IS THERE A DUTCH BOOK ARGUMENT FOR PROBABILITY KINEMATICS?*

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Dutch Book arguments have been presented for static belief systems and for belief change by conditionalization. An argument is given here that a rule for belief change which under certain conditions violates probability kinematics will leave the agent open to a Dutch Book.

Paul Teller (1973) has reported David Lewis' argument that under certain conditions an agent whose degrees of belief in some propositions are changed from less than 1 to 1 by his experience must change his other beliefs by simple conditionalization in order to avoid being the victim of a Dutch Book made by a bookie who knows no more than himself (the agent and the bookie know the agent's rule for changing his beliefs). The significance of Dutch Book arguments of any kind has been questioned and debated; I will not contribute to the debate here. I will examine the possibility of giving a Dutch Book argument for the generalization of simple conditionalization provided by Richard Jeffrey's probability kinematics. I will first show that certain gross violations of probability kinematics leave an agent open to a Dutch Book under assumptions little different from those of Lewis' argument (section **2**). I will then argue in sections **3** and **4** that for those beliefs which satisfy an additional independence condition, any departure from probability kinematics will leave the agent open to a Dutch Book by a bookie who knows the rule which leads to that departure. The statement and implementation of the additional condition involve the use of second order degrees of belief, and some readers may find this objectionable. However, they may still find the argument instructive.

1. It is important to recall that the formula for probability kinematics is equivalent to a condition stating that the agent does not change his conditional degrees of belief in certain propositions: Suppose an agent with a coherent system of beliefs *prob* has experience which directly affects his degrees of belief in propositions $B_1, \ldots, B_n$ (his new system of beliefs will be written *PROB*), where for all B_i, $prob(B_i) > 0$. Let $\{E_1, \ldots, E_{2^n}\}$ be the set of propositions of the form $C_1 \& \ldots \& C_n$, where each C_j is B_j or $\sim B_j$. Let $S = \{E_1, \ldots, E_m\}$ be the set whose elements are all the elements of the above set such that $prob(E_i) > 0$. By the probability axioms the following condition

(1) $(\forall E_i \in S)(\forall A)(\, prob(A / E_i) = PROB(A / E_i)\,)$

is equivalent to Jeffrey's formula

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(2) $(\forall A)(\,PROB(\,A) = \Sigma_m\, prob(\,A \,/\, E_i)\; PROB(\,E_i)\,).$

So if (1) holds and (2) is violated, *PROB* will be incoherent and open to a Dutch Book of the standard kind. So if (2) is violated and *PROB* is coherent, (1) was also violated.

2. Now suppose the rule by which the agent moves from *prob* to *PROB* violates (1) and (2). That is, for the given changes of belief in the given *B*'s, his rule is such that for some *A*'s and some E_i's, $prob(\,A \,/\, E_i) \neq PROB(\,A \,/\, E_i)$. Let $\mathcal{A}$ be the set of *A*'s for which (2) is violated. Suppose further that the agent's rule is such that for the particular set of *B*'s, there exists an E_i in *S* such that for some *A* in $\mathcal{A}$, $PROB(\,A \,/\, E_i) < prob(\,A \,/\, E_i)$, no matter what the new degrees of belief *PROB* in the E_j's may be. Then a bookie who knows the rule will be able to make a Dutch Book against the agent by buying bets on $A \,/\, E_i$ at the cheaper rate, selling them at the higher rate, and selling a side bet. The bookie will

$$\text{sell} \quad \left[\begin{array}{ll} 1 & \text{if } A \,\&\, E_i \\ 0 & \text{if not} \end{array}\right. , \quad \text{and} \quad \left[\begin{array}{ll} prob(\,A \,/\, E_i) & \text{if } {\sim}E_i \\ 0 & \text{if } E_i \end{array}\right.$$

for $prob(\,AE_i) + prob(\,A \,/\, E_i)\; prob({\sim}E_i) = prob(A \,/\, E_i)$

$$\text{and buy} \quad \left[\begin{array}{ll} 1 & \text{if } A \,\&\, E_i \\ 0 & \text{if not} \end{array}\right. , \quad \text{and} \quad \left[\begin{array}{ll} PROB(\,A \,/\, E_i) & \text{if } {\sim}E_i \\ 0 & \text{if } E_i \end{array}\right.$$

for $PROB(\,A \,/\, E_i)$.

