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Indisspensability and Explanation

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ABSTRACT

The question as to whether there are mathematical explanations of physical phenomena has recently received a great deal of attention in the literature. The answer is potentially relevant for the ontology of mathematics; if affirmative, it would support a new version of the indispensability argument for mathematical realism. In this article, I first review critically a few examples of such explanations and advance a general analysis of the desiderata to be satisfied by them. Second, in an attempt to strengthen the realist position, I propose a new type of example, drawing on probabilistic considerations.

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1 Introduction

The indispensability argument (IA) for mathematical realism was advanced by Quine and Putnam many years ago, but has been recently revived in a more specific form, which can be called 'explanationist'. This version was first intimated, curiously, by the arch anti-realist Hartry Field ([1989], pp. 15–7); the underlying idea is to combine the powerful and more general scientific realist argument from 'inference to the best explanation' (IBE) with the view that mathematics is indispensable to science. Field, however, has never fully

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articulated an argument to this effect; the insight has been developed by Alan Baker in two recent articles ([2005], [2009][1]).

But why do realists need a new version of the traditional IA? A good way to understand their motivation is to review, briefly, the traditional IA as well as some of the criticisms levelled against it.

Mark Colyvan ([2001], [2008]) presents the argument as follows:

(i) Mathematical entities are indispensable to our best scientific theories.

(ii) We ought to have ontological commitment to all and only the entities that are indispensable to our best scientific theories.[2]

From (i) and (ii), the conclusion is drawn that we ought to have ontological commitment to mathematical entities. More precisely, a scientific realist must be a mathematical realist: she must take numbers to be part of her ontology, along with electrons and genes. Thus, in light of IA, a scientific realist who doesn't acknowledge herself as a mathematical realist is operating with a 'double standard' with regard to ontology (Quine [1980], p. 45), and thus is guilty of 'intellectual dishonesty' (Putnam [1979], p. 347).

Its validity being beyond question, the criticisms of the argument focused on the credibility of the premises. Notably, Field ([1980]) tried to show that (i) is false.[3] Penelope Maddy ([1997]) doubted the 'all' part of premise (ii), i.e. confirmational holism.[4] More recently, a novel criticism against IA was launched by Joseph Melia ([2000], [2002]). Melia, a scientific realist, accepts (i) and attacks the 'all' part of (ii) as well, but from a somewhat different angle. He argues that one is not intellectually dishonest when refraining from making ontological commitments to everything that our theories quantify existentially over.[5] Mere indispensability of a posit is not enough, according to Melia; a posit must be indispensable 'in the right kind of way', as Baker ([2009], p. 613) put it, and one specific role a posit must necessarily play is in formulating explanations.

Accordingly, the question Melia raises is to what extent mathematical posits (can) fulfil this role. In essence, he proposes that to grant ontological rights to mathematical objects only when positing them 'results in an increase in the

1 In this latter article, Baker calls it 'the Enhanced Indispensability Argument'. Steiner ([1978a], [1978b]) is perhaps the first philosopher to reflect on this issue systematically. (See Baker [2009] and Leng ([2005], [2010]) for recent reactions to his work.)

2 This premise incorporates Quine's well-known criterion for ontological commitment: 'a theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true' ([1948], p. 33). For a recent discussion of this criterion, see Azzouni ([1998], [2004]).

3 Many remain unconvinced, for a variety of reasons. As far as I can tell, Malament's ([1982]) objections still stand.

4 Sober ([1993]) objects to confirmational holism too, on the basis of his views about confirmation as being contrastive.

