Brit. J. Phil. Sci. **50** (1999), 401–416

Countable Additivity and Subjective Probability

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ABSTRACT

While there are several arguments on either side, it is far from clear as to whether or not countable additivity is an acceptable axiom of subjective probability. I focus here on de Finetti's central argument against countable additivity and provide a new Dutch book proof of the principle, to argue that if we accept the Dutch book foundations of subjective probability, countable additivity is an unavoidable constraint.

Largely due to Kolmogorov's influence ([1933]), axiomatizations of the mathematical theory of probability now include the principle of countable additivity, or an equivalent principle, as standard. But de Finetti, one of the pioneers of the subjective interpretation of probability, argued against its acceptance (de Finetti [1970]). I shall give a brief overview of some of the central arguments for and against the adoption of countable additivity for subjective probability, and I will argue that they are inconclusive. I will then go on to look more closely at de Finetti's view, before presenting a simple Dutch book argument in favour of the principle, which will mean that we cannot reject countable additivity without abandoning the simplest and most intuitive foundations for subjective probability: the betting set-up and the ensuing Dutch book argument. Ironically from de Finetti's point of view, accepting countable additivity ensures that the interpretation retains its subjective flavour, for the principle prevents subjective probability being bolstered into a logical interpretation.

1 Guess the number

I am thinking of a particular natural number. For any natural number *n*, how confident are you that it is the one I am thinking of?

A distinctive feature of this scenario is that the alternatives about which I am asking you to form beliefs constitute a countably infinite partition. There is no great conceptual difficulty in dealing with the infinite here—perhaps you do not assign any credence to the event that I am thinking of n, for any n, or perhaps you are more confident that I am thinking of a small number than a large one, the point is that it is easy enough to form such beliefs. As a consequence, any adequate model of rational belief should be able to model beliefs over a countably infinite partition such as the one above.

2 The betting set-up

The subjective probability framework provides one approach to a normative model of belief. The principal idea is that one can measure the degree to which an agent believes an event a will occur (or has occurred) by the kind of bet on a she will find acceptable. We can encapsulate this relation between belief and betting in the following definition:[1]

> $bel_X^\tau(a) = x$ (X's degree of belief in a at time τ is x) iff at time τ, our agent X—Xenelda, say—is willing to bet $x\Delta\Theta$ on event a occurring, with return $\Delta\Theta$ if a does occur, where $\Delta\Theta$ is an unknown stake (either monetary or in terms of some measure of utility) which may depend on $bel_X^\tau(a)$, $\Theta \in \mathbf{R}_{\geq 0}$ being the magnitude and $\Delta = \pm 1$ the direction of the stake; and
>
> $bel_X^\tau(a|b) = x$ (X's degree of belief in a given b at time τ is x) iff she is prepared to bet $x\Delta\Theta$ on a occurring, with return $\Delta\Theta$ if both a and b occur but with the bet being called off if b fails to occur.

Ramsey ([1926]) and de Finetti ([1937]) provided the normative element—we can deem X's belief function bel_X^τ to be *coherent* if no stake-maker can choose stakes which make X lose money whatever happens—and according to them, coherence provides a criterion of rationality. By their Dutch book argument (Ramsey [1926]; de Finetti [1937]), bel_X^τ is coherent if and only if it is a probability function. Here, a probability function is any function *bel* that satisfies the following axioms of probability:

P1: if a is the certain event, $bel(a) = 1$;
P2: if a and b are mutually exclusive, $bel(a \cup b) = bel(a) + bel(b)$
P3: $bel(b) \neq 0 \Rightarrow bel(a|b) = bel(a \cap b)/bel(b)$.

Now any probability function in this sense is finitely additive, in that if A is a finite set of mutually exclusive events, $bel(\cup A) = \Sigma_{a \in A} bel(a)$. Given

[1] This definition is based on that of de Finetti ([1937]).

countable contexts like ‘guess the number’, the question naturally arises as to whether *countable additivity* also holds:

CA: if A is a countable set of mutually exclusive events, $bel(\cup A) = \Sigma_{a \in A} bel(a)$.

