometrical figures as much as possible in the way of the French translation in [T2].

Turning to al-Ṭūsī's ideas, first consider Eq. (5), that is $x^3 + c = ax^2 + bx$, to which al-Ṭūsī devotes the last 58 pages of Arabic text [T2, 70–127].

Fix segments $BC = a$ and $AB = \sqrt{b}$, as in Fig. 1. Al-Ṭūsī discusses the three cases $a = \sqrt{b}$, $a > \sqrt{b}$, and $a < \sqrt{b}$ separately. I omit the relatively easy case $a = \sqrt{b}$ [T2, 70–76]. First suppose $a < \sqrt{b}$ [T2, 104–127]. Al-Ṭūsī is interested in the relationship between x and c. Let $x = BE$ as in Fig. 1. Al-Ṭūsī sometimes uses a technical term *baqīya ḍilᶜ* BE ("the remainder for side BE") [2] for the quantity $BC \cdot BE^2 + AB^2 \cdot BE - BE^3$, and I therefore feel entitled to write this quantity as $f(BE)$. Then (5) can be written as $f(BE) = c$.

Al-Ṭūsī interprets AB^2 and BE^2 as real squares $ABH\alpha$, $EBK\varepsilon$, and the difference $AB^2 - BE^2$ as a "gnomon" (Arabic: ᶜalam) $A\alpha HK\varepsilon E$ as in Fig. 1, in the manner of Book II of Euclid's *Elements* (see [Heath 1956 I, 370–372]). I write the squares and the gnomon as $[B\alpha]$, $[B\varepsilon]$, and $[\varepsilon\alpha]$, respectively. Then

$$f(BE) = BC \cdot [B\varepsilon] + BE \cdot [\varepsilon\alpha]. \tag{10}$$

If D is a point between E and C, then similarly

$$f(BD) = BC \cdot [B\delta] + BD \cdot [\delta\alpha]. \tag{11}$$

Al-Ṭūsī investigates the difference between $f(BD)$ and $f(BE)$, but he does not use zero or negative quantities. For the sake of brevity I will use the minus sign in the modern way; thus I use "$a - b = c - d$" to shorten expressions like "if $a > b$ then $c > d$ and $a - b = c - d$; if $a = b$ then $c = d$; if $a < b$ then $c < d$ and $b - a = d - c$."

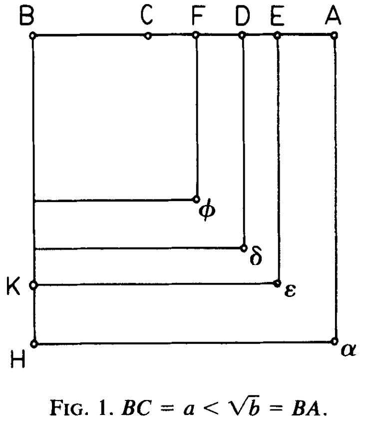

FIG. 1. $BC = a < \sqrt{b} = BA$.

Al-Ṭūsī simplifies $f(BD) - f(BE)$ by decomposing all squares and gnomons as far as possible. We have $[B\varepsilon] = [B\delta] + [\delta\varepsilon]$ and $[\delta\alpha] = [\delta\varepsilon] + [\varepsilon\alpha]$. Therefore by (10) and (11)

$$\begin{aligned} f(BD) - f(BE) &= (BC \cdot [B\delta] + BD \cdot [\delta\alpha]) - (BC \cdot [B\varepsilon] + BE \cdot [\varepsilon\alpha]) \\ &= BD \cdot [\delta\alpha] - (BC \cdot [\delta\varepsilon] + BE \cdot [\varepsilon\alpha]) \\ &= BD \cdot [\delta\varepsilon] - (BC \cdot [\delta\varepsilon] + DE \cdot [\varepsilon\alpha]) \\ &= CD \cdot [\delta\varepsilon] - DE \cdot [\varepsilon\alpha]. \end{aligned}$$

Al-Ṭūsī calls $CD \cdot [\delta\varepsilon]$ the *characteristic* (khāṣṣa) of $f(BD)$ and $DE \cdot [\varepsilon\alpha]$ the characteristic of $f(BE)$ (see the index in [T2, 159]). Note that both characteristics depend on D and E.