With these bets the bookie will come out ahead if E_i (by $prob(\,A \,/\, E_i) - PROB(\,A \,/\, E_i)$) and will tie if ${\sim}E_i$. To come out ahead in either event, he sells a side bet

$$\left[\begin{array}{ll} [\,prob(\,A \,/\, E_i) - PROB(\,A \,/\, E_i)] \,/\, 2 & \text{if } E_i \\ 0 & \text{if } {\sim}E_i \end{array}\right.$$

for $\tfrac{1}{2}\, prob(\,E_i)\; [prob(\,A \,/\, E_i) - PROB(\,A \,/\, E_i)]$.

For E_i's such that the inequality goes in the other direction, $PROB(\,A \,/\, E_i) > prob(\,A \,/\, E_i)$, the bookie will reverse the buying and selling of the first two pairs of bets and then sell a similar side bet.

3. In this section we will suppose the agent's rule leads to violations of Jeffrey's formula in a more complicated way. Suppose that for some A in $\mathscr{A}$, and for some E_i in S, the new degree of belief $PROB(A / E_i)$ is sometimes greater than, sometimes less than the old degree of belief $prob(A / E_i)$, depending upon the value of the new degree of belief $PROB(E_i)$. Note that here we suppose that the rule is such that only one value for $PROB(A / E_i)$ is associated with each possible value of $PROB(E_i)$. Let $\boldsymbol{X_i}$ be the set of all possible values of $PROB(E_i)$, and let $\boldsymbol{X_{i,<}}$ be the subset of $\boldsymbol{X_i}$ whose elements are such that $PROB(A / E_i)$ is less than $prob(A / E_i)$. Define $\boldsymbol{X_{i,>}}$ similarly. If a bookie knows the agent's rule to the extent the he knows the contents of $\boldsymbol{X_{i,<}}$ and $\boldsymbol{X_{i,>}}$, and if the following independence condition (C_3 in the second appendix to Skyrms (1979), interpreted as degree of belief $prob$ of future degrees of belief $PROB$) holds for A and E_i,

(3) $$prob[A / (E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}})] = prob(A / E_i)$$

(similarly for $\boldsymbol{X_{i,>}}$),

where $prob(PROB(E_i) \in \boldsymbol{X_{i,<}})$ is assumed to be nonzero, then the bookie will be able to make a Dutch Book by doing the following:

$$\text{sell} \quad \left[\begin{array}{ll} 1 & \text{if } A \mathbin{\&} E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}} \\ 0 & \text{if not} \end{array}\right.$$

$$\text{and} \quad \left[\begin{array}{ll} prob[A / (E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}})] & \text{if } {\sim}(E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}}) \\ 0 & \text{if } E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}} \end{array}\right.$$

for $prob[A / (E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}})]$.

Then if $PROB(E_i) \in \boldsymbol{X_{i,<}}$,

$$\text{buy} \quad \left[\begin{array}{ll} 1 & \text{if } A \mathbin{\&} E_i \\ 0 & \text{if not} \end{array}\right. , \quad \text{and} \quad \left[\begin{array}{ll} PROB(A / E_i) & \text{if } {\sim}E_i \\ 0 & \text{if } E_i \end{array}\right.$$

for $PROB(A / E_i)$.

