# ON THE JUSTICE OF DECISION RULES*

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Abstract. Which decision rules are the most efficient? Which are the best in terms of maximin, or maximax? We study these questions for the case of a group of individuals faced with a choice from a set of alternatives. First, we show that the set of optimal decision rules is well-defined, particularly simple, and well-known: the class of scoring rules. Second, we provide the optimal decision rules for the three different ideals of justice under consideration: utilitarianism (efficiency), maximin, and maximax. We show that plurality, arguably the most widely used voting system, is optimal in terms of maximax, while the best way to achieve maximin is by means of negative voting, and the optimal utilitarian decision rule depends on the culture of the society. We then provide the mapping between cultures and optimal decision rules.

**Keywords:** Decision rules, Scoring Rules, Voting, Utilitarianism, Maximin, Maximax.
**JEL classification numbers:** D00, D63, D71, D72.

## 1. Introduction

Should I implement a plurality voting rule if I aim to maximize efficiency? If I am persuaded by the maximin ideal of justice, should I implement the Borda count? What should I use if it is maximax that I wish to optimize? These are the type of questions addressed in this paper. Our aim is to identify the best decision rules in terms of these three ideals of justice.

In our setting, individuals' preferences are cardinal and interpersonally comparable utility random variables. The realization of the individual utility values depends on the *culture* of the society, which is simply a probability distribution over the range of utility intensities. We consider three prominent ideals of justice defined in cardinal

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terms: utilitarianism (or efficiency), maximin, and maximax. In short, utilitarianism evaluates an alternative in terms of the average of the individual utility values. The maximin principle disregards the utility values of the most favored individuals to evaluate an alternative on the basis of the utility value of the worst-off agent. In contrast with maximin, the maximax rule focuses on the best-off individuals. Maximax as an ideal of justice may appear a mere formal curiosity. However, we will see that, because of its close connection to plurality voting, the maximax principle plays a more important role in democratic political institutions than might be expected.

In our model, $n$ individuals must choose an alternative from a set of $k \geq 3$ alternatives. Decision rules serve precisely this purpose. A decision rule transforms the ordinal preferences of the individuals into a selection of alternatives. We impose no restriction on the possible set of decision rules, other than assuming that it uses the actual ordinal preferences of the individuals.

The question arises of what the optimal decision rules are in terms of a particular cardinal ideal of justice. That is, we explore the connection between a given ideal of justice and different decision rules, with the aim of identifying those that best serve the pursued ideal of justice. There are different options for measuring decision rule performance. A natural index can be obtained by evaluating decision rules in terms of their *expected attainable value* for a given ideal of justice. Another index emerges by judging decision rules on the basis of the *probability of their selecting the best alternative* for a given theory of justice. Yet a third possibility is to measure the *probability of avoiding the worst alternatives*. The three criteria approach the same problem from different angles, and hence in principle complement each other. It could be argued that the first index adopts a risk-neutral approach to measure the success of decision rules, while the second and third indices could be understood as representative of risk-loving and risk-averse positions, respectively.

Firstly, we show that a particularly prominent class of decision rules emerges as optimal: the class of *scoring rules*. A scoring rule is a vector of fixed points that individuals assign to the different alternatives. The plurality scoring rule, the negative scoring rule and Borda's scoring rule are especially salient. This result significantly restricts the class of decision rules from which to seek the optimal ones. A further point is that scoring rules are relatively easy to implement and widely used.

Secondly, our results indicate that the message for each ideal of justice is robust across indices. That is, what matters is the nature of the ideal of justice, not the particular way of measuring decision rule performance.

Thirdly, we provide the optimal decision rules for the three ideals of justice. In utilitarian terms, we show that there is a mapping from the culture of the society to the optimal decision rule. Moreover, we find the exact shape of the optimal decision rule, conditional on the culture of the society. We then illustrate the result with a number of examples. In particular, we show that for a range of cultures satisfying a certain regularity condition, a prominent scoring rule emerges as optimal: *Borda*. When the culture follows the normal distribution, the optimal decision rule for utilitarianism is not Borda but a scoring rule where the differences in the values given to consecutive alternatives is a symmetric and strictly convex function. That is, with the normal distribution, the optimal scoring rule strongly discriminates between the very best alternatives and also between the very worst alternatives. Alternatives in between are treated as almost alike. We explore the connection between the optimal utilitarian decision rule and other prominent cultures, such as the exponential or the logistic distribution.

In contrast to utilitarianism, maximax offers a unique optimal decision rule for every possible culture. It proves to be a remarkably important decision rule: *plurality*. Although plurality is probably the most widely used voting system in the democratic world, our results show that, instead of maximizing efficiency or the well-being of the worst-off agents in a society, plurality *maximizes the well-being of the best-off*, a property that may be considered of questionable merit. Finally, maximin is best approached with the negative scoring rule. Given the link between plurality and maximax, it is not surprising that negative, the symmetric scoring rule to plurality, is the best decision rule in terms of maximin, which can be shown to be in a way symmetric to maximax. Interestingly, the results for the three ideals of justice point to the three best-known scoring rules: plurality, Borda, and negative.

We are certainly not the first to evaluate decision rules. There is a large and still growing literature examining decision rules in terms of the properties they satisfy. In other words, decision rules are judged on the basis of their capacity to meet certain desirable properties such as anonymity, independence of irrelevant alternatives, strategy-proofness, consistency of the social preference ordering, Pareto-dominance, path-independence, probability of selecting the Condorcet winner, etc.[1]

In this paper, we focus not on the specific properties that voting systems may or may not satisfy, but on how well they perform in terms of an ideal of justice. If there is agreement over which ideal of justice is to be pursued, it seems much more relevant to evaluate voting systems on the basis of how well they serve that ideal, than it is to calculate the probability of their electing the Condorcet winner, for example. In other words, in this paper we shift the decision rule evaluation criterion from an ordinal to a cardinal approach. The problem therefore reduces to inferring the relevant utility intensities from ordinal rankings.

There is very little work on evaluating decision rules on the basis of some cardinal theory of justice. Notable exceptions are the early simulation studies of Bordley (1983) and Merrill (1984), and the theoretical work of Weber (1978). Bordley and

[1]Early studies are Brams and Fishburn (1978), Caplin and Nalebuff (1988), DeMeyer and Plott (1970), and Nurmi (1983). See also Benoit and Kornhauser (2007), Dasgupta and Maskin (2008), Gehrlein (1997), Levin and Nalebuff (1995), Myerson (2002), and Saari (1999).

Merrill use simulations to analyze the efficiency of different voting systems, including plurality and Borda. Consistent with our results, they show that plurality is easily outperformed in utilitarian terms by other decision rules. Weber (1978) studies the performance of scoring rules for the case of utilitarianism and for cultures following the uniform distribution. He shows that, asymptotically, Borda is the best scoring rule in this case. This, again, is consistent with our results.

In another related strand of literature, there are papers that study how to select a voting rule in a constitutional setting where there are two options, the status quo and a second alternative, and individual preferences are uncertain. A voting rule is characterized by the number of votes needed to accept the second alternative over the status quo. The papers that comprise this literature examine issues such as which voting rules maximize efficiency, which are self-stable, how to weight votes in heterogeneous contexts, self-enforcement voting rules, etc. For examples in this vein, see Rae (1969), Barbera and Jackson (2004, 2006), Maggi and Morelli (2006) and papers cited therein.

Finally, there is a growing literature addressing the question of the transmission of utility intensities in collective decision problems (see Casella, 2005; Jackson and Sonnenschein, 2007; and Hortala-Vallve, 2007). The innovation of these papers is to consider a collective decision situation problem repeated over $T$ times and endow individuals with a maximum number of votes to allocate over the $T$ problems. By so doing, individuals may transmit utility intensities.

The remainder of the paper is organized as follows. Section 2 introduces in detail the setting in which we will be working. Sections 3, 4, and 5 study the cases of utilitarianism, maximax and maximin, respectively. Finally, Section 6 presents some conclusions. All the proofs are given in Appendix A.

## 2. Environment

Let $N$ be a finite set of individuals with cardinality $n \geq 2$ and $K$ the set of alternatives with cardinality $k \geq 3$. Typical elements of $N$ are denoted by $i$ and $j$ and those of $K$ are denoted by $l, h$ and $q$. We then say that a *society* is composed of $n$ individuals with preferences over $k$ alternatives.

To address the question of finding the best ordinal decision rule to approach each given ideal of justice, we first present the cardinal environment (utilities and ideals of justice), then the ordinal setting (decision rules and ordinal preferences), and then the one that connects the two worlds (indices evaluating the success of a decision rule in terms of an ideal of justice).

2.1. **Cardinal Preferences.** Individual preferences over the set of alternatives are cardinal, interpersonally comparable utility random variables in the $[0, 1]$ interval. $U_i^l$ is the random variable representing the cardinal utility of individual $i$ for alternative

$l$. We adopt the convention that subindices refer to individuals and superindices refer to alternatives. Let *culture* $C$ be the probability distribution from which the cardinal individual utility values are drawn. We consider cultures with continuous probability distributions on $[0, 1]$. Hence, the probability that $U_i^l$ takes a particular value on $[0, 1]$ is zero, and then the probability of ties (i.e., of individual $i$ being indifferent between two alternatives) is also zero.[2] Moreover, we assume that variables $\{U_i^l\}_{i\in N, l\in K}$ are iid. We say that a culture is symmetric whenever its density function is symmetric.

Given the random variables $U_i^1, U_i^2, \ldots, U_i^k$, the order statistics $U_i^{(1)} \leq U_i^{(2)} \leq \cdots \leq U_i^{(k)}$ are also random variables, defined by sorting the realizations of $U_i^{(1)} \leq U_i^{(2)} \leq \cdots \leq U_i^{(k)}$ in increasing order of magnitude. $U_i^{(l)}$ denotes the $l$-th order statistic of individual $i$, representing the utility value for individual $i$ of that alternative having $l-1$ alternatives with lower utility values. Analogously, we will be using order statistics given an alternative $l$. Consider the collection of random variables $U_1^l, U_2^l, \ldots, U_n^l$, and arrange them in order of magnitude by denoting $U_{(1)}^l \leq U_{(2)}^l \leq \cdots \leq U_{(n)}^l$. Here $U_{(i)}^l$ denotes the utility value of the $i$-th order statistic for alternative $l$, representing the utility value of $l$ for that individual who ranks $l$ higher than exactly $i-1$ individuals.

A Social Welfare Function SWF is a mapping $W$ from $[0, 1]^{n\times k}$ to $[0, 1]^k$, where $W^l(U^l) \in [0, 1]$ denotes the social value of alternative $l$, given the realization of random variables $U^l = (U_1^l, \ldots, U_n^l)$.[3] The three SWFs we consider here are utilitarianism, maximin, and maximax. *Utilitarianism* evaluates an alternative by taking the average of individual cardinal utilities. Formally, a SWF is utilitarian if $W = W_{UT}$ with $W_{UT}^l(U^l) = \sum_{i=1}^n U_i^l/n$. The *maximin* principle evaluates an alternative on the basis of the utility value of the worst-off agent, disregarding any other utility value. In other words, a SWF is of the maximin type if $W = W_{MN}$ with $W_{MN}^l(U^l) = U_{(1)}^l$. Consider also the *maximax* rule. In contrast to maximin, the maximax rule focuses on the best-off individuals. That is, a SWF is of the maximax type if $W = W_{MX}$ with $W_{MX}^l(U^l) = U_{(n)}^l$.