3 Consequences versus foundations

There are two ways of arguing for (or against) countable additivity. One can argue that if we adopt countable additivity, our model of belief is better (or worse) off thanks to favourable (or unfavourable) effects of countable additivity in the model. Thus we accept (or reject) the principle according to how we judge its consequences. Alternatively one can appeal to the foundations of subjective probability and argue that countable additivity is a natural (or unnatural) constraint in the context of the betting set-up.

To date, the bulk of the debate about countable additivity has involved arguments from its consequences. My plan is to give a brief flavour of these arguments and the resulting impasse, before presenting a foundational argument for the principle. The benefit of appealing to foundations is threefold. Firstly, arguments from consequences have shown that while an acceptance of countable additivity leads to a noticeably different model of rational belief to that without the axiom, and while both the resulting models appear to have advantages and disadvantages, neither is blatantly inadequate, so neither can be dismissed on these grounds alone. Thus, at least at the moment, an appeal to foundations may be our only hope of resolving the issue. Secondly, I believe I can make a strong argument in favour of countable additivity by appealing to foundations, in which case the principle can only sensibly be denied if the foundations are rejected too, and there is currently little hope of providing such plausible foundations that will discriminate against an adoption of the axiom.

Furthermore, even if one accepts arguments about the mathematical convenience of countable additivity, one may deny that this provides enough grounds to accept the principle itself, given the demands that might be placed on subjective probability as a model of belief. Kelly adopts such a position when he gives an argument for scientific realism which relies on countable additivity. Does meta-scientific convenience lend any weight? As he notes,

> If probabilistic convergence theorems are to serve as a philosophical antidote to the logical reliabilist’s concerns about local underdetermination and inductive demons, then countable additivity is elevated from the status of a mere technical convenience to that of a central epistemological axiom favoring scientific realism. Such an axiom should be subject to the highest degree of philosophical scrutiny. Mere technical convenience cannot justify it. Neither can appeal to its ‘fruits,’ insofar as they include precisely the convergence theorems at issue (Kelly [1996], p. 323).

The idea is that if one considers subjective probability to have foundations (such as the betting and Dutch book foundations) then presumably even if one accepts or rejects countable additivity on the grounds of its consequences, the question remains as to whether it is a valid constraint according to the foundations. Thus, in a sense, arguments from the consequences of countable additivity are irrelevant to whether countable additivity is an appropriate axiom for subjective probability. At most they can tell us what one can do with subjective probability and how applicable it is.[2]

There is a broad range of countable additivity arguments from its consequences. Some (see for example de Finetti [1972], pp. 98–100, and Kadane *et al*. [1986], section 6) are directed at other principles—continuity, countable disintegrability and countable conglomerability—which are equivalent (Hill and Lane [1986]; Dubins [1975]; de Finetti [1972], p. 99) to countable additivity in the presence of the other probability axioms. More arguments proceed from a decision-theoretic viewpoint, where non-conglomerability leads to a violation of admissibility, a useful principle in decision theory (Seidenfeld and Schervish [1983]; Heath and Sudderth [1978]). Two other arguments from consequences have been particularly influential, yet remain inconclusive, as we shall see now and in the next section.

A popular attitude has been that countable additivity should be adopted as an axiom of probability because it leads to a stronger theory, and thus extra mathematical power can be applied to deriving probability theorems—for instance, it is required for the proof of the strong law of large numbers. Kolmogorov adopted countable additivity for his influential work on the mathematical theory of probability because of its success:

> *We limit ourselves, arbitrarily, to only those models which satisfy [countable additivity]*. This limitation has been found expedient in researches of the most diverse sort (Kolmogorov [1933], p. 15).

Likewise Fishburn:

> The present wisdom seems to be that countable additivity can keep one out of trouble that might arise in its absence even if it is arbitrary, or at best uncompelling, as a principle of rational choice. My own attitude towards the principle is pragmatic. Much like the Axiom of Choice in set theory, if I can go without countable additivity to get where I want to go, so much the better. But I will not hesitate to invoke it when its denial would create mathematical complexities of little interest to the topic at hand (Fishburn [1983], p. 358).