Using $[\delta\varepsilon] = DE \cdot (BD + BE)$ we obtain

$$f(BD) - f(BE) = DE \cdot (CD \cdot (BD + BE) - [\varepsilon\alpha]). \tag{12}$$

Similarly, if F is between C and D

$$f(BF) - f(BD) = FD \cdot (CF \cdot (BF + BD) - [\delta\alpha]). \tag{13}$$

We now try to find D such that $f(BD)$ is maximal. Then by (12) and (13) D must be a point such that for all E between D and A

$$CD \cdot (BD + BE) > [\varepsilon\alpha] \tag{14}$$

and for all F between D and C

$$CF \cdot (BF + BD) < [\delta\alpha]. \tag{15}$$

Since $CD \cdot (BD + EB) > 2CD \cdot DB$, and $[\delta\alpha] > [\varepsilon\alpha]$, (14) is true if $2CD \cdot DB \geq [\delta\alpha]$. Since $CF \cdot (BF + BD) < CD \cdot (BD + BD)$, (15) is true if $2CD \cdot BD \leq [\delta\alpha]$. Therefore, if D is such that

$$2CD \cdot BD = [\delta\alpha], \tag{16}$$

then $f(BD)$ is maximal. (Al-Ṭūsī shows that for D defined by (16) and for all relevant points P not between C and A also $f(BP) < f(BD)$.)

Putting $m = BD$, (16) can be reduced to

$$m^2 = (\tfrac{2}{3})m \cdot BC + (\tfrac{1}{3})AB^2. \tag{17}$$

Al-Ṭūsī defines m algebraically by (17) and he then derives (16). The rest of his argument is based exclusively on (16) and the ideas of the present analysis.

I now investigate the possible relationships between al-Ṭūsī's definition of D and the derivative. We have $f'(m) = 3m^2 - 2ma - b = 2m(m - a) - (b - m^2) = 2\,CD \cdot DB - [\delta\alpha] = 0$ (cf. (16)). However, for $x = BE, f'(x) = 2CE \cdot BE - [\varepsilon\alpha]$, but this quantity does not occur in al-Ṭūsī's argument. This means that al-Ṭūsī does not find m by computing the derivative f' and by putting $f'(x)$ equal to zero. Therefore the concept of derivative is not implicit here.

To return to al-Ṭūsī's ideas, it is now clear that the original Eq. (5) has no solution if $c > f(m)$ and one solution, namely $x = m$, if $c = f(m)$.

Now let $c < f(m)$, write $x_1 = BE$, and put $y_1 = DE$, then $y_1 = x_1 - m$. We have $CD \cdot (BD + BE) - [\varepsilon\alpha] = CD \cdot DE + [\delta\varepsilon]$, and therefore by (12) $f(m) - c = f(BD) - f(BE) = DE \cdot (CD \cdot DE + [\delta\varepsilon]) = y_1((m - a)y_1 + y_1(2m + y_1)) = y_1^2(3m - a + y_1)$. Therefore $y = y_1$ is the (unique positive) root of $y^3 + y^2(3m - a) = f(m) - c$, that is (6).

Similarly, if we let $x_2 = BF$, and put $z_2 = FD$, then $z_2 = m - x_2$ and $f(m) - c = FD \cdot ([\delta\alpha] - CF \cdot (BF + BD)) = z_2 \cdot (CD \cdot FD + [\phi\delta]) = z_2^2(3m - a - z_2)$, and therefore $z = z_2$ is the unique positive root of $z^3 + f(m) - c = z^2(3m - a)$, that is (9), such that $z < m$.

These are the essential ideas in the solution of (5). The parts labeled A, B, D2, F, and G in Section 1 are lengthy elaborations of these ideas (see [T2, 104–127]).

The preceding reasoning answers the question: for which c does a root x exist? Al-Ṭūsī also studies the similar question: For which x does $c > 0$ exist; i.e., what x can be roots of an equation of type (5) for fixed a and b? Such x should satisfy $c = f(x) > 0$, that is to say $x^2 > ax + b$. The further details (in parts C and E in Section 1) are mathematically trivial.