2.2. **Ordinal Preferences.** The following describes the ordinal rankings of the individuals over the set of alternatives. $M$ is an $n \times k$ matrix with properties: (1) $m_i^l \in \{1, \ldots, k\}$, and (2) $m_i^l \neq m_i^h$ for all $i \in N$ and for all $l, h \in K$, $l \neq h$. Entry $m_i^l$ denotes the position of alternative $l$ in the preferences of individual $i$, where the higher $m_i^l$ is, the higher alternative $l$ is ranked by individual $i$.[4] $M_i$ denotes the $i$-th row of matrix $M$. That is, $M_i$ represents the ordinal preferences of individual $i$. $M^l$ denotes the $l$-th column of matrix $M$, representing the ordinal preferences of all

[2]Given our assumption that cultures are *continuous* probability distributions on $[0, 1]$, all our results naturally extend to other ranges such as $(0, 1)$, $(-\infty, \infty)$, or $[0, \infty)$.

[3]Throughout the paper we use the terms "social welfare function" and "ideal of justice" indistinctly.

[4]Note that given the assumptions on continuous probability distributions, there are no ties, and hence property (2) above follows.

individuals with respect to alternative $l$. The collection of all possible matrices $M$ is denoted by $\mathcal{M}$. Finally, by $\mathbf{l}^{(h)}$ we refer to the number of individuals that place alternative $l$ exactly above $h-1$ alternatives. That is, $\mathbf{l}^{(h)} = |\{i \in N : m_i^l = h\}|$.

A multi-valued decision rule $f$ is a correspondence from $\mathcal{M}$ to $K$. Denote by $\mathcal{F}$ the set of all decision rules $f$. That is, a decision rule takes ordinal information on preferences as input to determine a set of alternatives as social choice. We assume that $f$ uses true information. That is, the particular matrix $M$ over which $f$ is applied is assumed to be known.

Scoring rules are a particularly interesting class of decision rules. They are typically simple to implement in practice and they encompass a number of widely-used decision rules. Formally, a scoring rule $S$ can be represented by a vector $S \in [0,1]^k$, with $S^l$ denoting the strength of an individual's vote for whichever alternative she ranks higher than exactly $l-1$ alternatives. We normalize strength of vote by assigning a value 1 to the best alternative, and a value 0 to the worst. That is, we set $S^k = 1$ and $S^1 = 0$. Note that this normalization of the extreme values is without loss of generality. We denote by $S(M) \subseteq K$ the set of alternatives selected by scoring rule $S$ with preferences $M$. An alternative $l$ is selected by scoring rule $S$ when $M$, if $l$ gets the highest number of points across individuals when ordinal preferences are $M$. The especially salient scoring rules are plurality, Borda, and negative. A scoring rule is *plurality* if $S = S_{Pl}$ with $S_{Pl}^l = 0$ for every $l < k$. It is *negative* if $S = S_{Ng}$ with $S_{Ng}^l = 1$ for every $l > 1$, and it is *Borda* if $S = S_{Bd}$ with $S_{Bd}^l = \frac{l-1}{k-1}$ for every $l$.

2.3. **Cardinal and Ordinal Preferences: Evaluating Decision Rules.** We will be using three different indices to evaluate the performance of alternatives in terms of the ideals of justice under consideration.

The first, which we denote by the $\alpha$-index, judges an alternative $l$ on the basis of the *expected value* that $l$ provides in terms of an ideal of justice $W$. That is,

$$\alpha_W^l(M) = \mathbb{E}[W^l(U^l) \mid M].$$

Recall that $W_{UT}^l(U^l) = \frac{\sum_{i=1}^n U_i^l}{n}$, $W_{MN}^l(U^l) = U_{(1)}^l$, and $W_{MX}^l(U^l) = U_{(n)}^l$. For the case of utilitarianism and given ordinal preferences $M$, alternative $l$ is better than alternative $q$ in terms of the $\alpha$-index, if the expected average utility value of $l$ is larger than the expected average value of $q$. Therefore, the aim is to find decision rules that select alternatives with the highest expected average utility value. Now consider maximin. Then, for a given $M$, alternative $l$ is better than alternative $q$ in terms of the $\alpha$-index, if $l$ provides a higher expected value of the lowest order statistic, than $q$. Finally, $l$ is better than $q$ for maximax in terms of the $\alpha$-index, if $l$ provides a higher expected value of the highest order statistic than $q$, given ordinal preferences $M$.

The $\beta$-index, on the other hand, measures for a given SWF $W$, the *probability of an alternative $l$ being optimal*:

$$\beta_W^l(M) = P(W^l(U^l) \geq W^h(U^h) \text{ for all } h \in K \setminus \{l\} \mid M).$$

Instead of looking to the expected value of an alternative, the $\beta$-index considers the probability of that alternative providing the highest social welfare value, given ordinal preferences $M$. Under utilitarianism, maximin, and maximax, we will be talking about the probability of selecting those alternatives that give the highest average utility value, the highest value among the first order statistics, and the highest value among the highest order statistics, respectively.

Finally, for a given SWF $W$ the $\gamma$-index evaluates alternatives in terms of the *probability of not choosing the worst alternative*:

$$\begin{aligned}\gamma_W^l(M) &= 1 - P(W^l(U^l) < W^h(U^h) \text{ for all } h \in K \setminus \{l\} \mid M) \\ &= P(W^l(U^l) \geq W^h(U^h) \text{ for some } h \in K \setminus \{l\} \mid M).\end{aligned}$$

The three indices are different in nature. It could be argued that the $\alpha$-index adopts a risk neutral approach to measure the success of decision rules. Indices $\beta$ and $\gamma$, however, could be understood as adopting risk-loving and risk-averse positions, respectively.

Indices and ideals of justice are intimately linked. There is a natural link between utilitarianism and the $\alpha$-index, between maximax and the $\beta$-index, and between maximin and the $\gamma$-index. It is in fact the case that these natural links translate into analytical tractability. We will see that utilitarianism is best approached analytically by the $\alpha$-index, while the $\beta$-index is poorly suited to the formal study of utilitarianism. With respect to maximax, in contrast, great analytical progress can be made with the $\beta$-index, but the tractability of maximax with regard to $\alpha$ becomes complex. For intuition, consider for instance the $\alpha$-index of an alternative $l$ in utilitarian terms. The value of the $\alpha$-index is the average of the expected value of the different order statistics that make up alternative $l$. In terms of maximax (and similarly for maximin), one has to obtain the expected value of the highest order statistic of $l$. This is a challenge. If there is an individual who ranks $l$ as her best alternative, then this individual is a good candidate to represent the highest order statistic of $l$. However, if for example there are many individuals who ranks $l$ as their second best alternative, then it is likely that the highest order statistic of $l$ is among the latter group of individuals. Thus, there is a complex dependent structure that forces consideration of the interrelation between the random utility variables of all the individuals in the society for all the alternatives.

An $X_W$-optimal decision rule, with $X_W = \{\alpha_W, \beta_W, \gamma_W\}$, is a decision rule $f$ that always selects alternatives with the largest $X$-index for ideal of justice $W$. That is,

$f$ is $X_W$-optimal decision rule if for all $M \in \mathcal{M}$:

$$l \in f(M) \Rightarrow X_W^l(M) \geq X_W^h(M) \text{ for all } h \in K.$$

It is straightforward to see that, given an $X_W$-optimal decision rule $f$, any other decision rule in the family $\delta(f) = \{g \in \mathcal{F} : \emptyset \neq g(M) \subseteq f(M) \text{ for all } M \in \mathcal{M}\}$ is also an $X_W$-optimal decision rule. In order to characterize the set of $X_W$-optimal decision rules, it is sufficient to identify the maximum $X_W$-optimal $f$. We will say that $f$ is the *maximum $X_W$-optimal decision rule* if whenever $f$ and $g$ are $X_W$-optimal, then it must be that $g \in \delta(f)$.

## 3. Utilitarianism

We first show that for every $n \times k$ society and every culture, the optimal decision rule to maximize the utilitarian expected value is a scoring rule. This is good news. It implies that if the interest is to maximize the expected value of utilitarianism, it is advisable to implement a scoring rule, which is a relatively simple decision rule.

Furthermore, we provide the exact form of the optimal scoring rule, conditional on the culture under consideration. More specifically, a culture determines the expected value of the order statistics, which in turn determines the optimal value of the scoring rule. Knowledge of the culture of the society provides information on the expected value of the order statistics, which can be used to design the optimal scoring rule. In particular, we show that for a range of cultures with symmetric density functions, a prominent scoring rule emerges as the maximum optimal: *Borda.* The intuition goes as follows. The jumps in strength that Borda assigns to consecutive order statistics are constant. Hence, cultures in which the expected value of the order statistics has this property are the most closely linked to Borda. Also, when the culture follows the normal distribution, we show that the optimal decision rule is a scoring rule where the differences in the values given to consecutive alternatives is a symmetric convex function. That is, under the normal distribution, it is optimal to discriminate between the very best alternatives and also between the very worst alternatives, and less important to discriminate between the intermediate alternatives. Interestingly, when the culture of the society follows a Cauchy distribution then the pattern becomes even more pronounced. In this case, the optimal scoring rule tends to discriminate between the very best alternative and the very worst alternative, treating all remaining alternatives in almost the same manner.

We then turn to the analysis of the $\beta$- and $\gamma$-indices. The $\beta$-index (as well as the $\gamma$) is less well suited than the $\alpha$-index to the analytical study of utilitarianism. The source of difficulty lies in the dependency across alternatives. To determine the probability of an alternative being the utilitarian alternative, it is necessary to take the whole society into account. This dependency drastically limits the analytical tractability of utilitarianism in terms of the $\beta$-index. For this reason, we begin with the formal study of a small society, a $2\times3$ society, for a complete range of cultures with symmetric density functions, and then we run a set of simulations to approach larger societies. Our analytical results for small societies and the computational analysis for larger societies provide analogous results as those obtained in the formal study of the $\alpha$-index. This is reassuring since it indicates that there is a great deal of consistency across indices. What matters is the social welfare function, not the particular way of measuring success. Finally, our analysis of the $\gamma$-index exploits a symmetry relation with the $\beta$-index, allowing us to immediately reach the same conclusions as those reached with the $\beta$-index.

We organize the study of utilitarianism by first giving the general results obtained with the $\alpha$-index, and we then turn to the study of the $\beta$- and $\gamma$-indices.

3.1. **The $\alpha_{UT}$-index: Evaluating Decision Rules in terms of Expected Value.** Theorem 3.1 below provides the main result for utilitarianism. It first identifies a scoring rule as the maximum optimal decision rule, and then it characterizes the exact shape of the scoring rule. Importantly, it does so for any $n \times k$ society and for any culture.

**Theorem 3.1.** *For $n \times k$ societies and for every culture, the maximum $\alpha_{UT}$-optimal rule is a scoring rule with $S^{*l} = \frac{\mathbb{E}[U^{(l)}]-\mathbb{E}[U^{(1)}]}{\mathbb{E}[U^{(k)}]-\mathbb{E}[U^{(1)}]}$, $l \in \{1, \ldots, k\}$.*

Theorem 3.1 shows that there is a mapping from the culture of the society to the optimal decision rule. A culture determines the first moment of the order statistics, and this in turn characterizes the optimal decision rule. This raises the question of what the optimal decision rules of especially prominent cultures are like. The following corollary represents a first illustration.