With these bets the bookie will come out ahead if $E_i \mathbin{\&} PROB(E_i) \in \boldsymbol{X_{i,<}}$ and will break even if not, since by equality (3) the price of the first pair of bets is $prob(A / E_i)$, which is greater than the price of the second pair, $PROB(A / E_i)$, if $PROB(E_i) \in \boldsymbol{X_{i,<}}$. His guaranteed gain if E_i & $PROB(E_i) \in \boldsymbol{X_{i,<}}$ is given by $prob(A / E_i) - \alpha$, where $\alpha = max[PROB(A / E_i)$ given that the value of $PROB(E_i)$ is in $\boldsymbol{X_{i,<}}$].[1] To be sure of winning in every case the bookie sells the side bet

$$\begin{cases} [\,prob(A / E_i) - \alpha] / 2 & \text{if } E_i \,\&\, PROB(E_i) \in \boldsymbol{X_{i,<}} \\ 0 & \text{if not} \end{cases}$$

for $[prob(A / E_i) - \alpha]\, prob(E_i \,\&\, PROB(E_i) \in \boldsymbol{X_{i,<}}) / 2$. It is not necessary to construct the similar bets (with the buying and selling reversed) for $\boldsymbol{X_{i,>}}$ unless the assumption that $prob(PROB(E_i) \in \boldsymbol{X_{i,<}}) \neq 0$ fails, in which case the above bets would not have been made.

4. In this section we will first consider a generalization of the argument given in section **3**. In that section it was assumed that the agent's rule is such that the values of $PROB(A / E_i)$ vary as a function of the values of $PROB(E_i)$. This need not be the case; for some possible values x of $PROB(E_i)$, his rule may provide a set of possible values of $PROB(A / E_i)$, depending on the values taken by $PROB(E_j)$ for some or all of the E_j's in S. We can deal with this in a straightforward way by considering the function f_i mapping m-tuples $(x_1, \ldots, x_m)$, where each x_k is an element of $\boldsymbol{X_k}$, onto the set of possible values of $PROB(A / E_i)$ in the obvious way: $f_i(x_1, \ldots, x_m) = y$ iff the agent's rule is such that if $PROB(E_1) = x_1$ & … & $PROB(E_m) = x_m$, then $PROB(A / E_i) = y$. Call the set of all the m-tuples $\boldsymbol{Z}$ and define $\boldsymbol{Z_{i,<}}$ as the subset of $\boldsymbol{Z}$ such that if $(x_1, \ldots, x_m) \in \boldsymbol{Z_{i,<}}$, then $f_i(x_1, \ldots, x_m) = PROB(A / E_i) < prob(A / E_i)$, and similarly for $\boldsymbol{Z_{i,>}}$. Then if the following independence condition holds for A and E_i ,

(3') $\quad prob[A / (E_i \,\&\, (PROB(E_1), \ldots, PROB(E_m)) \in \boldsymbol{Z_{i,<}})] = prob(A / E_i)$

where $prob[(PROB(E_1), \ldots, PROB(E_m)) \in \boldsymbol{Z_{i,<}})]$ is assumed non-zero, a Dutch Book can be made like that described in section **3**. (Substitute the statement "$(PROB(E_1), \ldots, PROB(E_m)) \in \boldsymbol{Z_{i,<}}$" wherever "$PROB(E_i) \in \boldsymbol{X_{i,<}}$" occurs in the description of the bets.)

Now it is conceivable that further generalization of the argument is called for. If the agent's rule is such that something in addition to all the new degrees of belief in the E_i's helps determine the values of all the $PROB(A / E_i)$'s which are different from $prob(A / E_i)$'s (what could the extra factor plausibly be?), then the arguments above do not supply a Dutch Book against the rule. If the additional factor is such that (3') can be changed to

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[1] $\boldsymbol{X_{i,<}}$ may be such that α does not exist. And perhaps values of $PROB(A / E_i)$ are not bounded away from $prob(A / E_i)$. If α does not exist take the least upper bound of the values of $PROB(A / E_i)$; if the least upper bound is $prob(A / E_i)$, a different set $\boldsymbol{X}^{\alpha}_{i,<}$ can be chosen such that its members yield values for $PROB(A / E_i)$ less than some $\alpha < prob(A / E_i)$. As long as $prob(PROB(E_i) \in \boldsymbol{X}^{\alpha}_{i,<})$ is nonzero, the above argument works when $\boldsymbol{X}^{\alpha}_{i,<}$ is substituted for $\boldsymbol{X_{i,<}}$.