This concludes the discussion of the case $a < b$, so suppose $BC = a > \sqrt{b} = AB$, as in [T2, 76–104], and let the notation be as in Fig. 2. Then

$$f(BD) = BC \cdot BD^2 + AB^2 \cdot BD - BD^3 = BC \cdot [B\delta] - BD \cdot [\alpha\delta], \tag{18}$$

and by a similar reasoning as above

$$f(BD) - f(BE) = DE \cdot ([\alpha\delta] - EC \cdot (BE + BD)) \tag{19}$$

and

$$f(BF) - f(BD) = FD \cdot ([\alpha\phi] - DC \cdot (BD + BF)). \tag{20}$$

We now wish to find D such that $f(BD)$ is maximal. Then for all E between D and C

$$[\alpha\delta] > EC \cdot (BE + BD) \tag{21}$$

and for all F between D and A

$$[\alpha\phi] < DC \cdot (BD + BF). \tag{22}$$

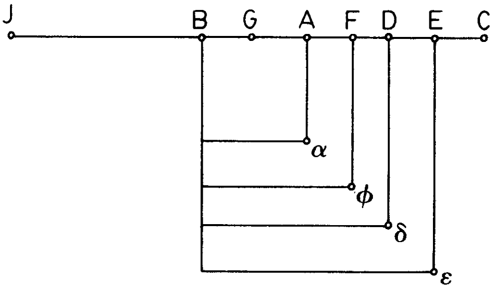

FIG. 2. $BC = a > \sqrt{b} = AB$.

First consider (21). The term $[\alpha\delta]$ does not depend on E. We now determine the maximum of $EC \cdot (BE + BD)$ for E a variable point between C and D. If we choose J on DB extended such that $BJ = BD$, then $EC \cdot (BE + BD) = EC \cdot JE$.

Suppose that the midpoint of segment JC lies between J and D. Then by Euclid, *Elements* II:6 [Heath 1956 I, 385] $EC \cdot JE < DC \cdot JD = DC \cdot 2DB$. Therefore (21) holds for all E between C and D if

$$2DC \cdot DB \leq [\alpha\delta]. \tag{23}$$

Note that if $2DC \cdot DB \leq [\alpha\delta]$, then $2DC \cdot DB < [B\delta]$, so that $DC < BD/2$, hence the midpoint of JC is in fact between D and B.

At first sight the analysis of (22) seems more complicated, because both terms increase monotonically if F tends to D. The difficulty disappears if we guess (with (23) in mind) that D should also be defined by (16), that is, $2DC \cdot DB = [\alpha\delta]$, and if we then consider the differences $[\alpha\delta] - [\alpha\phi] = [\phi\delta] = (BD + BF) \cdot FD$ and $2DC \cdot DB - DC \cdot (BD + BF) = DC \cdot FD$.

By (16), $DC < BD < BD + BF$, so that $DC \cdot FD < [\phi\delta]$, and (22) follows.

Thus if D is defined by (16), $f(BD)$ is maximal. Everything else is the same as in the case $a < \sqrt{b}$.

This concludes my analysis of al-Ṭūsī's solution of Eq. (5).

Al-Ṭūsī treats Eq. (2), that is $x^3 + c = bx$, and (3), that is $x^3 + ax^2 + c = bx$, in the same way as $x^3 + c = ax^2 + bx$, case $a < \sqrt{b}$. For (2), C coincides with B in Fig. 1, and in (3), C is chosen on AB extended such that $|BC| = a$.

The treatment of Eq. (4), that is $x^3 + bx + c = ax^2$, resembles that of $x^3 + c = ax^2 + bx$, case $a > \sqrt{b}$. For (4), al-Ṭūsī draws a segment $BA = \sqrt{b}$ perpendicular to BC (Fig. 3).

In (19) and (20) one obtains instead of gnomons $[\alpha\delta]$ and $[\alpha\phi]$ quantities $AB^2 + BD^2$ and $AB^2 + BF^2$, respectively (which al-Ṭūsī interprets geometrically as the squares of hypotenuses of right-angled triangles). Thus $f(BD) > f(BE)$ and $f(BD) > f(BF)$ are seen to be equivalent to

$$AB^2 + BD^2 > EC \cdot (BE + BD) \tag{24}$$

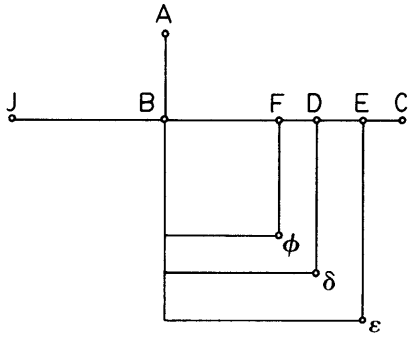

FIG. 3. $x^3 + bx + c = ax^2$. $AB = \sqrt{b}$, $BC = a$.

and

$$AB^2 + BF^2 < DC \cdot (BD + BF), \tag{25}$$

respectively.

The inequalities (24) and (25) can be investigated in similar ways as (21) and (22), leading to the result that $f(BD)$ is maximal if D is such that $AB^2 + BD^2 = 2DC \cdot DB$.