**Corollary 3.2.** *The maximum $\alpha_{UT}$-optimal rule is Borda:*

- *For $n \times k$ societies and for every culture such that $\mathbb{E}[U^{(l+1)}] - \mathbb{E}[U^{(l)}] = c$, with $1 \leq l \leq n-1$.*
- *In particular for the uniform distribution.*
- *For $n \times 3$ societies and for every culture with a symmetric density function.*

Corollary 3.2 points to a prominent decision rule, Borda, as the optimal rule for a wide range of cultures with density functions satisfying a regularity condition. But there are many other relevant cultures for which to obtain the optimal scoring rule. Consider the normal distribution, for example. The expected value of the order statistics from the normal distribution are tabularized (see, e.g., Teichroew, 1956). From inspection of these values, it immediately emerges that the $\alpha_{UT}$-optimal decision rule in this case is a scoring rule where $S^{*l+1} - S^{*l}$ is a symmetric and convex function. A more extreme version of the above emerges when considering the Cauchy distribution (see, e.g., Barnett (1966) for the expected values of the Cauchy order statistics). In the latter case, the optimal scoring rule tends to discriminate only between the very best alternative, the very worst alternative and all the rest.

We can further apply our Theorem 3.1 to other classes of probability distributions via known results in the literature of order statistics. By way of illustration consider the following class of probability distributions $F$ (see Kamps, 1991, 1992; see also Balakrishnan and Sultan 1998):

$$\frac{d}{dt}F^{-1}(t) = \frac{1}{d}t^p(1-t)^{q-p-1},$$

$t \in (0,1)$, where $p$ and $q$ are integers and $d > 0$. The values of parameters $p$ and $q$ determine the probability distribution. For example, if $p = q = 0$ we get the exponential distribution, if $p = q = -1$ we get the logistic distribution, if $p > -1$ and $q = p + 1$ we get the power function, etc. We can then establish the following result for this class of probability distributions.

**Theorem 3.3.** *For $n \times k$ societies and for the class of cultures $F$, the maximum $\alpha_{UT}$-optimal rule is a scoring rule with*

$$S^{*l} = \frac{\sum_{h=2}^{l} \mathbb{E}[U^{(h)}] - \mathbb{E}[U^{(h-1)}]}{\sum_{h=2}^{k} \mathbb{E}[U^{(h)}] - \mathbb{E}[U^{(h-1)}]},$$

$l \in \{1, \ldots, k\}$, *with* $\mathbb{E}[U^{(h)}] - \mathbb{E}[U^{(h-1)}] = \frac{\binom{k}{h-1}}{(h+p)\binom{k+q}{h+p}}$.

The introduction of the values of the parameters into the equations of Theorem 3.3 immediately gives the optimal scoring rule. By way of illustration, let $k = 3$. It is immediate that when there are three alternatives, the optimal scoring rule is well-defined by the value assigned to the middle alternative.

**Corollary 3.4.** *For $n \times 3$ societies and for the class of cultures $F$, the maximum $\alpha_{UT}$-optimal rule is a scoring rule with $S^{*2} = \frac{1-p+q}{3+q}$, where $p$ and $q$ are integers.*

The above corollary provides the optimal scoring rule contingent on the values of parameters $p$ and $q$, and hence contingent on the culture of the society.

3.2. **The $\beta_{UT}$-index: Evaluating Decision Rules in terms of the Probability of Selecting an Optimal Alternative.** Due to the analytical complexity of the study of utilitarianism through the lens of the $\beta_{UT}$-index, we begin with the formal study of a small society, a $2\times3$ society, for a complete range of cultures with symmetric density functions, and we then run a set of simulations to approach larger societies. In societies with $n = 2$ and $k = 3$ there are 5 different scoring rules, since every $S^2 \in (0, 1/2)$ represents the same scoring rule, and similarly every $S^2 \in (1/2, 1)$ also represents the same scoring rule.[5] Hence, the set of different scoring rules when $n = 2$ and $k = 3$ comprises plurality with $S^2_{Pl} = 0$, Borda with $S^2_{Bd} = 1/2$, negative with $S^2_{Ng} = 1$, an explosion rule with any value $S^2_{Ex} \in (1/2, 1)$, and an implosion rule with any value $S^2_{Im} \in (0, 1/2)$.

We study a collection of cultures with symmetric density functions. Take the parabolic function $f(x) = ax^2 + bx + c$, with $x \in [0, 1]$. Now, in order for $f(x)$ to represent a symmetric density function it must be the case that $a = -b = -(1-c)$, with $c$ taking values in $[0, 3]$. Note that the $c = 1$ case corresponds to the uniform distribution. Values of $c \in [0, 1)$ correspond to strictly concave density functions. That is, lower values of $c$ represent higher levels of consensus on the evaluation of an alternative. Finally, values of $c \in (1, 3]$ correspond to strictly convex density functions. Hence, higher values of $c$ represent higher levels of extremism.

**Theorem 3.5.** *For $2 \times 3$ societies and for any culture with a symmetric parabolic function:*

*(1) The maximum $\beta_{UT}$-optimal rule is the implosion scoring rule.*

*(2) The intersection between the alternatives selected by Borda and those selected by the maximum $\beta_{UT}$-optimal rule is always non-empty.*

The analysis of the $2 \times 3$ case shows that any scoring rule with $S^2 \in (0, 1/2)$ is $\beta_{UT}$-optimal. Moreover, it also shows that although Borda is not $\beta_{UT}$-optimal, it comes close to being so. For every type of society $M$, the alternatives selected by Borda and implosion coincide, except for one (which in the proof of Theorem 3.5 we call societies of type 4) where Borda randomizes between the 3 alternatives while implosion randomizes between 2 alternatives.

We extend the study of the $\beta_{UT}$-optimal decision rule by using computational methods. We generate random cardinal societies from three different probability distributions: the uniform distribution on the interval $[0, 1]$, the standard normal distribution on the interval $(-\infty, \infty)$, and the exponential distribution on the interval $[0, \infty)$. We study societies of $2, 3, 5, 10$ and $100$ individuals with preferences over $3, 4, 5$ and $7$ alternatives. The technical details are contained in Appendix B.

[5] We say that two scoring rules $S$ and $S'$ are the same if $S(M) = S'(M)$ for all $M$.

The main conclusion of the computational analysis is very clear. *The results across probability distributions and sizes of societies are fully in line with the analytical results we presented for the case of the $\alpha_{UT}$-index.* That is, for each of the three probability distributions and sizes of societies scrutinized, the scoring rule providing the highest $\beta_{UT}$-value is the corresponding one as stated in Theorem 3.1. In particular, for the uniform distribution the highest $\beta_{UT}$-value is given by Borda; for the normal distribution it is given by a scoring rule where $S^{*l+1} - S^{*l}$ is a symmetric and convex function; and finally for the exponential distribution, the highest $\beta_{UT}$-value is given by a scoring rule where $S^{*l+1} - S^{*l}$ grows exponentially.[6]

3.3. **The $\gamma_{UT}$-index: Evaluating Decision Rules in terms of the Probability of Not Choosing the Worst Alternative.** It is not difficult to appreciate that for utilitarianism and symmetry of the culture $C$, the $\gamma$-index is symmetric to the $\beta$-index in the following sense: take a society $\{U_i^l\}_{i \in N, l \in K}$, and let $l$ be an alternative giving the highest utilitarian value. Write the following symmetric society $\{U_i'^l\}_{i \in N, l \in K}$ with $U_i'^l = 1 - U_{\sigma(i)}^l$ where $\sigma(i) = n - i + 1$. It follows that alternative $l$ in society $\{U_i'^l\}_{i \in N, l \in K}$ is one of the worst alternatives in utilitarian terms. Then, the analytical and simulation results that we obtained for the $\beta$-index and for symmetric cultures can be extrapolated to the case of the $\gamma$-index. To complete the picture, we run an extra set of simulations for the exponential distribution, an asymmetric distribution, that points to, once again, that the particular index to measure performance is not essential (see Appendix B). We may conclude, therefore, that at least for the cases computationally scrutinized, the optimal decision rules in terms of $\gamma_{UT}$ are well captured in Theorem 3.1.

## 4. Maximax

We start the analysis of maximax by showing formally that for any $n \times k$ society and for any culture, the maximum optimal decision rule in terms of the *probability of selecting the maximax alternatives* is a scoring rule. In fact, it is a particularly simple and widely used scoring rule: *plurality.* The intuition is very simple. Maximax evaluates an alternative $l$ in terms of that individual that most values alternative $l$. It is very likely that such an individual ranks $l$ as her most preferred alternative. Consequently, those alternatives that are considered by the highest number of individuals as best are the most likely maximax alternatives. Plurality selects precisely these alternatives. The analytical and computational study of the two other indices reinforce the connection between plurality and maximax.

[6]For the sake of comparison, Appendix B reports the $\beta_{UT}$-values of the optimal scoring rules as predicted by Theorem 3.1, together with the three prominent scoring rules–plurality, Borda, and negative–for each of the societies and cultures under scrutiny.

We start the analysis of the maximax case by first giving the general results reached with the $\beta$-index and then we turn to the study of the $\alpha$ and $\gamma$-indices.

4.1. **The $\beta_{MX}$-index: Evaluating Decision Rules in terms of the Probability of Selecting an Optimal Alternative.** Theorem 4.1 below shows that something as simple as the plurality decision rule offers the highest probability of selecting the maximax alternative. Theorem 4.1 does not assume any particular probability distribution over the individual utility values, or any society size.

**Theorem 4.1.** *For $n \times k$ societies and for any culture, the maximum $\beta_{MX}$-optimal rule is the plurality scoring rule.*

Corollary 4.2 reinforces the role of plurality in the maximax ideal of justice. It shows that, as the society grows, plurality is the unique optimal scoring rule.

**Corollary 4.2.** *For any given $k$ and for any culture, when $n$ tends to infinity plurality is the only scoring rule in the set of $\beta_{MX}$-optimal scoring rules.*

Theorem 4.1 and Corollary 4.2 give a very clear message: if the interest is in maximax, then the decision rule that should be implemented is as simple as plurality, irrespective of the culture and size of the society.

4.2. **The $\alpha_{MX}$-index: Evaluating Decision Rules in terms of Expected Value.** We start the analysis of maximax in terms of the $\alpha$-index by studying the contribution made to the maximax expected value of an alternative $l$ by adding one extra best order statistic, as opposed to the value added by a second best order statistic, when the culture follows the uniform distribution. We explore the extreme case where everybody ranks alternative $l$ as the best. It is in this case that the extra expected maximax value of adding one best alternative is intuitively the smallest. We show that, even in this case, the contribution made to the expected maximax value by adding one extra best order statistic is of a higher order of magnitude than that made by adding a second best order statistic. Then, plurality is the clear candidate to be the optimal maximax decision rule in terms of the $\alpha$-index, since it gives weight only to the best random variables.

Consider the $n \times k$ society $M$ such that $m_i^l = k$ for every $i \in \{1, \ldots, n\}$. That is, all $n$ individuals in society $M$ rank alternative $l$ as the best. Consider also the following two $(n+1) \times k$ societies, $M'$ and $\overline{M}$, such that $m_i'^l = k$ for every $i \in \{1, \ldots, n, n+1\}$, and $\overline{m}_i^l = k$ for every $i \in \{1, \ldots, n, n+1\} \setminus \{j\}$ and $\overline{m}_j^l = k - 1$. In $M'$, all $n + 1$

individuals rank $l$ as the best alternative, while in $\overline{M}$, there are $n$ individuals who rank $l$ as best, and one that ranks $l$ as second best.

Now, taking the $n \times k$ society $M$ as the baseline, we ask what is the marginal contribution to the $l$-th maximax expected value of adding one extra individual who ranks $l$ as best (society $M'$). Also, we wonder about the marginal contribution of adding one extra individual who ranks $l$ as second-best (society $\overline{M}$). Denote these marginal contributions by $\Delta_n^{(k)}$ and $\Delta_n^{(k-1)}$, where $\Delta_n^{(k)} = \mathbb{E}[U^l_{(n+1)}|M'^l] - \mathbb{E}[U^l_{(n)}|M^l]$ and $\Delta_n^{(k-1)} = \mathbb{E}[U^l_{(n+1)}|\overline{M}^l] - \mathbb{E}[U^l_{(n)}|M^l]$. The ratio $\frac{\Delta_n^{(k)}}{\Delta_n^{(k-1)}}$ represents the ratio of the marginal contributions of an extra order statistic in the top positions $k$ and $k-1$.