apply to conditionalization upon statements about it (the factor), then suitable modification of the bets in section **3** will give a Dutch Book if kinematics is violated:

(3") $prob[\, A / (E_i$ & the value of the factor is such that $PROB(\, A / E_i) < prob(\, A / E_i))]$

$$= prob(\, A / E_i).$$

If (3") is not satisfied, then violations of kinematics by the imagined rule will not leave the agent open to a Dutch Book. Of course, without knowledge of the nature of the suggested factor, we cannot judge the propriety of the change of beliefs nor whether a Dutch Book should be available.

5. What do the above arguments and their assumptions tell us about violations of probability kinematics? Let us review the assumptions. First we have the usual assumptions about the agent's betting practices—for example, that he is willing to accept any bet he regards as fair. These are of interest in assessing the significance of Dutch Book arguments in general, but they are not special assumptions of the kinematics Dutch Book argument. Next we have assumed (in sections **2**, **3**, and **4**) that the agent and the bookie know where the change of belief will originate, that they know the identities of the B's. There are several things to say about this. The assumption is also made in Lewis' argument for simple conditionalization, where it is imagined that a fully detailed partition describing the possible future experiences of the agent is available. The assumption need not be too strong—the exact identities need not be known, just some set of B's which contains those where the change originates. But we cannot remove the assumptions by blowing up the set of B's to include everything: *a*) that set gives us our partition of E_i's, and the larger the set of E_i's is, the stronger the independence assumptions used in sections **3** and **4** become; and *b*) if the set of B's includes *everything* then probability kinematics is trivialized—any A will be an element of the partition and (2) will be useless. In many practical situations, however, agent and bookie may have very good information about the origin of the agent's belief change (outcomes of experiments, elections, etc.). And if we see any significance in Dutch Book arguments generally, we may be willing to think that solid arguments for the desirability of a rule (in this case, kinematics) in ideal situations give us at least some reason to like the rule in other situations.

What about conditions (3) and (3')? What sort of independence conditions are they? They assert that when information about the new *degree of belief* in E_i (or in all the E_i's) is combined with information about the *truth* of E_i, only the latter and not the former influences the degree of belief in A. This is plausible, it seems, for most A's. For example, if in a series of coin tosses $PROB$ is the agent's system of beliefs after the ninth toss, $prob$[heads on toss 10 / (coin is fair & $PROB$(coin is fair) $\in$ (.2, .3))] = $prob$(heads on toss 10 / coin is fair) = ½. But there are clear cases in which it should fail, cases in which A is not independent from the agent's beliefs in the E_i's. A might, for example, be "$PROB(\, E_i) \in (.5, .6)$", or more generally, it might be a proposition about some part of the world which is correlated with features of that part of the world which includes the agent's belief states.[2] In these cases, then, the failure of the equality in condition (1) may be accompanied by a failure of the new degree of belief in *A*, *PROB*(*A*), to satisfy (2), Jeffrey's formula for kinematics. So what we have shown is that rules containing violations of kinematics for independent *A*'s lead to Dutch Books, while violations for correlated *A*'s need not.

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[2] For further discussion of (3'), see Brian Skyrms (forthcoming) on his "sufficiency condition" where he notes that the condition guarantees that second order conditionalization is equivalent to first order probability kinematics. He discusses the condition's plausibility under various interpretations of the distribution *PROB*. The relevant point here is that in the application we are making, *prob* represents present degrees of belief about, among other things, future final degrees of belief *PROB*. The condition seems very plausible in many situations under this interpretation. It is implausible if *PROB* is given a propensity or an observational degree of belief interpretation.