The equation (1), $x^3 + c = ax^2$, is treated in the same way as the case $a > \sqrt{b}$ of $x^3 + c = ax^2 + bx$, with A coinciding with B in Fig. 2. For (1), al-Ṭūsī derives the quadratic equation $q^2 + q(a - x_1) = x_1(a - x_1)$, where $x_1 > m$ and $x_2 < m$ are the two positive roots and $x_2 = a - x_1 + q$ (see Section 1, part D1) in the following manner. Referring to Fig. 2, put $x_1 = BE$, $x_2 = BF$, $a = BC$. From

$$c = ax_1^2 - x_1^3 = BE^2 \cdot CE = ax_2^2 - x_2^3 = BF^2 \cdot CF$$

we get, subtracting from $BE^2 \cdot CF$,

$$BE^2 \cdot EF = CF \cdot [\varepsilon\phi],$$

hence

$$BE^2 = CF \cdot (BE + BF) \tag{26}$$

hence

$$BF \cdot (BE + BF - CB) = BE \cdot CE. \tag{27}$$

In order to cast (27) in a nice geometrical form, al-Ṭūsī defines G on BC such that $BG = CE$. Then (27) can be written as

$$BF \cdot GF = BE \cdot CE. \tag{28}$$

If B, E, and C (and hence G) are known, the construction of F is a standard Euclidean problem: to apply to BG a rectangle, equal in area to $BE \cdot CE$, and exceeding by a square (GF^2). Or, in other words, $GF^2 + BG \cdot GF = BE \cdot CE$ (this is the equation used in [T2, 7]). The fact that al-Ṭūsī uses GF and not BF (in (27)) as the unknown shows that his method is basically geometrical.

The preceding summary contains the essence of the second part of the *Algebra*, with the exception of trivialities and the Ruffini–Horner process (see the next section). Al-Ṭūsī discusses each equation in such an elaborate way that his *Algebra* resembles the *Cutting-off of a Ratio* of Apollonius of Perga. Unlike Apollonius, al-Ṭūsī sometimes makes his proofs more complicated than necessary by introducing useless proportions. Suter also noted complications of this kind in another text of al-Ṭūsī [Suter 1907–1908]. My analysis does not take account of such complicating factors.

3. AL-ṬŪSĪ'S INITIAL MOTIVATION

In the preceding section we have seen that certain identities for a cubic polynomial f, such as $f(BD) - f(BE) = DE \cdot (CD \cdot (BD + BE) - [\varepsilon\alpha])$ (that is (12)), play a cardinal role in the reasoning of al-Ṭūsī. Clearly al-Ṭūsī discovered many of the results in the *Algebra,* such as (16) and (17), after he had found identities such as (12). Thus one wonders for what reasons al-Ṭūsī initially studied (12).

A possible reason may have been his search for geometrical proofs of numerical algorithms for the approximation of roots of cubic equations. A proof of this kind appears in [T2, 15–18], in connection with the approximation of the smallest positive root of Eq. (1), that is $x^3 + c = ax^2$.

The algorithm is essentially the method of Ruffini–Horner (see [Luckey 1948]). This method was used for the computation of cube roots before the middle of the third century A.D. in China [Wang and Needham 1955; Vogel 1968, 41–42, 113–119] and in the 10th century A.D. in the Islamic world [Kūshyār 1965, 26–28, 100–104]. The extraction of cube roots was apparently well known in the time of al-Ṭūsī, who does not even bother to explain the details [T1, 24]. The generalization to arbitrary cubic equations is straightforward (see [Luckey 1948, 220–221, 229–230]) and may have been used in the early 11th century A.D. by al-Bīrūnī for the computation of the roots of $x^3 + 1 = 3x$ and $x^3 = 1 + 3x$ [Schoy 1927, 19, 21]. In the first part of the *Algebra,* al-Ṭūsī describes the generalized algorithm for all cubic equations of the form $x^3 + rax + sbx = c$ with r and s equal to -1, 0, or 1, not both zero. In these cases al-Ṭūsī adds numerical examples and a verbal explanation of why the algorithm is correct. It seems that he felt more uncertain about (1), that is $x^3 + c = ax^2$, perhaps because a (positive) root does not always exist. This may have prompted him to develop the geometrical proof in [T2, 13–15], which will now be rendered in modern notation.