Lemma 4.3 below shows that, for the uniform distribution, when $\mathbf{l}^{(k)} = n$, the additional expected value provided by one extra order statistic in the same position $k$ is considerably higher than that provided by one extra order statistic in position $k-1$.

**Lemma 4.3.** *For $n \times k$ societies and for the uniform distribution, $\frac{\Delta_n^{(k)}}{\Delta_n^{(k-1)}} = \frac{k(n+1)}{k-1}$.*

Intuitively the situation where everybody ranks $l$ as best is the worst-case scenario for the expected marginal contribution of adding one extra individual who rank $l$ the best alternative. Even in this case, Lemma 4.3 shows that the order of magnitude in the expected maximax value of an extra best random variable is considerably higher than that of an extra second best random variable. This clearly points to plurality as the candidate for the optimal maximax decision rule. Plurality gives weight only to best random variables, and hence selects alternatives with the highest number of best random variables.

In the context of $n$ individuals and 3 alternatives, and for the uniform distribution, Proposition 4.4 below first provides the exact formula for the computation of the maximax expected value of any alternative $l$. Then, it shows that only under extreme circumstances does plurality prove not to be the $\alpha_{MX}$-optimal decision rule.

**Proposition 4.4.** *For $n \times 3$ societies and for the uniform distribution,*

(1) $\mathbb{E}[U^l_{(n)} \mid M] = \sum_{a=0}^{\mathbf{l}^{(2)}} \sum_{b=0}^{\mathbf{l}^{(1)}} \sum_{c=0}^{3b} \binom{\mathbf{l}^{(2)}}{a}\binom{\mathbf{l}^{(1)}}{b}\binom{3b}{c}(-1)^{a+b+c}3^{\mathbf{l}^{(2)}-a}2^a \frac{3\mathbf{l}^{(3)}+2\mathbf{l}^{(2)}+a+c}{3\mathbf{l}^{(3)}+2\mathbf{l}^{(2)}+a+c+1}$

(2) *Let $h$ be an $\alpha_{MX}$-optimal alternative and let $p \in S_{Pl}(M)\setminus\{h\}$. Then, $M$ must satisfy that either*

(a) $\mathbf{h}^{(3)} = \mathbf{p}^{(3)}$ *and* $\mathbf{h}^{(2)} \geq \mathbf{p}^{(2)}$, *or*

(b) $\mathbf{h}^{(3)} = \mathbf{p}^{(3)} - 1$ *and* $\mathbf{h}^{(2)} > \frac{n-1}{2}$

Proposition 4.4 shows that plurality is very difficult to beat in terms of its capacity to optimize the maximax expected value. The plurality alternative can be surpassed by another alternative in terms of $\alpha_{MX}$ if about the same people rank it as the best alternative and (many) more people rank it as second-best. Clearly, these are low-probability events, and hence plurality is very likely to be the $\alpha_{MX}$-optimal decision rule.

In accordance with the above, the computational studies for the uniform, normal, and exponential distributions show that, for every single size of society scrutinized, plurality or a scoring rule in the neighborhood of plurality are always the $\alpha_{MX}$-optimal decision rule among the set of scoring rules evaluated (see Appendix B for details).

4.3. **The $\gamma_{MX}$-index: Evaluating Decision Rules in terms of the Probability of Not Choosing the Worst Alternative.** We start with the characterization of a small society.

**Theorem 4.5.** *For $2 \times 3$ societies and for any culture with a symmetric parabolic function:*

*(1) Implosion is a $\gamma_{MX}$-optimal decision rule.*

*(2) The intersection between the alternatives selected by plurality and those selected by the maximum $\gamma_{MX}$-optimal decision rule is always non-empty.*

Note that implosion is not the *maximum* $\gamma_{MX}$-optimal decision rule since, in societies where the two individuals share the same ranking over the alternatives, implosion selects the single best alternative, instead of the two better alternatives.

However, it is not difficult to see that, for general $n \times k$ societies, plurality is the natural candidate to select $\gamma_{MX}$-optimal alternatives. Below, in Proposition 4.6, we show that, for any $n \times k$ society and for any culture, whenever there is a set of alternatives Pareto-dominating some other alternatives, any decision rule selecting this set of Pareto-dominating alternatives is optimal in the sense of maximizing the probability of avoiding the selection of worst maximax alternatives. Clearly, plurality alternatives tend to be Pareto-dominating, and hence plurality voting is the most obvious candidate to be the optimal decision rule in terms of the $\gamma_{MX}$-index.

We write $M^l > M^h$ whenever $m_i^l > m_i^h$ for every $i$. That is, every individual $i$ ranks alternative $l$ higher than alternative $h$. This is equivalent to Pareto-dominance. Then, consider the following set $P(M) = \{l \in K$ such that there exists $h \in K$ with $M^l > M^h\}$.

**Proposition 4.6.** *For $n \times k$ societies and for any culture, whenever the set $P(M)$ is non-empty, $P(M)$ is the maximum $\gamma_{MX}$-optimal rule.*

Finally, the simulations also generally select plurality as the best decision rule, across society sizes and cultures (see Appendix B for a discussion).

## 5. Maximin

For the analysis of the maximin case we exploit a symmetry between maximin and maximax. This allows us to directly extrapolate results that we obtained from the maximax case to the maximin case. In particular, the results point to *negative*, the symmetric scoring rule of plurality, as the obvious candidate to be the optimal decision rule for maximin in terms of the three indices of consistency, $\alpha$, $\beta$, and $\gamma$.

Denote by $\mathbf{1}$ an $n \times k$ matrix with all entries equal to 1. Then, consider the following Theorem.

**Theorem 5.1.** *For $n \times k$ societies, symmetric cultures, every $M \in \mathcal{M}$, and $l \in K$:*

(1) $\alpha^l_{MN}(M) = 1 - \alpha^l_{MX}((k+1)\mathbf{1} - M)$,
(2) $\beta^l_{MN}(M) = 1 - \gamma^l_{MX}((k+1)\mathbf{1} - M)$, *and*
(3) $\gamma^l_{MN}(M) = 1 - \beta^l_{MX}((k+1)\mathbf{1} - M)$.

Theorem 5.1 allows the extrapolation of all analytical and simulations results obtained in the previous section for maximax that involve symmetric cultures to the case of maximin. This means that negative is the decision rule best suited to optimize maximin. For asymmetric cultures, Theorem 5.1 does not apply. To test the robustness of our theoretical findings when the culture is asymmetric, we estimate the value of the three indices for the case of the exponential distribution. Table 3 in Appendix B reports the results. Once more, it can be appreciated that negative generally emerges as the optimal scoring rule in the computational analysis (see Appendix B for a more detailed account of the results of the simulations).

## 6. Conclusions

This paper explores the relation between ideals of justice and decision rules. Whereas ideals of justice are typically presented in cardinal terms, decision rules are primarily constructed on the basis of ordinal information. We study the cardinal consequences of using ordinal-based decision rules.

We have shown that the optimal choice of decision rules depends on the criterion of justice that one wishes to follow. Among our specific findings we emphasize that our results identify a particularly prominent set of decision rules as optimal: the set of scoring rules. Moreover, the three most prominent scoring rules, plurality, Borda, and negative, are optimal for the three different ideals of justice. Maximax is best approached by plurality for any possible culture, maximin is best approached by negative, and, for a range of cultures, utilitarianism is best approached by Borda.

Moreover, in the latter case of utilitarianism we provide the mapping between cultures and optimal decision rules.

## Appendix A. Proofs

***Proof of Theorem 3.1.*** Recall that $\alpha^l_{UT}(M) = \mathbb{E}[\frac{\sum_i U^l_i}{n} \mid M]$. The independence of the random variables $U^l_i$ across individuals allows us to write $\alpha^l_{UT}(M) = \sum_i \mathbb{E}[U^l_i \mid M_i]/n$. In addition, since, for every individual $i$, all the random variables are identically distributed, we can write $\alpha^l_{UT}(M) = \sum_i \mathbb{E}[U^l_i \mid M^l_i]/n$. Finally, since $U^l_1, \ldots, U^l_n$ are also identically distributed, we can write the last expression in terms of the number of order statistics. First, note that, since $U^{(h)}_i$ and $U^{(h)}_j$ are identically distributed, we may simply write $U^{(h)}$. Then, $\alpha^l_{UT}(M) = \frac{\sum_h \mathbf{l}^{(h)}\mathbb{E}[U^{(h)}]}{n}$.

The value of the difference between two alternatives $l$ and $q$ is $\alpha^l_{UT}(M) - \alpha^q_{UT}(M) = \frac{\sum_h \mathbf{l}^{(h)}\mathbb{E}[U^{(h)}]}{n} - \frac{\sum_h \mathbf{q}^{(h)}\mathbb{E}[U^{(h)}]}{n} = \frac{\sum_h (\mathbf{l}^{(h)} - \mathbf{q}^{(h)})\mathbb{E}[U^{(h)}]}{n}$. Note that $\sum_h \mathbf{l}^{(h)} = \sum_h \mathbf{q}^{(h)} = n$, and therefore $\frac{\sum_h \mathbb{E}[U^{(1)}](\mathbf{l}^{(h)} - \mathbf{q}^{(h)})}{n} = 0$. We then write $\alpha^l_{UT}(M) - \alpha^q_{UT}(M) = \frac{\sum_h (\mathbf{l}^{(h)} - \mathbf{q}^{(h)})\mathbb{E}[U^{(h)}]}{n} - \frac{\sum_h \mathbb{E}[U^{(1)}](\mathbf{l}^{(h)} - \mathbf{q}^{(h)})}{n}$ and hence $\alpha^l_{UT}(M) - \alpha^q_{UT}(M) = \frac{\sum_h (\mathbf{l}^{(h)} - \mathbf{q}^{(h)})(\mathbb{E}[U^{(h)}] - \mathbb{E}[U^{(1)}])}{n}$. Note that $\mathbb{E}[U^{(k)}] - \mathbb{E}[U^{(1)}] > 0$. Then, multiplying and dividing by $\mathbb{E}[U^{(k)}] - \mathbb{E}[U^{(1)}]$, we have

$$\begin{aligned}
\alpha^l_{UT}(M) - \alpha^q_{UT}(M) &= \frac{\mathbb{E}[U^{(k)}] - \mathbb{E}[U^{(1)}]}{n} \frac{\sum_h (\mathbf{l}^{(h)} - \mathbf{q}^{(h)})(\mathbb{E}[U^{(h)}] - \mathbb{E}[U^{(1)}])}{\mathbb{E}[U^{(k)}] - \mathbb{E}[U^{(1)}]} \\
&= \frac{\mathbb{E}[U^{(k)}] - \mathbb{E}[U^{(1)}]}{n} (\sum_h [\mathbf{l}^{(h)} - \mathbf{q}^{(h)}] S^{*h}).
\end{aligned}$$

Since $\mathbb{E}[U^{(k)}] - \mathbb{E}[U^{(1)}] > 0$, it follows that, for every $M$, $S^*$ gives exactly the same ranking over the alternatives as the one given by utilitarianism. Hence, the two claims of the theorem follow. □

***Proof of Corollary 3.2.*** The first two points in the corollary follow directly from Theorem 3.1. To see the third claim, consider that a well known fact in order statistics theory is that $\sum_l \mathbb{E}[U^{(l)}] = k\mathbb{E}[U^h]$ for any alternative $h$. Consequently, since the symmetry of the density function implies that $\mathbb{E}[U^h] = \mathbb{E}[U^{(2)}] = \frac{1}{2}$, it follows immediately that $\mathbb{E}[U^{(l+1)}] - \mathbb{E}[U^{(l)}] = c$ with $1 \leq l \leq 2$. Then, from Theorem 3.1, Borda is the $\alpha_{UT}$-optimal decision rule. □

***Proof of Theorem 3.5.*** We report the proof for the uniform distribution. The corresponding proof for any other parabolic function is analogous, and hence is omitted. We organize the proof by types of societies in ordinal terms.