[2] Of course, one may reject foundations for subjective probability, in which case one will find arguments from consequences more convincing than foundational arguments. But my line of argument here is that *if* one accepts the Dutch book foundations for subjective probability, then one will have to accept that subjective probability is countably additive.

And Sudderth, in his comments on the above, concurs:

> The requirement of coherence does not imply countable additivity as de Finetti has often emphasized [...] in general I agree with Kolmogorov ([1933]) that the assumption of countable additivity, although expedient, is arbitrary (Fishburn [1983], p. 358).

But this type of argument becomes more important when we extend it from a mathematical to a physical setting—mathematical expediency can be bolstered to scientific indispensability. Van Fraassen ([1980], Ch. 6) argues that the mathematical power provided by countable additivity is indispensable for science, so there are physical reasons for adopting the principle.

In fact, however, Chen ([1977]) shows that more can be derived from finite additivity than is normally thought—he gives versions of the strong laws for example—so the argument from mathematical convenience is questionable. At the very least more work needs to be done before we can be sure whether the apparent strength of countable additivity is real or theorems can be proved using finite additivity which are as useful as those requiring countable additivity. There are further avenues for doubting the convenience argument: de Finetti ([1972], sections 5.21–5.23) claims that countable additivity is mathematically inelegant, in that some mathematical event spaces do not admit of countably additive probability distributions unless perfect additivity[3] also holds, and that countably additive distributions do not form a closed set. He also notes that a set of admissible probability distributions is convex, and claims that it is inappropriate that some individual cannot have a limit point of this set as an admissible probability distribution, which is a consequence of adopting countable additivity.

Thus there are arguments on both sides which rely on mathematical considerations. In my view, none of these can really be considered to be knockdown.

De Finetti himself has been the main proponent against assuming countable additivity across the board, and he has an argument from consequences which is particularly interesting, so we shall take a closer look at his views before I give a foundational justification of countable additivity.

4 Uniform distributions

De Finetti argues that in a 'guess the number' scenario, one may have no information favouring one number over another and so should assign each number the same degree of belief q. Now q cannot be positive for otherwise

[3] A distribution is perfectly additive if countably additive and is zero on all but a countable subset of any uncountable partition.

one would be obliged to assign degree of belief greater than one that the number is within any finite set whose cardinality is greater than $1/q$, which contradicts probability axioms P1-2. Thus q must be zero. But then countable additivity cannot hold, for the sum of countably many zeros is zero, so $1 = bel(\cup_{i=1}^{\infty} i) \neq \Sigma_{i=1}^{\infty} bel(i) = 0$ (where i represents the event that I am thinking of number i). In summary, de Finetti would argue, one should be able to bet according to a uniform distribution in the 'guess the number' example, in which case countable additivity cannot possibly hold. This is an example of an argument against countable additivity involving an appeal to its (supposedly counterintuitive) consequences.

It will prove useful to distinguish two types of position one can take here. Suppose an agent X is quite indifferent as to which element of a partition occurs. For instance, she does not favour any number over any other in the guess the number scenario. Then the positions are as follows. In the subjective probability model of belief,

Weak uniformity: it is quite consistent that X assigns a uniform belief distribution over the partition,
Strong uniformity: *X should* assign a uniform belief distribution over the partition.

If one accepts either position then one will have to abandon countable additivity as a constraint of subjective probability, but the second position is itself a strong constraint, while the first is not. In our discussion of the uniformity argument we shall first look at, and reject, an objection to de Finetti's position, before focusing on his reasoning. We shall see that the case for uniformity is not as strong as de Finetti makes out, by showing that both strong uniformity and weak uniformity only apply in exceptional circumstances. However, I shall maintain, de Finetti still has a case for us to answer, which motivates the Dutch book result of the next section.