Suppose x_0 is the smallest positive root of (1). (We assume $c \leq (4/27)a^3$, so that x_0 exists.) Let $x_0 = n_1 \cdot 10^k + n_2 \cdot 10^{k-1} + \ldots$ be the decimal expression, with $n_1 \neq 0$. We can estimate k using $x_0 \approx \sqrt{(c/a)}$ (see [T2, 15] and [1] below). We then find by trial and error $x_1 = n_1 \cdot 10^k$ as the maximal number $X = n \cdot 10^k$ such that n is an integer and $aX^2 \leq X^3 + c$. We then compute the following quantities:

$$\begin{aligned} &a' = a - x_1, && a'' = a' - x_1, && a_1 = a'' - x_1 \\ &b' = x_1 a', && b_1 = b' + x_1 a'', && \\ &c_1 = c - x_1 b' && && \end{aligned}$$

(note that $x_0 = x_1 + y$ with $y(b_1 + y(a_1 - y)) = c_1$).

We now find by trial and error $y_1 = n_2 \cdot 10^{k-1}$ as the maximum number $Y = n \cdot 10^{k-1}$ such that n is an integer and $Y(b_1 + Y(a_1 - Y)) \leq c_1$.

We then compute

$$\begin{aligned} &a_1' = a_1 - y_1, && a_1'' = a_1' - y_1, && a_2 = a_1'' - y_1 \\ &b_1' = b_1 + y_1 a_1', && b_2 = b_1' + y_1 a_1'', && \\ &c_2 = c_1 - y_1 b_1 && && \end{aligned}$$

(note that $y = y_1 + z$ with $z(b_2 + z\,(a_2 - z)) = c_2$) and so on. With each step we find one further decimal of the root; the successive approximations of x_0 are x_1, $x_1 + y_1$, etc.

FIGURE 4

Al-Ṭūsī proves the correctness of this procedure in a somewhat obscure passage [T2, 15–18], which we paraphrase as follows (Fig. 4). The algebraical notation a, b, c, x_i, y_i, z and the symbols K, K_1 are mine. Let $AB = a$, $BD = x_0$, $BE = x_1$, $ED = y$. Then $c = ax_0^2 - x_0^3 = DA \cdot BD^2 = DA \cdot BE^2 + DA \cdot (BD^2 - BE^2) = EA \cdot BE^2 - ED \cdot BE^2 + DA \cdot (2\,BE \cdot ED + ED^2)$. Therefore

$$c_1 = c - (a - x_1)x_1^2 = DA \cdot BD^2 - EA \cdot BE^2 = ED \cdot K$$

with

$$\begin{aligned} K &= DA \cdot (2BE + ED) - BE^2 & (29)\\ &= 2(DA + ED) \cdot BE - 2\,ED \cdot BE + DA \cdot ED - BE^2 \\ &= 2\,EA \cdot BE + DA \cdot ED - BE^2 - 2ED \cdot BE \\ &= EA \cdot BE + (EA \cdot BE - BE^2) + DA \cdot ED - 2BE \cdot ED \\ &= EA \cdot BE + (EA - BE) \cdot BE + (EA - BE - BE - ED) \cdot ED. & (30) \end{aligned}$$

Thus $c_1 = y \cdot K$ with $K = b_1 + y(a_1 - y)$ as desired.

Similarly, let $y_1 = EI$, $z = ID$, $x_2 = x_1 + y_1 = BI$. The text is very concise, but the underlying line of thought seems to be as follows ([T2, 17 line 20–18 line 6]:

We have in the algorithm $c_2 = c_1 - y_1 \cdot (b_1 + y_1(a_1 - y_1))$, or geometrically $c_2 = c_1 - EI \cdot K_1$ with $K_1 = EA \cdot BE + (EA - BE) \cdot BE + (EA - BE - BE - EI) \cdot EI$ (cf. (30)). Hence, as above, $c_2 = c_1 - EI \cdot (IA \cdot (2BE + EI) - BE^2)$ (cf. (29)). Thus $c_2 = c_1 + EI \cdot BE^2 - IA \cdot (2EI \cdot BE + EI^2)$, as stated in the text. Therefore $c_2 = c_1 + EA \cdot BE^2 - IA \cdot BI^2 = c - IA \cdot BI^2 = c - x_2^2(a - x_2)$. It is also easily verified that $b_2 = x_2(a - x_2) + x_2(a - 2x_2)$ and $a_2 = a - 3x_2$.

We can now apply the proof of (30) to a_2, b_2, c_2, x_2, z instead of a_1, b_1, c_1, x_1, y. It follows that

$$c_2 = z(b_2 + z(a_2 - z))$$

as desired.