Type 1: Perfect correlation. This is the case when $M_1 = M_2$. Let, w.l.o.g., $M_1 = (3, 2, 1)$. Utilitarianism always selects alternative 1. Hence, $\beta_{UT}(M) = (1, 0, 0)$.

Type 2: Only best-correlation. Both individuals agree on the best alternative, but disagree on the others. Let, w.l.o.g., $M_1 = (3, 2, 1)$ and $M_2 = (3, 1, 2)$. Utilitarianism selects alternative 1, and then $\beta_{UT}(M) = (1, 0, 0)$.

Type 3: Only worst-correlation. Both individuals coincide in the worst alternative, but disagree on the others. Let, w.l.o.g., $M_1 = (3, 2, 1)$ and $M_2 = (2, 3, 1)$. It is straightforward to see that $\beta_{UT}(M) = (1/2, 1/2, 0)$.

Type 4: Only middle-correlation. Both individuals agree on the middle alternative, but disagree on the others.[7] Let, w.l.o.g., $M_1 = (3, 2, 1)$ and $M_2 = (1, 2, 3)$. To compute the selection of utilitarianism we first need to derive the joint probability distribution associated to the continuous random variables $U_i^{(1)}$, $U_i^{(2)}$ and $U_i^{(3)}$.

The joint probability density function is

$$f(u_i^{(1)}, u_i^{(2)}, u_i^{(3)}) = \begin{cases} 6 & \text{if } 1 \geq u_i^{(3)} \geq u_i^{(2)} \geq u_i^{(1)} \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the density function for the two agents is

$$f(u_i^{(1)}, u_i^{(2)}, u_i^{(3)}, u_j^{(1)}, u_j^{(2)}, u_j^{(3)}) = \begin{cases} 36 & \text{if } 1 \geq u_i^{(3)} \geq u_i^{(2)} \geq u_i^{(1)} \geq 0, \text{ and} \\ & 1 \geq u_j^{(3)} \geq u_j^{(2)} \geq u_j^{(1)} \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We can now calculate the probability of utilitarianism resulting in the selection of alternative 2. Alternative 2 will be the utilitarian winner whenever the sum of $U_i^{(2)}$ and $U_j^{(2)}$ is greater than $U_i^{(1)} + U_j^{(3)}$ and $U_i^{(3)} + U_j^{(1)}$. Equivalently,

(A.1)
$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbf{H_a}} \int_{\mathbf{H_b}} f(u_i^{(1)}, u_i^{(2)}, u_i^{(3)}, u_j^{(1)}, u_j^{(2)}, u_j^{(3)}) du_i^{(3)} du_j^{(3)} du_i^{(2)} du_j^{(2)} du_i^{(1)} du_j^{(1)},$$

where $H_a$ and $H_b$ are the sets $H_a = \{t : t \leq u_i^{(2)} + u_j^{(2)} - u_i^{(1)}\}$ and $H_b = \{t : t \leq u_i^{(2)} + u_j^{(2)} - u_j^{(1)}\}$. Then, (A.1) can be expressed as

(A.2)
$$\int_0^1 \int_0^1 \int_{u_j^{(1)}}^1 \int_{u_i^{(1)}}^1 \int_{u_j^{(2)}}^{\min\{1, u_i^{(2)} + u_j^{(2)} - u_i^{(1)}\}} \int_{u_i^{(2)}}^{\min\{1, u_i^{(2)} + u_j^{(2)} - u_j^{(1)}\}} 36 du_i^{(3)} du_j^{(3)} du_i^{(2)} du_j^{(2)} du_i^{(1)} du_j^{(1)}.$$

It can be shown that the above is .3 (for any parabolic distribution, the value of the integral is strictly lower than 1/3). Clearly, the utilitarian choice is, with equal probability, either alternative 1 or 3. Then, $\beta_{UT}(M) = (.35, .3, .35)$ (for the remaining parabolic functions, $\beta_{UT}^1(M) = \beta_{UT}^3(M) > \beta_{UT}^2(M)$.

[7] This type of societies has attracted a good deal of attention. Interestingly, Börgers and Postl (2007) study whether for this class of societies there are incentive-compatible rules which elicit utilities and implement efficient decisions.

Type 5. No correlation. Individuals do not agree on the ranking of any alternative. Let, w.l.o.g., $M_1 = (3,2,1)$ and $M_1 = (2,1,3)$. Hence, alternative 1 dominates alternative 2, and consequently the utilitarian choice will never be alternative 2. We now show that alternative 1 has a higher probability of being the utilitarian choice than alternative 3. For any society where alternative 3 is better in utilitarian terms, a society can be injectively constructed such that: (i) it is of type 5, and (ii) the utilitarian choice is alternative 1. Simply write $u_1^{'1} = u_2^3$, $u_1^{'2} = u_2^1$, $u_1^{'3} = u_2^2$, $u_2^{'1} = u_1^2$, $u_2^{'2} = u_1^3$, and $u_2^{'3} = u_1^1$.[8] Clearly, the constructed society is of type 5, and to see that the utilitarian choice is alternative 1, notice that $u_1^{'1} + u_2^{'1} = u_2^3 + u_1^2 > u_2^3 + u_1^3 > u_1^1 + u_2^1 > u_1^1 + u_2^2 = u_2^{'3} + u_1^{'3}$. Then, $\beta_{UT}^1(M) > \beta_{UT}^3(M) > \beta_{UT}^2(M) = 0$.

Taking into consideration all five cases together, implosion is the $\beta_{UT}$-optimal generator. This proves the first part of the theorem. The second part follows immediately from the above analysis.□

***Proof of Theorem 4.1.*** First note that only alternatives ranked best by some agents can provide the highest maximax value. In fact, due to the i.i.d. nature of utility realizations across individuals, for all $M \in \mathcal{M}$ and for all $h, l \in K$

$$\beta_{MX}^l(M) \geq \beta_{MX}^h(M) \Leftrightarrow \mathbf{l}^{(k)} \geq \mathbf{h}^{(k)}.$$

It is immediate that every decision rule $f$ in $\delta(S_{Pl})$ is a $\beta_{MX}$-optimal decision rule since it selects precisely that subset of the alternatives that are ranked best by the largest number of individuals. Furthermore, for every $f \notin \delta(S_{Pl})$ there exists an $M \in \mathcal{M}$ and an alternative $h \in K$ such that $h \in f(M) \setminus S_{Pl}(M)$. Consequently, there is an $l \in K$ such that $\mathbf{l}^{(k)} > \mathbf{h}^{(k)}$ and hence $f$ is not a $\beta_{MX}$-optimal decision rule.□

***Proof of Corollary 4.2.*** We prove the corollary by showing that, for any given $K$, when $n$ tends to infinity, plurality is the only scoring rule in $\delta(S_{Pl})$. Take a scoring rule $f$ different from plurality. We now find a real number $n_f$ such that for every positive integer $n$ with $n > n_f$, there exists a profile $M \in \mathcal{M}$ with $f(M) \nsubseteq S_{Pl}(M)$.

Given that $f$ is not plurality, $S^{k-1} > 0$. Take $n_f = 2(\frac{1}{S^{k-1}} - 1)$, and $n > n_f$. If $n$ is odd, consider the following matrix $M$: $M_i = (k, k-1, k-2, \ldots, 2, 1)$ for all $1 \leq i \leq \frac{n+1}{2}$, and $M_j = (1, k, k-1, \ldots, 3, 2)$ for all $\frac{n+3}{2} \leq j \leq n$. Plurality selects alternative 1. Scoring rule $f$ assigns to alternative 1 a total score of $\frac{n+1}{2}$, while the total score of alternative 2 is $\frac{n-1}{2} + \frac{n+1}{2}S^{k-1}$. We now show that $\frac{n-1}{2} + \frac{n+1}{2}S^{k-1} > \frac{n+1}{2}$, or equivalently $\frac{n+1}{2}S^{k-1} > 1$. Our original assumption that $n > n_f = 2(\frac{1}{S^{k-1}} - 1)$ guarantees the latter inequality.

If $n$ is even, consider the matrix $M$ with $M_i = (k, k-1, k-2, \ldots, 2, 1)$ for all $1 \leq i \leq \frac{n}{2}$, and $M_j = (1, k, k-1, \ldots, 3, 2)$ for all $\frac{n+2}{2} \leq j \leq n-1$ and $M_n =$

[8] Note, that under i.i.d., this argument is distribution free.

$(1, k-1, k-2, \ldots, 2, k)$ . An argument analogous to the one above shows that plurality selects alternative 1 while $f$ selects alternative 2.

Then, we have shown that, for $n > n_f$, decision rule $f$ is not a $\beta_{MX}$-optimal scoring rule, which proves the result.□

***Proof of Lemma 4.3.*** The distribution function of a uniform $k$ order statistic is $F^{(k)}(t) = t^k$ and of a $(k-1)$ order statistic is $F^{(k-1)}(t) = t^k + kt^{k-1}(1-t)$. Now, the distribution function of $U^l_{(n)}$ when $M^l$ such that $m^l_i = k$ for every $i \in \{1, \ldots, n\}$, is $F(t) = t^{nk}$. Then, $\mathbb{E}[U^l_{(n)} \mid M] = \int_0^1 t \frac{dF(t)}{dt} dt = \frac{nk}{nk+1}$. Analogously, $\mathbb{E}[U^l_{(n+1)} \mid M'] = \frac{(n+1)k}{(n+1)k+1}$, where $m'^l_i = k$ for every $i \in \{1, \ldots, n, n+1\}$. Finally, the distribution function of $U^l_{(n+1)}$ when $\overline{M}^l$ with $\overline{m}^l_i = k$ for every $i \in \{1, \ldots, n, n+1\} \setminus \{j\}$ and $\overline{m}^l_j = k-1$, is $F(t) = t^{nk}(t^k + kt^{k-1}(1-t))$. Then, $\mathbb{E}[U^l_{(n+1)} \mid \overline{M}] = \int_0^1 t \frac{dF(t)}{dt} dt = \frac{k(n^2k+2nk+k-1)}{(nk+k)(nk+k+1)}$. Consequently, $\mathbb{E}[U^l_{(n+1)} | M'^l] - \mathbb{E}[U^l_{(n)} | M^l] = \frac{k}{(nk+1)(nk+k+1)}$ and $\Delta_n^{(k-1)} = \mathbb{E}[U^l_{(n+1)} | \overline{M}^l] - \mathbb{E}[U^l_{(n)} | M^l] = \frac{k(k-1)}{(nk+1)(nk+k)(nk+k+1)}$. Then, the result follows.□

***Proof of Proposition 4.4.*** We start by proving point (1). For every individual $i$, the distribution functions of her best, medium and worst random variables are $F^{(3)}(t) = t^3$, $F^{(2)}(t) = t^3 + 3t^2(1-t)$, and $F^{(1)}(t) = 1 - (1-t)^3$, respectively. The distribution function of the maximum of $\mathbf{l}^{(3)}, \mathbf{l}^{(2)}$ and $n - \mathbf{l}^{(3)} - \mathbf{l}^{(2)} = \mathbf{l}^{(1)}$ best, medium and worst random variables is

$$F(t) = (t^3)^{\mathbf{l}^{(3)}} (t^3 + 3t^2(1-t))^{\mathbf{l}^{(2)}} (1 - (1-t)^3)^{\mathbf{l}^{(1)}}.$$