Unfortunately, the example de Finetti ([1972], section 5.17) actually used to make his point revolved around guessing some number *chosen at random*, rather than guessing the number I am thinking of. This led Spielman ([1977], section 4) to doubt the possibility of being able to choose a number at random in the sense that each number has the same (zero) objective probability of being chosen. Spielman maintains that any way of explicating 'choose a number at random' yields an asymmetric distribution: 'I will be entirely unmoved by it until someone shows me a precisely defined non-metaphorical gedanken experiment which selects integers ''at random'' ' (Spielman [1977], p. 255).[4]

[4] See Kadane and O'Hagen ([1995]) for some possible interpretations of 'random natural number' such that each natural number is equally possible. They do not show, however, that there is any physical mechanism for generating such random numbers.

I claim that this misses the point—we are interested in determining whether *subjective*, not *objective*, probability satisfies countable additivity. Besides, it is easy enough to see that an objective, frequency, notion of probability must admit uniform distributions over countable partitions ('attribute spaces') and hence fail to satisfy countable additivity—just take any countable attribute space whose elements can only be instantiated a finite number of times. For example, if we examine engine numbers from a certain make of car that, we assume, never stops producing cars, then the frequency of any particular engine number occurring is zero, while the frequency of the cars having *some* engine number is one. One needs a great deal more than this to argue that one may be rational to believe to degree *0* that a particular car will have engine number n.

The point at issue is not the viability of some objective mechanism but rather an agent's epistemic state. The agent may have no idea as to which number will crop up and so be predisposed to a uniform distribution. The agent may even believe that no mechanism for picking a number according to an objective uniform distribution exists and that the number is practically certain to be in some finite subset, but not know which subset, and so again be predisposed to place bets according to a uniform distribution. Hence Spielman's argument has no force against de Finetti's claim in favour of uniform distributions.

De Finetti notes that countably additive distributions are skewed rather than uniform:

> By taking the sum of probabilities to be $= 1$[...], one necessarily has an inequality such that for any $\varepsilon > 0$, however small, a finite number of events [...] together have probability $> 1 - \varepsilon$, and the infinity of the others together have probability $< \varepsilon$. (In such circumstances, I am tempted to say that the events 'are not countably infinite' but 'a finite number—up to trifles') [...]
>
> From a mathematical standpoint this is obvious. What is strange is simply that a formal axiom, instead of being *neutral* with respect to the evaluations (or, for those who believe in them, with respect to the objective reasons), and only imposing formal conditions of coherence, on the contrary, imposes constraints of the above kind without even bothering about examining the possibility of there being a case against doing so (de Finetti [1970], p. 122).
>
> If this lack of symmetry does not reflect the actual judgement of the subject, perhaps because he is indifferent toward all the possible outcomes, how could we then include in the definition of consistency (in a purely formal sense) a condition which does not allow him to assign equal probabilities p_n? Should we force him, against his own judgement, to assign practically the entire probability to some finite set of events, perhaps chosen arbitrarily? Such limitations on the choice of the probabilities are altogether extraneous to the essence of the consistency condition (de Finetti [1972], pp. 91–2).

Suppose someone chooses his subjective probability distribution over a countable partition:

> Someone tells him that in order to be coherent he can choose the Pi in any way he likes, so long as the sum *=1* (it is the same thing as in the finite case, anyway!).
>
> The same thing?!!! You must be joking, the other will answer. In the finite case, this condition allowed me to choose the probabilities to be all equal, or slightly different, or very different; in short, I could express any opinion whatsoever. Here, on the other hand, the *content* of my judgements enter the picture: I am allowed to express them only if they are unbalanced to the extent illustrated [above]. Otherwise, even if I think they are equally probable [. . .] I am obliged to pick 'at random' a convergent series which, however I choose it, is in absolute contrast to what I think. If not, you call me *incoherent*! In leaving the finite domain, is it I who has ceased to understand anything, or is it you who has gone mad? (de Finetti [1970], p. 123)

Note that in the previous quotes de Finetti argues for weak uniformity, i.e. that uniform distributions are quite reasonable, but in this last quote he seems to making a case for strong uniformity—that adopting a skewed belief distribution is less rational than adopting a uniform distribution. Thus there are two claims to answer.