Differences such as $DA \cdot BD^2 - EA \cdot BE^2$ play an important role in this proof (cf. (29) and (30), or [T2, 16 line 5–17 line 19 (Arabic), 16 line 3–17 line 21 (French)]). Hence it is conceivable that al-Ṭūsī first studied the differences $f(BD) - f(BE)$ while he was searching for this proof, and possibly for similar proofs for Eqs. (2)–(5). In the beginning he may not have known that the roots of (2)–(5) can be found by solving (1) and $x^3 + ax^2 = c$. Anyhow, it would be natural for al-Ṭūsī to begin with (1), because the necessary and sufficient condition $c \leq (4/27)a^3$ for the existence of a root was known in his time. This condition had been derived geometrically by Archimedes, and it had been stated algebraically in the 10th century (see [Woepcke 1851, 96–103] = [Woepcke 1986 I, 168–175]). Note that it was important for al-Ṭūsī, who did not work with negative numbers, that the quantities $a_1 = a - 3x_1$, $a_2 = a - 3x_2$, etc., in the algorithm are all positive. This is only true if $x_0 \leq (\frac{1}{3})a$. For $x_0 > (\frac{1}{3})a$, one can use Fig. 4 for $BD = (\frac{2}{3})a$, $BE = x_0$, $ED = y$ to obtain $y^3 + [(\frac{4}{27})a^3 - c] = ay^2$ using methods which are even simpler than the proof of (30). Because $y \leq (\frac{1}{3})a$ one can now use the algorithm to compute y. Hence al-Ṭūsī may well have discovered the substitution $y = (\frac{2}{3})a - x$ ($z_2 = m - x_2$ in the notation of Section 2) in connection with his investigation of the proof of the algorithm for Eq. (1).

In conclusion, it seems to me that the *Algebra* of al-Ṭūsī can be explained as the result of a project that started with a more modest aim, namely the search for geometrical proofs of algorithms for approximating the roots of cubic equations. I believe that I have shown that al-Ṭūsī's motivation and ideas can be explained without the assumption that he drew cubic curves and determined their local maxima and minima by means of the method of P. de Fermat. And as we have seen in Section 2, there is no evidence that al-Ṭūsī used the derivative. The absence of traces of these concepts does not detract from the intrinsic value of al-Ṭūsī's work. On the contrary, al-Ṭūsī's ingenuity appears very clearly when one realizes that he used only traditional ancient and medieval mathematical methods.

4. NOTES TO THE TEXT AND TRANSLATION OF THE *ALGEBRA*

The following notes are intended for the reader who wishes to study the original text or the translation of the second part of al-Ṭūsī's *Algebra,* which has been analyzed in Sections 2 and 3 of this paper. I wish to stress here that the edition and translation in [T2] are in my opinion very good, and that my notes on details do not imply a qualification of this general judgment. This section contains notes to the Arabic text, followed by corresponding notes to the translation (not all notes to the text entail a change in the translation). A notation such as 98:2 refers to line 2 of page 98 of the Arabic text or the translation. In the transcription of the Arabic text I conform to the conventions in [T2]; thus letters denoting points in the geometrical figures are transcribed according to the system used in [T2] (therefore jīm = C, zāy = G, ṭāʾ = I), and angular brackets contain editorial additions to the Arabic text in the manuscripts. I also put the French translation of these words in angular brackets, even though such brackets do not appear in the translation.

Notes to the Arabic Text

1. 15:11 delete ⟨murabbaᶜ⟩.
2. 16:16 and 16:17 for *BE* read ⟨murabbaᶜ⟩ *BE.*
3. 17:22–18:1 ⟨*AI,* wa-ḍarabnā *EI* fī⟩: In view of the singular *mablagh* on 18:1 one should add here something like ⟨*AI,* wa-naqaṣnā al-mablagh min al-ᶜadad, wa-ḍarabnā *EI* fī⟩.
4. 30:5 for illā mālan read wa-illā mālan, and 30:7 for illā kaᶜban read wa-illā kaᶜban, as in the mss. (cf. the apparatus); illā functions as the minus sign. Compare 69:1–2 (wa-illā amwālan), 102:21–103:1 (wa-illā kaᶜban).
5. 35:11 delete ⟨murabbaᶜ⟩.

6. 38:11 for *EM* read *CM*. The reading in the footnote to 38:12 is preferable to the text in 38:12. The mathematical context requires that ka-dhālika in 38:13 be emended, for example to wa-dhālika.