The polynomial decomposition of the latter expression gives

$$F(t) = \sum_{a=0}^{\mathbf{l}^{(2)}} \sum_{b=0}^{\mathbf{l}^{(1)}} \sum_{c=0}^{3b} \binom{\mathbf{l}^{(2)}}{a} \binom{\mathbf{l}^{(1)}}{b} \binom{3b}{c} (-1)^{a+b+c} 3^{\mathbf{l}^{(2)}-a} 2^a t^{3\mathbf{l}^{(3)}+2\mathbf{l}^{(2)}+a+c}.$$

The density function is

$$f(t) = \sum_{a=0}^{\mathbf{l}^{(2)}} \sum_{b=0}^{\mathbf{l}^{(1)}} \sum_{c=0}^{3b} \binom{\mathbf{l}^{(2)}}{a} \binom{\mathbf{l}^{(1)}}{b} \binom{3b}{c} (-1)^{a+b+c} 3^{\mathbf{l}^{(2)}-a} 2^a (3\mathbf{l}^{(3)}+2\mathbf{l}^{(2)}+a+c) t^{3\mathbf{l}^{(3)}+2\mathbf{l}^{(2)}+a+c-1}.$$

Finally, the expected value is

$$\mathbb{E}[U^l_{(n)} | M] = \sum_{a=0}^{\mathbf{l}^{(2)}} \sum_{b=0}^{\mathbf{l}^{(1)}} \sum_{c=0}^{3b} \binom{\mathbf{l}^{(2)}}{a} \binom{\mathbf{l}^{(1)}}{b} \binom{3b}{c} (-1)^{a+b+c} 3^{\mathbf{l}^{(2)}-a} 2^a \frac{3\mathbf{l}^{(3)} + 2\mathbf{l}^{(2)} + a + c}{3\mathbf{l}^{(3)} + 2\mathbf{l}^{(2)} + a + c + 1}.$$

We start the proof of part (2) of the theorem by stating the following lemma.

**Lemma A.1.** *Let $\{x_a\}_{a \in T \cup \{b\}}$, $b \notin T \neq \emptyset$ be a set of independent random variables. Then $\mathbb{E}[\max\{x_a\}_{a \in T \cup \{b\}}] - \mathbb{E}[\max\{x_a\}_{a \in T}] \leq \mathbb{E}[\max\{x_a\}_{a \in S \cup \{b\}}] - \mathbb{E}[\max\{x_a\}_{a \in S}]$ for all $S \subseteq T$.*

The lemma is straightforward, and hence we omit its proof. Since $p \in S_{Pl}(M)$ it must be that $\mathbf{p}^{(3)} \geq \mathbf{h}^{(3)}$. Let $\mathbf{h}^{(3)} < \mathbf{p}^{(3)} - 1$. Vector $(x, y, z)$ represents the case where $x = \mathbf{p}^{(3)}$, $y = \mathbf{p}^{(2)}$, and $z = \mathbf{p}^{(1)}$. Then, it is immediate that $\mathbb{E}[U^p_{(n)} \mid M] \geq \mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)]$.[9] Since $\mathbf{h}^{(3)} < \mathbf{p}^{(3)} - 1$ it is immediate that $\mathbb{E}[U^h_{(n)} \mid M] \leq \mathbb{E}[(\mathbf{p}^{(3)} - 2, n - \mathbf{p}^{(3)} + 2, 0)]$. By Lemma A.1 $\mathbb{E}[(\mathbf{p}^{(3)} - 2, n - \mathbf{p}^{(3)} + 2, 0)] \leq \mathbb{E}[(\mathbf{p}^{(3)} - 2, 0, 0)] + (n - \mathbf{p}^{(3)} + 2)\Delta^{(2)}_{\mathbf{p}^{(3)}-2}$. We now prove that $\mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)] > \mathbb{E}[(\mathbf{p}^{(3)} - 2, n - \mathbf{p}^{(3)} + 2, 0)]$. By showing the latter, we are excluding the possibility of $h$ being an optimal alternative whenever $\mathbf{h}^{(3)} < \mathbf{p}^{(3)} - 1$.

Let us assume by way of contradiction that $\mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)] \leq \mathbb{E}[(\mathbf{p}^{(3)} - 2, n - \mathbf{p}^{(3)} + 2, 0)]$. Note $\mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)] = \mathbb{E}[(\mathbf{p}^{(3)} - 2, 0, 0)] + \Delta^{(3)}_{\mathbf{p}^{(3)}-2} + \Delta^{(3)}_{\mathbf{p}^{(3)}-1}$. Then it must be that $\Delta^{(3)}_{\mathbf{p}^{(3)}-2} + \Delta^{(3)}_{\mathbf{p}^{(3)}-1} \leq (n - \mathbf{p}^{(3)} + 2)\Delta^{(2)}_{\mathbf{p}^{(3)}-2}$. The latter leads to $n \geq \frac{(3\mathbf{p}^{(3)}-3)(3\mathbf{p}^{(3)}-2)}{(3\mathbf{p}^{(3)}+1)} + \mathbf{p}^{(3)} - 2$. Now, since $p$ is a plurality alternative, $n \leq 3\mathbf{p}^{(3)} - 2$. Then it must be that $3\mathbf{p}^{(3)} - 2 \geq \frac{(3\mathbf{p}^{(3)}-3)(3\mathbf{p}^{(3)}-2)}{(3\mathbf{p}^{(3)}+1)} + \mathbf{p}^{(3)} - 2$. This can only be the case if $\mathbf{p}^{(3)} \leq 5$. For all $2 \leq \mathbf{p}^{(3)} \leq 5$, and for all $\mathbf{p}^{(3)} \leq n \leq 3\mathbf{p}^{(3)} - 2$ the direct evaluation of the expression in point (1) of the proposition shows that $\mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)] > \mathbb{E}[(\mathbf{p}^{(3)} - 2, n - \mathbf{p}^{(3)} + 2, 0)]$, hence, a contradiction arises, and then it must be that $\mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)] > \mathbb{E}[(\mathbf{p}^{(3)} - 2, n - \mathbf{p}^{(3)} + 2, 0)]$ for all $n$ and all $p$ defined as in the proposition, as desired.

The latter shows that it can only be either $\mathbf{h}^{(3)} = \mathbf{p}^{(3)}$ or $\mathbf{h}^{(3)} = \mathbf{p}^{(3)} - 1$. In the first case it is immediate that, for $h$ to be an optimal alternative, then it can only be that $\mathbf{h}^{(2)} \geq \mathbf{p}^{(2)}$. For the second case, note that $\mathbb{E}[U^h_{(n)} \mid M] = \mathbb{E}[(\mathbf{p}^{(3)} - 1, \mathbf{h}^{(2)}, n - \mathbf{p}^{(3)} + 1 - \mathbf{h}^{(2)})]$. Now, by Lemma A.1 $\mathbb{E}[(\mathbf{p}^{(3)} - 1, \mathbf{h}^{(2)}, n - \mathbf{p}^{(3)} + 1 - \mathbf{h}^{(2)})] \leq \mathbb{E}[(\mathbf{p}^{(3)} - 1, 0, 0)] + \mathbf{h}^{(2)}\Delta^{(2)}_{\mathbf{p}^{(3)}-1} + (n - \mathbf{p}^{(3)} + 1 - \mathbf{h}^{(2)})\Delta^{(1)}_{\mathbf{p}^{(3)}-1}$. Now, since $p$ is the plurality alternative, it must be that $n \leq 3\mathbf{p}^{(3)} - 1$. Hence, $(n - \mathbf{p}^{(3)} + 1 - \mathbf{h}^{(2)}) \leq 2\mathbf{p}^{(3)} - \mathbf{h}^{(2)}$. For all $\mathbf{p}^{(3)} \geq 1$, $2\mathbf{p}^{(3)} - \mathbf{h}^{(2)} \leq 3\mathbf{p}^{(3)} - 1$. It can be checked that $\frac{\Delta^{(2)}_{\mathbf{p}^{(3)}-1}}{\Delta^{(1)}_{\mathbf{p}^{(3)}-1}} = 3\mathbf{p}^{(3)} - 1$. Hence, $(n - \mathbf{p}^{(3)} + 1 - \mathbf{h}^{(2)})\Delta^{(1)}_{\mathbf{p}^{(3)}-1} \leq \Delta^{(2)}_{\mathbf{p}^{(3)}-1}$. Then, $\mathbb{E}[(\mathbf{p}^{(3)} - 1, \mathbf{h}^{(2)}, n - \mathbf{p}^{(3)} + 1 - \mathbf{h}^{(2)})] \leq \mathbb{E}[(\mathbf{p}^{(3)} - 1, 0, 0)] + (\mathbf{h}^{(2)} + 1)\Delta^{(2)}_{\mathbf{p}^{(3)}-1}$. For alternative $p$, $\mathbb{E}[U^p_{(n)} \mid M] \geq \mathbb{E}[(\mathbf{p}^{(3)}, 0, 0)] = \mathbb{E}[(\mathbf{p}^{(3)} - 1, 0, 0)] + \Delta^{(3)}_{\mathbf{p}^{(3)}-1}$. Hence, for alternative $h$ to be an $\alpha_{MX}$-optimal alternative, it must be that $(\mathbf{h}^{(2)} + 1)\Delta^{(2)}_{\mathbf{p}^{(3)}-1} \geq \Delta^{(3)}_{\mathbf{p}^{(3)}-1}$. The latter implies that $\mathbf{h}^{(2)} \geq \frac{3\mathbf{p}^{(3)}}{2} - 1$ and hence $\mathbf{h}^{(2)} \geq \frac{n-1}{2}$, as desired.$\square$

***Proof of Proposition 4.6.*** We start by noticing that, for every $l \in P(M)$, index $\gamma^l_{MX} = 1$. This follows trivially from the fact that, for every $l \in P(M)$, there is an $h \in K$ (possibly a different $h$ for every $l$) such that $M^l > M^h$ and hence $U^l_{(n)} > U^h_{(n)}$.

We now show that for every $l \in K \setminus P(M)$, index $\gamma^l_{MX} < 1$. Suppose, w.l.o.g. that $l = 1$. Since $1 \notin P(M)$ for every $h > 1$ there is an agent $i_h$ such that $m^1_{i_h} < m^h_{i_h}$.

[9] Note that vector $(\mathbf{p}^{(3)}, 0, 0)$ may not be feasible in $M$.

Denote all these agents by $I = \cup_{h\in K\setminus\{1\}}\{i_h\}$. $P(U^1_{(n)} < U^h_{(n)}$ for all $h > 1 \mid M) \geq P(U^1_{(n)} = U^1_j$ for any $j \in I \mid M) \cdot P(U^2_{(n)} = U^2_{i_2} \mid M) \cdot \ldots \cdot P(U^k_{(n)} = U^k_{i_k} \mid M)$. Clearly, the latter is strictly above 0, and the result follows.□

***Proof of Theorem 4.5.*** Take the structure of the proof of Theorem 3.5. Here, too, we provide the proof for the uniform distribution. The values of $\gamma_{MX}(M)$ for any other parabolic function are exactly the ones provided below.

Then, the values of $\gamma_{MX}(M)$ for the different types of societies are:

Type 1: It is straightforward that $\gamma_{MX}(M) = (1, 1, 0)$.

Type 2: It is straightforward that $\gamma_{MX}(M) = (1, 1/2, 1/2)$.

Type 3: It is straightforward that $\gamma_{MX}(M) = (1, 1, 0)$.