At first sight de Finetti's position seems to be quite watertight: it appears to be incongruous to just override a strong intuition about rational belief like uniformity. But there are various reasons why this position is not as strong as at first sight.

Let us tackle strong uniformity first. It is well known that a principled application of uniform distributions can lead to problems analogous to paradoxes of the 'principle of indifference' such as Bertrand's paradox (Bertrand [1888], pp. 4–5) and von Mises' water–wine paradox (von Mises [1964], p. 161). The trouble is that in some situations there is more than one partition over which one can ascribe a uniform (zero) belief distribution, and different solutions arise according to which partition is chosen. Moreover, there appears to be no general way of choosing a best partition and disregarding the others, so this is really a problem for those who advocate uniform distributions.[5]

Yet these paradoxes are not as far-reaching as to dispose of strong uniformity altogether. The proponent of uniform distributions may well agree that they lead to difficulties in some cases, but go on to argue for their use in situations where there is a unique partition worthy of a uniform distribution. The problem with such a move is that it often takes considerable analysis to determine whether an application of a uniform distribution is problematic or

[5] Jaynes ([1973]) suggests a solution to Bertrand's paradox but acknowledges that many similar paradoxes cannot be solved.

not. But while strong uniformity becomes a rather unattractive position, its die-hard proponent can maintain that still countable additivity cannot be adopted for fear these harmless applications of uniform distributions would be prevented.

Alternatively, the proponent of uniformity can abandon strong uniformity in favour of the weaker version. But even weak uniformity suffers from a lack of applicability, as we shall now see.

One criticism of intuitions towards uniformity might go: a uniform prior is adopted for simplicity, and does not correspond to anyone's beliefs, since people are practically certain that the outcome in question lies in an interval. For instance you may believe I am thinking of a smaller rather than a larger number, and practically certain that I am not thinking of any number greater than $10^{10^{10}}$.

But while complete indifference is no doubt an over-simplification in most cases, it is not always, and even if the outcome is practically certain to lie in an interval, one may not know which interval it is likely to lie in. De Finetti puts it thus:

> Initially, *X* has a uniform distribution over the real numbers between *0* and *1* [...]
>
> If one thought of actually interpreting the problem geometrically, one might perhaps doubt the judgement of all the rationals as equally probable, considering as 'rather special' the end points, mid-point, fractions with small denominator, decimal fractions with only a few figures, etc.
>
> This effect is lessened if one thinks of taking the 'distance between two points chosen at random' [...]
>
> It disappears altogether if one thinks in terms of a circle obtained by rolling up the segment without indicating which is the 'zero' point (de Finetti [1970], p. 121).

De Finetti's position is somewhat weakened because it becomes clear that one has to come up with quite contrived examples before one can find complete indifference over a countable partition. However, his point still goes through: while perhaps in most cases one will be practically certain the outcome lies in some interval, and know which that interval is, there are some cases where one may be faced with complete indifference and still want to set uniform beliefs, so the uniform distribution problem does not entirely vanish.

Other criticisms of uniform distributions revolve around claims that paradoxes arise from their use (see Stone [1976]). However, these paradoxes have been contested (Hill [1980]). It is important to note that paradoxes are often formulated in the Bayesian statistics framework where Bayesian conditionalization is the only way to change rational belief. Thus any paradoxes may be due to other parts of the framework rather than just the component of finite additivity. One way of obtaining a 'paradox' is by showing that a finitely additive Bayesian statistical solution to a problem gives a different answer to a more conventional countably additive statistical solution. But this disagreement may be attributed to the strict adherence to Bayesian conditionalization, rather than finite additivity, because even if countable additivity holds in the Bayesian framework, the Bayesian solution to a problem can differ from the classical statistical solution. For instance an agent's subjective probability may differ from a known frequency, since between τ and $\tau + 1$ agent X may come to know $freq(R(x)) = q$, but $bel_X^{\tau+1}(R(a)) = bel_X^{\tau}(R(a)|freq(R(x)) = q)$ may not equal q, the observed frequency.