7. 40:6 delete ⟨maᶜlūm⟩.

8. 40:17 for wa-⟨huwa⟩ mithl ḍiᶜf read wa-ḍiᶜf, the word mithl in the manuscript should be deleted from the text and put in the apparatus, because it is a scribal error.

9. 46:1, 2 for *BC* read *MC*.

10. 49:16 delete ⟨wa-quṭruhā *AB*⟩; the words ᶜalā *AB* indicate that *AB* is the diameter.

11. 51:11–12 for fa-lā yuᶜraḍu . . . li-l-istiḥāla read: fa-lā yaᶜriḍu . . . . al-istiḥāla (al-istiḥāla and li-l-istiḥāla are indistinguishable in the London manuscript). Delete the footnote to 51:11.

12. On p. 64 interchange yā and ṣād in the figure.

13. 67:3 for ⟨fī *BE*⟩ read ⟨fī *EG*⟩.

14. 73:16 if *DA* is emended to *BA*, the additions ⟨wa-huwa musāwin li-murabbaᶜ *AB*⟩ and ⟨*DK* wa-huwa⟩ can be omitted.

15. 74:16 the emendation must be incorrect, because the quantity in question does not in fact have a (positive) lower bound, as al-Ṭūsī proves in the subsequent passage (75:1–5). Perhaps li-bayān should be emended not to li-l-bayān ⟨lahu⟩, but to laysa lahu (the final nūn in the manuscript being a trace of lahu).

16. 77:5 for *BG* read *AB* as in the mss. (see the apparatus).

17. 78:7 delete ⟨wa-huwa⟩, and for wa-huwa read huwa.

18. 79:10 note that [maᶜa] is evidently a trace of ⟨murabbaᶜ⟩ in 79:11.

19. 84:21 fa-ḍarb: the fa- makes no sense here, and the text is much clearer if we emend wa-⟨huwa⟩ ḍarb; this takes care of the difficulty mentioned in the footnote to 85:1. In 85:3 delete ⟨huwa⟩ and for bi-mukaᶜ ᶜab read mukaᶜ ᶜab.

20. 85:10 emend *CO* to *DJ*, delete ⟨maḍrūban fī *OM*⟩, for li-kawn read lākin as in the mss. (see the apparatus). Note $CM = DJ$.

21. 94:5–6 delete ⟨*BE* . . . li-ḍilᶜ⟩, instead of the footnotes to 94:6 and 94:6–9 put: 94:6–9 *BD* ᶜalā mukaᶜ ᶜabihi . . . ḍilᶜ: nāqiṣa L.

22. 98:3 al-awwal: there is no need for this emendation, read al-thānī as in the mss. (see the apparatus).

23. 98:17 delete ⟨wa l-ashyāʾ wa l-māl⟩ (the gnomon is $AE(EB + BA)$), 98:19 delete ⟨ziyāda⟩.

24. 108:12 for *DC* read *GC*.

25. 109:1 for *BD* read *BG*.

26. 116:11 delete ⟨*DB* maᶜa *BE*⟩, 116:12 for *DC* read *EC* as in the mss. (see the apparatus), 116:15 for *DC* read *EM* (cf. the apparatus), 116:16 for *DC* read *EC* as in the mss. (see the apparatus). In 116:14–16 note $DC + EK = DC + EM - MK = EM$, because $KM = DC$ (cf. 110:12). The emendation ⟨alladhī⟩ in 116:13 is mathematically correct, but ⟨wa-murabbaᶜ *DE* fī *EK*⟩ is perhaps more plausible from a paleographical point of view.

Notes to the French Translation

1. For 15:12 *et il peut*–15:15 *inférieur* I suggest the following alternative translation *et il peut convenir qu'il n'ait pas d'écart* (*pour le premier chiffre*), *et que l'écart ait seulement lieu pour les autres chiffres cherchées* (*du quotient*). *Le premier chiffre de ce quotient* (*par AB*) *sera donc le* (*premier chiffre*) *exact* (*du quotient par AD*) *ou un nombre voisin qui lui est inférieur*. The following *il* refers to the *premier chiffre de ce quotient* (*par AB*). Footnote 47 on page 15 and footnote 1 on p. xix of the commentary are misleading. In 15:16 for *le carré du nombre* read *le nombre*.

2. 16:22 and 16:23 for *BE* read *le carré de BE*.

On p. 17 footnote 50 read *petit* for *grand;* thus the translation in 17:24 is correct.