Type 4: It is easy to appreciate that alternative 2 can never be the maximax alternative, and hence $\gamma_{MX}(M) = (1, 0, 1)$.

Type 5: It is straightforward that $1 = \gamma^1_{MX}(M) > \gamma^3_{MX}(M) > \gamma^2_{MX}(M)$.

Finally, taking into consideration all five cases together, explosion is an optimal decision rule. The second part of the Theorem follows trivially from the above analysis. □

***Proof of Theorem 5.1.*** (1) Recall that we denote the distribution function of the $l$-th order statistic by $F^{(l)}(t)$. First note that $P(U^l_{(1)} \leq t|M) = 1 - P(U^l_{(1)} > t|M) = 1 - (1 - F^{(k)}(t))^{\mathbf{1}^{(k)}} \cdot \ldots \cdot (1 - F^{(1)}(t))^{\mathbf{1}^{(1)}}$. Given the symmetry of the culture, it follows that, for every $l \in K$, $F^{(l)}(t) = 1 - F^{(k-l+1)}(1 - t)$. Therefore, $P(U^l_{(1)} \leq t|M) = 1 - F^{(1)}(1 - t)^{\mathbf{1}^{(k)}} \cdot \ldots \cdot F^{(k)}(1 - t)^{\mathbf{1}^{(1)}}$. The latter expression is $1 - P(U^l_{(n)} \leq t|(k + 1)\mathbf{1} - M)$. It follows that $\mathbb{E}[U^l_{(1)} \mid M] = \mathbb{E}[U^l_{(n)} \mid (k + 1)\mathbf{1} - M]$, and hence $\alpha^l_{MN}(M) = 1 - \alpha^l_{MX}((k + 1)\mathbf{1} - M)$.

(2) Given $M$, $\beta^l_{MN}(M)$ is the probability that alternative $l$ is the best maximin alternative. That is, $\beta^l_{MN}(M) = P(U^l_{(1)} > U^h_{(1)}$ for all $h \in K \setminus \{l\} \mid M)$. Equivalently, $\beta^l_{MN}(M) = 1 - P($There exists $h \in K \setminus \{l\}$ such that $U^l_{(1)} < U^h_{(1)} \mid M)$. For every symmetric culture we then have that $\beta^l_{MN}(M) = 1 - P($There exists $h \in K \setminus \{l\}$ such that $U^l_{(n)} > U^h_{(n)} \mid (k + 1)\mathbf{1} - M)$. The latter expression is equivalent to $1 - \gamma^l_{MX}((k + 1)\mathbf{1} - M)$.

(3) Given $M$, $\gamma^l_{MN}(M)$ is the probability that alternative $l$ is not the worst maximin alternative. That is, $\gamma^l_{MN}(M) = 1 - P(U^l_{(1)} < U^h_{(1)}$ for all $h \in K \setminus \{l\} \mid M)$. Equivalently, $\gamma^l_{MN}(M) = P($There exists $h \in K \setminus \{l\}$ such that $U^l_{(1)} > U^h_{(1)} \mid M)$. For every symmetric culture we then have that $\gamma^l_{MN}(M) = P($There exists $h \in K \setminus \{l\}$ such that $U^l_{(n)} < U^h_{(n)} \mid (k + 1)\mathbf{1} - M)$. The latter expression is equivalent to $1-P(U^l_{(n)} > U^h_{(n)}$ for all $h \in K\setminus\{l\} \mid (k+1)\mathbf{1}-M) = 1-\beta^l_{MX}((k+1)\mathbf{1}-M)$.□

## Appendix B. Computational Analysis[10]

In this section we provide the technical details of the computational analysis, and report some further results.

We study societies of size $n \times k$, with $n \times k \in \{2, 3, 5, 10, 100\} \times \{3, 4, 5, 7\}$, and three different cultures. First, we consider societies where the utility values of the individuals are drawn independently from a *uniform distribution* on the interval $[0, 1]$. Second, we consider the standard normal distribution in the interval $(-\infty, \infty)$. Finally, to study societies with cultures having asymmetric density functions, we study the case where the utility values are drawn independently from an *exponential distribution* with $\lambda = 1$.

For every single size of society and culture, we independently draw $3,841,600$ cardinal societies. Then, for every combination of size of society and culture, we check the performance of a set of scoring rules across all the $3,841,600$ simulated cardinal societies. The set of scoring rules is defined as follows. We randomly generate 47 scoring rules by independently drawing values from a uniform distribution on the interval $[0, 1]$. Apart from this 47 randomly-generated scoring rules, our analysis always includes the three most prominent ones: *plurality*, *Borda*, and *negative*. Finally, for the case of the analysis of utilitarianism, we also include the optimal scoring rules as predicted in Theorem 3.1.

Since the $\beta$ and $\gamma$ indices measure probabilities of success, we use interval estimation methods for binomial distributions. Our sample size provides a confidence interval of length .001 around the value of the corresponding index,[11] assuming that the variance of the value of the index is the highest possible for a binomial distribution (i.e., .25). This is a conservative method to compare significant differences across scoring rules.[12] For the $\alpha-$index, we estimate both the value of the index and its variance in order to construct confidence intervals that depend on the estimated variance.[13]

The simulations complement the theoretical results obtained in the text by studying the following cases:

- Utilitarianism: $\beta_{UT}$ for the uniform, normal and exponential distributions, and $\gamma_{UT}$ for the exponential distribution.

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[10]We are grateful to the Advanced Computing Center for Research and Education at Vanderbilt University for providing the necessary computing resources.

[11]To determine that the difference between two values is significantly different than zero, this method provides a confidence interval of length $.001 \times 2^{1/2}$ for such difference.

[12]The use of this method might provide unnecessarily wide intervals; however, it allows us to find significant differences across scoring rules without having to estimate their covariance.

[13]Given our sample size, with the exponential distribution and the normal distribution the length of these intervals varies between .002 and .004 for the case of maximax and between .0002 and .002 for the case of maximin. With the uniform distribution, the length of the interval varies between .00001 and .0005. Note that the differences across distributions are due to their different ranges.

- Maximax: $\alpha_{MX}$ for the uniform (with $k > 3$), normal and exponential distributions, and $\gamma_{MX}$ for the exponential distribution.
- Maximin: $\beta_{MN}$ for the uniform, normal and exponential distributions, $\alpha_{MN}$ for the exponential distribution, and $\gamma_{MN}$ for the exponential distribution.

Table 1 reports the $\beta_{UT}$ and $\gamma_{UT}$ values of the optimal scoring rules according to Theorem 3.1. Moreover, for the sake of comparison, the table also includes the $\beta_{UT}$ and $\gamma_{UT}$-values that the three most prominent scoring rules, *plurality*, *Borda* and *negative*, attain in the simulations. Accordingly, Tables 2 and 3 report the corresponding values attained by *plurality*, *Borda* and *negative*.

As shown in Tables 1, 2 and 3, for each of the three ideals of justice, each probability distributions, and each performance index, the scoring rule that provides the highest value is generally the one predicted in the theoretical results presented in the text. This means that the scoring rule determined by Theorem 3.1 is the best in the case of utilitarianism, *plurality* is the best in the case of maximax, and *negative* is the best in the case of maximin, across probability distributions. There are only some exceptions for small society sizes, in which the frequency of ties in the number of points sometimes causes distortions.

More specifically, for utilitarianism the scoring rule predicted in Theorem 3.1 provides a higher value than not only *plurality* and *negative* (and *Borda*, when *Borda* is not the $S^*$) but also than the best randomly-generated scoring rule. In addition, we find that $S^*$ cannot be rejected as the best-performing scoring rule.[14]

For maximax, *plurality* has a significantly higher expected value and a significantly higher probability of not selecting the worst alternative than *Borda* and *negative*, as predicted. There are only a few exceptions for small societies, wherein the frequency of ties causes distortions. In addition, the best randomly-generated scoring rule is in a neighborhood of *plurality*. Moreover, as the size of the society increases (i.e., as the probability of ties decreases) the values of *plurality* and of the best randomly-generated scoring rule converge. In particular, for societies of 100 individuals we find that *plurality* is significantly better than the best randomly-generated scoring rule.

For maximin, when the number of individuals is larger than the number of alternatives, the results are analogous to the case of maximax. That is, *negative* has a significantly higher expected value, a significantly higher probability of selecting the best alternative, and a significantly higher probability of not selecting the worst alternative than *Borda* and *plurality*. Furthermore, the best randomly-generated scoring rule is in a neighborhood around *negative*. In contrast, notice that for societies with $n$ below or around $k$, it is likely than more that one alternative is not the worst for any of the individuals. In such cases, *negative* chooses randomly between these alternatives, which is not necessarily optimal and is the cause of some distortions. As the size of

[14]We find a few exceptions in small societies due to the frequency of ties.

the society increases, this effect vanishes. In particular, for societies of 100 individuals we find that *negative* is significantly better than the best randomly-generated scoring rule.

Table 1. Utilitarianism: Values of the $\beta_{UT}$ and $\gamma_{UT}$ obtained from the simulations analysis†

| | $\beta_{UT}$: | | | | | | | | | | | | $\gamma_{UT}$: | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | uniform | | | | normal | | | | exponential | | | | exponential | | | |
| Society | $S^*$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S^*$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S^*$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S^*$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ |
| $2 \times 3$ | .74204 | .64186 | .73928 | .65034 | .73416 | .64396 | .73416 | .64017 | .71313 | .65771 | .69450 | .57401 | .95382 | .92609 | .94449 | .92597 |
| $2 \times 4$ | .74209 | .57213 | .74209 | .42070 | .73562 | .58160 | .72691 | .44061 | .69153 | .60629 | .65990 | .37427 | .98928 | .97653 | .98960 | .92981 |
| $2 \times 5$ | .74761 | .52147 | .74761 | .30598 | .73552 | .53927 | .72065 | .30058 | .67637 | .57242 | .63473 | .27581 | .99783 | .99231 | .99832 | .93476 |
| $2 \times 7$ | .76053 | .45010 | .76053 | .19414 | .73243 | .48280 | .71278 | .19162 | .65902 | .52975 | .60122 | .18032 | .99981 | .99912 | .99997 | .94396 |
| $3 \times 3$ | .77371 | .70532 | .77371 | .63016 | .76684 | .70159 | .76684 | .62365 | .72969 | .69497 | .72037 | .55357 | .94861 | .90968 | .94978 | .87633 |
| $3 \times 4$ | .75332 | .60882 | .75332 | .49281 | .74518 | .61296 | .74199 | .48411 | .69016 | .62031 | .67112 | .41069 | .98705 | .95682 | .98693 | .92625 |
| $3 \times 5$ | .74798 | .53728 | .74798 | .35166 | .73650 | .54989 | .73026 | .34562 | .66896 | .56919 | .63761 | .29861 | .99658 | .97883 | .99666 | .93037 |
| $3 \times 7$ | .75031 | .43900 | .75031 | .21768 | .72953 | .46782 | .71805 | .21419 | .64443 | .50102 | .59018 | .19155 | .99968 | .99469 | .99976 | .93944 |
| $5 \times 3$ | .76119 | .67977 | .76119 | .64665 | .74988 | .67360 | .74988 | .63738 | .69934 | .66396 | .69014 | .55899 | .93771 | .91111 | .93682 | .87878 |
| $5 \times 4$ | .74555 | .60333 | .74555 | .51594 | .73244 | .60569 | .72912 | .50941 | .66686 | .61630 | .64420 | .42163 | .98223 | .96890 | .98170 | .90919 |
| $5 \times 5$ | .74186 | .54338 | .74186 | .41929 | .72539 | .55630 | .71977 | .41501 | .64821 | .58292 | .61361 | .33265 | .99461 | .98287 | .99445 | .93159 |
| $5 \times 7$ | .74523 | .45056 | .74523 | .25846 | .72018 | .48118 | .71070 | .25618 | .62678 | .52813 | .57102 | .21147 | .99945 | .99149 | .99947 | .94202 |
| $10 \times 3$ | .75836 | .68593 | .75836 | .64749 | .74416 | .67741 | .74416 | .63795 | .68773 | .67723 | .66521 | .55622 | .93123 | .91594 | .92860 | .86517 |
| $10 \times 4$ | .74451 | .59445 | .74451 | .53448 | .72742 | .59359 | .72480 | .52893 | .65434 | .60345 | .63270 | .43241 | .97903 | .95990 | .97719 | .90648 |
| $10 \times 5$ | .74042 | .52718 | .74042 | .44809 | .71877 | .53595 | .71471 | .44645 | .63567 | .56416 | .60268 | .34844 | .99330 | .98058 | .99213 | .92766 |
| $10 \times 7$ | .74276 | .43011 | .74276 | .32442 | .71512 | .45486 | .70746 | .32827 | .61672 | .50635 | .56279 | .24362 | .99925 | .99182 | .99900 | .94694 |
| $100 \times 3$ | .75851 | .67585 | .75851 | .66555 | .74129 | .66480 | .74129 | .65424 | .68084 | .65575 | .67190 | .56861 | .92356 | .90899 | .91975 | .86181 |
| $100 \times 4$ | .74446 | .58028 | .74446 | .56412 | .72250 | .57685 | .72149 | .55747 | .64815 | .59520 | .62760 | .45105 | .97454 | .95841 | .97056 | .90413 |
| $100 \times 5$ | .73949 | .50775 | .73949 | .48393 | .71422 | .51226 | .71174 | .48512 | .63176 | .55134 | .60044 | .37281 | .99113 | .97784 | .98835 | .92606 |
| $100 \times 7$ | .74170 | .40377 | .74170 | .37407 | .71082 | .42191 | .70487 | .38529 | .61763 | .48858 | .56558 | .27378 | .99883 | .99138 | .99780 | .94837 |