In sum, it is clear that both strong and weak uniformity are less applicable than might at first be thought, but they are not completely paradoxical positions, so a case still needs to be made for countable additivity.

Note that while it is incongruous to deny uniform beliefs, it is not so very outrageous, if one has good reason to do so. It must be remembered that the subjective interpretation of probability is a *normative* account of belief. It models *rational* belief, and is not intended to be an accurate description of human belief (it is quite clear that people do not associate each conceivable event with a real number, capturing their degree of belief in that event; nor are they logically omniscient; nor would they in any case be able to ensure their degrees of belief strictly adhere to the axioms of probability at each point in time). Thus even supposing people regularly have the same degree of belief in each event in a denumerable partition, they may well be some clear reason showing why they are wrong to do so, and thus why subjective probability should not allow uniform distributions over denumerable partitions.

Indeed, I claim that there is such a clear reason, motivated by the foundations of subjective probability. Given my Dutch book justification of the next section, a predisposition to a uniform distribution over a denumerable partition is irrational because the corresponding bets will lead to certain loss, and thus it is more rational for an agent to bet according to an arbitrary countably additive distribution in such a situation.

We have seen that de Finetti's criticism of countable additivity is based on his idea that uniform distributions are reasonable, yet incompatible with countable additivity. While uniform distributions may be descriptively accurate in a few situations, they are defeasible in the presence of a normative argument for countable additivity.

5 A Dutch book argument

On balance it is fair to say that the question of countable additivity is far from decided. As de Finetti claimed, 'No-one has given a real justification of countable additivity (other than just taking it as a ''natural extension'' of finite additivity)' (de Finetti [1970], p. 119). Of course, he was opposed to countable additivity, but we have seen that his own arguments against the principle, while making the issue of a real justification more pressing, do not preclude the possibility of a decisive justification.

Spielman ([1977]) claimed that a Dutch book argument for countable additivity might be possible, though it appears that Adams ([1964]) first gave such an argument. While I maintain that there are meaningful situations where one must form beliefs over countable partitions, and that the issue of countable additivity is of practical importance even for non-mathematicians, Adams' interest in countable additivity stems solely from mathematical curiosity: he thinks the principle is applicable only to 'extremely unrealistic' (Adams [1964], p. 8) infinite systems. His paper chiefly concerns the possibility of giving Dutch book arguments for the axioms in a framework where the direction of the bet, what I have called Δ, is fixed to be positive in advance. Consequently the proofs are rather more involved than de Finetti's original Dutch book proofs. It appears that these complications and remarks as to the inapplicability of countable additivity have disguised the clarity and appeal of the conclusion, so that the question of countable additivity is still unresolved for most people interested in the issue.

As von Plato says in his recent book,

> De Finetti seems to think that denumerable additivity would be an optional property. It would then be a question of the accuracy of our probabilistic vision to see whether denumerable additivity holds in a given case. Is de Finetti's attitude eclectic, or is there some principle, so far not found, that will go deeper into infinitary events and their probabilities, so as to decide in which cases measure theory's simple-minded extrapolation from the finite to the infinite is justified? At present it seems that the foundations of the topic remain as open as ever (von Plato [1994], p. 278).

Here I shall generalize de Finetti's proof of the finite case (de Finetti [1937]) to demonstrate that countable additivity is as obvious a constraint as the other axioms. I naturally assume only a finite amount of money can change hands after the truth value of any sentence on which a bet is placed is observed, for the betting situation would be quite meaningless if this were denied. Then we have the following:

bel **is coherent if and only if it satisfies countable additivity.**

Proof*: Suppose* $A = \{a_0, a_1, a_2, \ldots\}$ is a set of mutually exclusive and exhaustive sentences.