3. 17:28 for ⟨*AI, et multiplié AI*⟩ read ⟨*AI, et soustrait ce produit du nombre et que nous ayons multiplié EI par*⟩. 18:1 for *ces produits* read *ce produit*.

5. 35:15 delete ⟨*le carré*⟩.

6. 38:15–16 for ⟨*par EM*⟩ read ⟨*par CM*⟩. For 38:17 *le reste sera donc* ⟨*la difference du*⟩ *premier* ⟨*solide*⟩ *et du deuxième. Aussi puisque* read: *le premier reste sera donc plus grand que l'autre* (*reste*). *Car puisque*.

7. 40:9 for *le nombre des carrés est* ⟨*connu*⟩ read *est le nombre des carrés*.

8. The translation 40:23 corresponds to the text as I have corrected it.

9. 46:3 for *BC* read *MC* (*MC* in 46:1 is correct).

10. 49:21 for *nous . . . et* read *nous construisons sur AB un demi-cercle de centre G, et*.

11. 51:17 for *le problème* read *tel problème*, the reference is to the quadratic equation in 51:16. Footnote 59 is misleading.

12. On p. 64 interchange *J* and *U* in Fig. 59.

13. 67:5 for *BE* read *EG*.

14. 73:20–22 for *le carré de DA . . . serait* read *le carré de BA ou* (*un quantité*) *plus grand que lui par trois fois AB ou* (*un quantité*) *plus grand que lui, serait*.

15. 74:19 the translation is based on an emendation which must be incorrect, because the *nombre cherchée* does not in fact have a positive *limite en petitesse*, as is proved subsequently in 75:1–7.

16. 77:5 delete *alors*, 77:6 for *BC seraient* read *AB sont*.

19. 84:23 for *par CD. Le produit* read *par CD, c'est-à-dire le produit*.

20. 85:11 for *par CO,* ⟨*multiplié par OM,*⟩ *du fait que* read *par DJ, mais*. Note $CM = DJ$.

21. 94:6–7 delete ⟨*BE* . . . côté⟩. 94:8 delete *ce qui*.

22. 98:4 for *premier* read *deuxième*.

23. 98:24 delete ⟨*les choses et le carré*⟩ (the gnomon is $AE(EB + BA)$), 98:27 for *qui reste de l'augmentation du cube* read *qui reste du cube*.

24. 108:16 for *DC* read *GC*.

25. 109:2 for *BD* read *BG*.

26. 116:13 delete (*DB plus BE*), 116:14 for *DC* read *EC*, 116:17 for *Mais* read *Donc*, 116:18 for *DC* read *EM*, 116:19 for *DC* read *EC*.

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NOTES

1. The following references are to the Arabic text of [T2]. The French translation has the same pagination as the Arabic text, but the line numbers may be different. A notation such as 3:8 refers to line 8 of page 3. (1) A 1:1–5:9, B 5:10–6:18, D1 and G 7:1–8:2 and 10:6–18:22, F 8:3–10:5. (2) A 19:1–23:14, B 27:1–28:20, D2 23:15–26:13, F 29:1–30:16, G 31:1–32:4, H 32:5–34:2. (3) A 34:3–40:4, B 40:5–41:22, D2 42:1–43:19, F 44:1–45:10, G 45:11–46:14, H 47:1–48:20. (4) A 49:1–58:4, B 58:4–60:7, D2 60:8–62:2, C and E 63:1–66:7, F 66:8–67:19, G 68:1–69:9, H 69:10–70:13. (5) case $a = \sqrt{b}$, A 70:17–72:9, B 72:9–73:11, C 73:12–73:19, D2 73:20–74:15, E 74:16–75:6, F 75:7–75:15, G 75:16–76:3, H 76:4–76:15. (5) case $a > \sqrt{b}$, A 76:16–84:4, B 84:5–89:8, C 89:9–90:12, D2 90:13–95:15, F 95:16–99:13, G 99:14–103:16, H 103:17–104:15. (5) case $a < \sqrt{b}$, A 104:16–110:10, B 110:10–114:12, C 114:13–115:16, D2 116:1–119:18, F 119:19–123:11, G 123:12–126:21, H 127:1 -127:17.

2. Compare [T2, 41 lines 11, 15–20; 43 lines 11, 15; 65 line 6]; al-Ṭūsī also uses variant expressions such as "the remainder which is together with *BE*" (al-baqīya alladhī maʿa *BE*) on [T2, 52 line 12].

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