† $S^*$, $S_{Pl}$, $S_{Bd}$, $S_{Ng}$ denote the optimal scoring rule according to Theorem 3.1, plurality, Borda, and negative, respectively.

TABLE 2. Maximax: Values of the $\alpha_{MX}$ and $\gamma_{MX}$ obtained from the simulations analysis†

| | $\alpha_{MX}$: | | | | | | | | | $\gamma_{MX}$: | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | uniform | | | normal | | | exponential | | | exponential | | |
| Society | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ |
| $2 \times 3$ | .79997 | .80471 | .76420 | 1.02883 | 1.01675 | .88800 | 2.08285 | 2.06077 | 1.87436 | .93346 | .94451 | .90028 |
| $2 \times 4$ | .83337 | .82640 | .74331 | 1.16314 | 1.12777 | .81335 | 2.28311 | 2.22064 | 1.77465 | .98207 | .98953 | .90719 |
| $2 \times 5$ | .85718 | .84581 | .72962 | 1.26771 | 1.20770 | .76762 | 2.44959 | 2.34409 | 1.71641 | .99524 | .99832 | .91528 |
| $2 \times 7$ | .88889 | .87109 | .71271 | 1.42397 | 1.32054 | .71214 | 2.71864 | 2.52740 | 1.65127 | .99967 | .99997 | .92934 |
| $3 \times 3$ | .85006 | .85154 | .81445 | 1.23547 | 1.23684 | 1.07893 | 2.39920 | 2.39541 | 2.14759 | .89832 | .91557 | .83322 |
| $3 \times 4$ | .86958 | .86981 | .80995 | 1.32769 | 1.31996 | 1.05859 | 2.55332 | 2.53054 | 2.11495 | .95810 | .96997 | .88068 |
| $3 \times 5$ | .88437 | .88201 | .79862 | 1.40352 | 1.37869 | 1.01631 | 2.68501 | 2.63062 | 2.05359 | .98276 | .98874 | .89327 |
| $3 \times 7$ | .90583 | .89756 | .78528 | 1.52329 | 1.45877 | .96713 | 2.90205 | 2.77205 | 1.98592 | .99717 | .99821 | .91374 |
| $5 \times 3$ | .89472 | .89355 | .87289 | 1.45266 | 1.44320 | 1.33512 | 2.76755 | 2.74869 | 2.55782 | .86773 | .86821 | .79803 |
| $5 \times 4$ | .91048 | .90489 | .86842 | 1.54352 | 1.50629 | 1.31333 | 2.93556 | 2.86207 | 2.52192 | .94776 | .93903 | .84610 |
| $5 \times 5$ | .92099 | .91265 | .86532 | 1.61221 | 1.55162 | 1.29968 | 3.06743 | 2.94601 | 2.49861 | .97517 | .96949 | .87660 |
| $5 \times 7$ | .93400 | .92258 | .85710 | 1.71006 | 1.61470 | 1.26214 | 3.26589 | 3.06767 | 2.43879 | .99241 | .99118 | .90423 |
| $10 \times 3$ | .93816 | .93626 | .92609 | 1.73677 | 1.72056 | 1.64567 | 3.31847 | 3.28412 | 3.13255 | .82751 | .81654 | .75654 |
| $10 \times 4$ | .94517 | .94170 | .92462 | 1.79633 | 1.76395 | 1.63587 | 3.44237 | 3.37264 | 3.11435 | .91072 | .89793 | .81953 |
| $10 \times 5$ | .95076 | .94535 | .92326 | 1.84743 | 1.79450 | 1.62767 | 3.55181 | 3.43597 | 3.09803 | .95322 | .93846 | .85579 |
| $10 \times 7$ | .95821 | .95025 | .92127 | 1.92517 | 1.83806 | 1.61486 | 3.72422 | 3.52891 | 3.07485 | .98458 | .97415 | .89593 |
| $100 \times 3$ | .99143 | .99128 | .99080 | 2.55784 | 2.55162 | 2.53311 | 5.32864 | 5.31107 | 5.25782 | .72474 | .71785 | .69600 |
| $100 \times 4$ | .99189 | .99156 | .99074 | 2.57703 | 2.56319 | 2.53113 | 5.38295 | 5.34403 | 5.25495 | .81729 | .80422 | .77360 |
| $100 \times 5$ | .99228 | .99176 | .99071 | 2.59362 | 2.57112 | 2.52978 | 5.43006 | 5.36731 | 5.25030 | .87128 | .85421 | .81956 |
| $100 \times 7$ | .99288 | .99202 | .99063 | 2.62156 | 2.58319 | 2.52797 | 5.51008 | 5.39864 | 5.24392 | .92983 | .90851 | .87164 |

† $S_{Pl}$, $S_{Bd}$, $S_{Ng}$ denote plurality, Borda, and negative, respectively.

TABLE 3. Maximin: Values of the $\alpha_{MN}$, $\beta_{MN}$ and $\gamma_{MN}$ obtained from the simulations analysis[†]

| | $\alpha_{MN}$: | | | $\beta_{MN}$: | | | | | | | | | $\gamma_{MN}$: | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | exponential | | | uniform | | | normal | | | exponential | | | exponential | | |
| Society | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ | $S_{Pl}$ | $S_{Bd}$ | $S_{Ng}$ |
| $2 \times 3$ | .74980 | .82758 | .79152 | .61695 | 73946 | .70038 | .61697 | .73886 | .69963 | .61688 | .73934 | .70004 | .83333 | .94463 | 1 |
| $2 \times 4$ | .79979 | .95218 | .72462 | .53244 | .73396 | .44047 | .53225 | .73400 | .44061 | .53197 | .73407 | .44047 | .87469 | .98948 | 1 |
| $2 \times 5$ | .83309 | 1.05261 | .68330 | .47321 | .73200 | .31387 | .47322 | .73172 | .31363 | .47272 | .73178 | .31294 | .89991 | .99837 | 1 |
| $2 \times 7$ | .87518 | 1.20970 | .63375 | .39467 | .73151 | .19526 | .39448 | .73148 | .19503 | .39476 | .73165 | .19552 | .92860 | .99997 | 1 |
| $3 \times 3$ | .49047 | .52856 | .52270 | .57923 | .66663 | .66581 | .57929 | .66633 | .66530 | .57883 | .66584 | .66494 | .81457 | .88397 | .92577 |
| $3 \times 4$ | .50756 | .59796 | .53992 | .47285 | .64017 | .54584 | .47315 | .63994 | .54577 | .47297 | .64019 | .54589 | .85420 | .94154 | 1 |
| $3 \times 5$ | .51552 | .65827 | .50393 | .40178 | .63022 | .38201 | .40113 | .62997 | .38165 | .40164 | .63022 | .38175 | .88004 | .96825 | 1 |
| $3 \times 7$ | .52048 | .75646 | .45978 | .31197 | .62372 | .22735 | .31211 | .62385 | .22718 | .31193 | .62368 | .22753 | .91153 | .98905 | 1 |
| $5 \times 3$ | .27205 | .29872 | .30312 | .50243 | .58753 | .60990 | .50249 | .58831 | .61031 | .50269 | .58815 | .61029 | .77750 | .84153 | .87663 |
| $5 \times 4$ | .27872 | .33341 | .33440 | .40382 | .54802 | .56296 | .40375 | .54774 | .56276 | .40374 | .54749 | .56255 | .83164 | .90637 | .95332 |
| $5 \times 5$ | .28172 | .36313 | .34452 | .33883 | .52575 | .48873 | .33941 | .52600 | .48829 | .33876 | .52569 | .48844 | .86419 | .93847 | .99229 |
| $5 \times 7$ | .28188 | .41080 | .31519 | .25831 | .50289 | .29765 | .25800 | .50294 | .29791 | .25845 | .50298 | .29770 | .90085 | .96769 | 1 |
| $10 \times 3$ | .12583 | .13781 | .14100 | .44597 | .51092 | .53296 | .44636 | .51112 | .53310 | .44592 | .51112 | .53308 | .74596 | .79361 | .81454 |
| $10 \times 4$ | .12623 | .15058 | .15788 | .34641 | .45550 | .49875 | .34657 | .45554 | .49868 | .34662 | .45514 | .49823 | .80643 | .86522 | .90012 |
| $10 \times 5$ | .12593 | .16094 | .17173 | .28474 | .42189 | .47694 | .28499 | .42229 | .47680 | .28474 | .42174 | .47659 | .84404 | .90332 | .94654 |
| $10 \times 7$ | .12515 | .17796 | .18671 | .21114 | .38323 | .41707 | .21071 | .38303 | .41758 | .21110 | .38342 | .41646 | .88673 | .94142 | .98962 |
| $100 \times 3$ | .01076 | .01129 | .01148 | .36482 | .38768 | .39623 | .36479 | .38722 | .39614 | .36580 | .38826 | .39688 | .69113 | .70834 | .71499 |
| $100 \times 4$ | .01073 | .01166 | .01216 | .27635 | .31092 | .32968 | .27587 | .31014 | .32931 | .27619 | .31015 | .32941 | .76759 | .78869 | .80016 |
| $100 \times 5$ | .01071 | .01195 | .01278 | .22213 | .26307 | .29221 | .22231 | .26335 | .29232 | .22274 | .26332 | .29231 | .81321 | .83567 | .85065 |
| $100 \times 7$ | .01064 | .01237 | .01399 | .16004 | .20766 | .25507 | .16029 | .20790 | .25577 | .15993 | .20775 | .25543 | .86605 | .88717 | .90592 |

[†] $S_{Pl}$, $S_{Bd}$, $S_{Ng}$ denote plurality, Borda, and negative, respectively.

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