Let $q_i = bel(a_i)$ and $\Delta_i\Theta_i$, where $\Delta_i = \pm 1$, $\Theta_i \in \mathbf{R}_{\geq 0}$ be the stakes corresponding to the betting quotients q_i, for $i = 0, 1, 2 \ldots.$

Then the loss X would incur on a_k having been found to be true is $L_k = \Sigma_{i=0}^{\infty} q_i\Delta_i\Theta_i - \Delta_k\Theta_k$. A finite amount of money changing hands is equivalent to the condition $|L_k| < \infty$ for each natural number k, which in turn is equivalent to condition $\mathbf{C}$: $|\Sigma_{i=0}^{\infty} q_i\Delta_i\Theta_i| < \infty$.

We need to prove: $L_k \leq 0$ for some $k \Leftrightarrow \Sigma_{i=0}^{\infty} q_i = 1$.

[⇒] Suppose $\Sigma_{i=0}^{\infty} q_i < 1$ (it cannot be greater than one by the first probability axiom). Then for each $i \in \mathbf{N}$ let $\Delta_i \Theta_i = \Delta\Theta$, a constant, where $\Delta = \pm 1$, $\Theta \in \mathbf{R}_{\geq 0}$.
Now C is satisfied because $|\Sigma_{i=0}^{\infty} q_i \Delta_i \Theta_i| = |\Delta\Theta||\Sigma_{i=0}^{\infty} q_i| = \Theta\Sigma_{i=0}^{\infty} q_i < \Theta < \infty$, and so we have $L_k = \Sigma_{i=0}^{\infty} q_i \Delta\Theta - \Delta\Theta = \Delta\Theta(\Sigma_{i=0}^{\infty} q_i - 1)$.
So setting $\Delta = -1$, we get that $L_k > 0$ for all $k \in \mathbf{N}$.
[⇐] Conversely suppose $\Sigma_{i=0}^{\infty} q_i = 1$.
Then

$$\sum_{i=0}^{\infty} q_i L_i = \sum_{i=0}^{\infty} q_i \left[\sum_{j=0}^{\infty} q_j \Delta_j \Theta_j - \Delta_i \Theta_i \right]$$

$$= \sum_{i=0}^{\infty} q_i \sum_{j=0}^{\infty} q_j \Delta_j \Theta_j - \sum_{i=0}^{\infty} q_i \Delta_i \Theta_i \text{ (since C holds)}$$

$$= 1 \sum_{j=0}^{\infty} q_j \Delta_j \Theta_j - \sum_{i=0}^{\infty} q_i \Delta_i \Theta_i = 0.$$

But $q_i \geq 0$ and $q_k > 0$ for some $k \in \mathbf{N}$, so $L_k \leq 0$ for some $k \in \mathbf{N}$, as required.

Note that the assumption that only a finite amount of money changes hands is a restriction on the stake-maker rather than the agent Xenelda, because by P1,2, $\Sigma_{i=0}^{\infty} q_i \leq 1$ so the stakes would have to be very awkward for C to fail. When thought of in utility terms it amounts to the empirically well-confirmed fact that however rational your beliefs are about the possible outcomes of a single event, they will never gain you an infinite amount of utility when the actual outcome of that event is decided.

Countable additivity should be accepted because otherwise bets would lead to certain loss. Further, this crucial point can override intuitions about adopting uniform distributions. There is an analogy to be found in the game of roulette. One would be foolish to place the same stake on every number on a roulette wheel because one will lose whatever happens. The smart thing to do, if one has to play at all, is to adopt a skewed distribution and quit if and when one is ahead.[6] Thus, whether a uniform distribution is appropriate depends primarily on the betting situation rather than general intuitions about uniform beliefs in the face of ignorance of outcome. In 'think of a number' the set-up is such that if one does not want to lose money whatever happens, one cannot hold a countably additive uniform distribution, and the smart thing to do is to adopt a (countably additive) skewed distribution, even if one is indifferent as to which way to skew it.

[6] Of course, the rules of roulette are formulated so that a player will tend to lose money in the long run, however wise her bets. But this difference makes the analogy closer in the short term, because a uniform distribution over the finite number of options on the roulette wheel will lead to certain loss just as a uniform distribution over a denumerable partition in the subjective probability betting set-up does.

6 Subjectivity ensured

We shall now see that adopting countable additivity ensures that the subjective interpretation of probability remains truly subjective.

A complaint against the subjective interpretation of probability is precisely that it is subjective. In some areas of reasoning, such as medical diagnosis, we need the objectivity provided by the frequency interpretation (where we can presume there is a frequency of patients with certain symptoms having a certain disease) together with the wide applicability of the subjective approach (due to lack of data, reliable estimates of these frequencies may be unavailable, so guesses may have to be made to complete a diagnosis). Even in areas where there are no repeatable experiments (such as the prediction of the performance of stocks and shares), and thus where frequencies might not be assumed to exist and any guesses cannot be interpreted as estimates of frequencies, it is not implausible that one might make a guess that is best in the sense that it maximizes possible gain for minimum possible loss in a betting situation. In our search for objective probabilities in such situations we might naturally ask whether bel_X^τ might be further constrained to a unique *objectively* most rational belief function—perhaps $bel_X^\tau(a)$ could be constrained by some syntactic 'principle of indifference' which gives a best degree of belief in a in the face of insufficient information about any frequency that may be associated with it. If the function was unique then room for subjectivity would be precluded in X's belief function. It would provide an objective interpretation of probability, analogous to Keynes' logical interpretation (Keynes [1921]).

However, the principle of countable additivity implies that an objective best-belief interpretation is unattainable using syntactic constraining principles. Suppose we have a countable set of mutually exclusive events $A = \{a_i | i \in \mathbf{N}\}$. Then as we have seen, *bel* can not be uniform over A. So there must be an a_m and an a_n such that $bel(a_m) > bel(a_n)$. But what determines which event is to receive greater belief? If the value of $bel(a_i)$ is determined by i we can just permute A to change the distribution—but clearly any rational constraining principle should be independent of the order of the events to be constrained. So maybe there are other syntactic features of the events that determine the 'best' distribution for *bel*. But there need be no syntactic differences between the events other than an arbitrary index. So any discrimination between the events will have to have a semantic basis which may not always be available, ruling out the goal of the logical interpretation, a unique 'most rational' *bel*. Subjectivity is here to stay.

Ironically, although de Finetti is considered to be champion of the subjective interpretation of probability his arguments regarding countable additivity bring an objective logical interpretation one step closer, while my acceptance of countable additivity gives rise to a strong argument in favour of his subjective standpoint. De Finetti's arguments do this in two ways. Firstly, his rejection of countable additivity means that his concept of subjective probability can be bolstered to a logical theory, whereas this is not possible on my account. Secondly, he argued against countable additivity by claiming that when faced with total indifference one should adopt a uniform distribution. While no doubt he was striving for a weak position in which uniform distributions are merely consistent with subjective probability, his arguments also motivate strong uniformity. If this principle were to be accepted then there may indeed be hope for an objective best-belief interpretation.

7 Summary

There is no shortage of arguments for and against countable additivity, especially those focusing on the consequences of the principle. But by and large they are rather inconclusive, and in any case there is some demand for foundational rather than consequential arguments. A Dutch book argument, I claim, fulfils this role and is as compelling as the arguments for the other, rarely disputed, axioms of probability.

I may have disagreed with de Finetti over countable additivity, but it is only through adopting his clear betting approach and Dutch book methods that I have been able to present a firm argument in favour of the principle. I have also backed de Finetti's aim of subjectivity, and through examples like 'think of a number', agree that the issues involved, far from being esoteric talk of infinity, must be confronted. As his conclusion notes:

> I would like to have succeeded in convincing the reader in one thing; that we are dealing with a complex of problems, connected and meaningful, concerning which there are many things to be discussed under various headings: the conceptual, the mathematical, the practical. It is not just, as might seem logical at first sight, a question of arbitrary conventions for the subtleties involved, having no connection with real problems (de Finetti [1970], p. 127).

Acknowledgements

Thanks to the British Academy for financial support, and to Donald Gillies and David Corfield for helpful comments.

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