5 Melia's strategy is called 'weaselling'; see (Melia [2000]). (Colyvan [2010]) is a recent criticism of this strategy.

same kind of utility as that provided by the postulation of theoretical entities' ([2002], p. 75), where, as noted, explanation is a primary example of such utility. Now, one reading of this proposal is that it advances a criterion for ontological commitment linking existence with causal explanation—precisely because theoretical entities are typically taken to play a causal role in explanations.[6] Thus, the criterion for ontological commitment would change into 'to be real is to fall within the range of an existential quantifier and to have a causal explanatory role'. But, for a naturalist–indispensabilist–realist, this appeal to causality marks an unwelcome return to metaphysics (or 'first-philosophy')[7], hence it is a *prima facie* dubious move; on the other hand, if the causal requirement is dropped from it, the criterion can be accepted by the indispensabilist–realist. So, I submit, it is this construal of Melia's challenge that should be dealt with and this is something that realists attempted to do, as we'll see below.

Returning to the main argument, recall that at the very beginning I attributed the proto-explanationist version of IA to Hartry Field. More concretely, the line of thinking he envisages is as follows: Suppose we hold a belief about a physical phenomenon, and we present the best explanation of that phenomenon. Furthermore, suppose that a certain claim *K* is part of this explanation, and no explanation of the phenomenon is possible without holding claim *K*. Field notes:

> [i]f a belief [*K*] plays an ineliminable role in explanations of our observations, then other things being equal we should believe it, regardless of whether that belief is itself observational, and regardless of whether the entities it is about are observable. ([1989], p. 15).

The relevance of this idea for the issue of mathematical realism is clear. If a physical phenomenon is best explained by making several assumptions, and at least one is an ineliminable *mathematical* claim *K*, then IBE entitles us to believe that *K* is true, and that the mathematical posits making it true exist.

Yet, as I pointed out previously, this insight had not been worked into a full explanationist–indispensabilist argument until very recently, when Baker

6 To use a well-known example, the neutrino was postulated (by W. Pauli) to explain the missing amount of energy in a beta decay. This explanation can be said to be 'causal', in so far as it appeals to the 'causal power' of the particle (its mass–energy).

7 As Quine points out, 'the notion of cause is out of place in modern physics [. . .] Clearly the term plays no role at the austere levels of the subject' ([1974], p. 6) and '[s]cience at its most austere bypasses the notion [of cause]' ([1990], p. 76). By overlooking this aspect of the indispensabilist position, Field's (and others') considerations from the role of causality in the decision to believe in the existence of mathematical entities are beside the point. (See Field ([1980], p. 43), ([1989], pp. 18–20), etc.). Burgess and Rosen ([1997]) also highlight the difficulties to pin down the distinction between concrete and abstract, when this distinction is spelled out as the distinction between causally efficacious versus causally inert.

did it. He maintains that we ought to believe in the existence of mathematical objects because

(i) we ought to believe in the existence of any entity that plays an indispensable explanatory role in our best scientific theories, and

(ii) mathematical entities do play such a role. (Baker [2009]).

This version of the argument is of course valid, yet two aspects of it require clarification.

First, how do mathematical entities enter the picture? The answer an explanationist–indispensabilist gives is that they are the truthmakers of the true mathematical statements appearing among the explanans. Belief in the existence of the mathematical objects is required by assuming the connection between what Shapiro ([1997], [2000]) calls 'realism in truth-value' and 'realism in ontology'. Shapiro notes that the preliminary conclusion for which the indispensabilist argues is that we 'must accept mathematics as true' ([2000], p. 216). An explanationist can make this more precise: we must accept as true the mathematical components of the explanans of the best explanation. Accepting mathematics as true amounts to realism in truth value and, as Shapiro puts it,

> we get to realism in ontology by insisting that the mathematics be taken at face value, just as we take physics at face value. Mathematical assertions refer to (and have variables ranging over) entities like real numbers, geometric points, and sets. Some of these assertions are literally true. So numbers, points, and sets exist. ([2000], p. 216)[8]

The second important aspect to be clarified is what sense of explanation is used in this debate. Generally speaking, it is assumed that (i) the explanations have the form of an argument in which the explanandum is the conclusion to be derived from the premises (the explanans), and (ii) the explanans of good explanations have to be true. Claim (ii) is potentially contentious because not everyone agrees that these mathematical statements have to be true to serve their role (those arguing along these lines typically endorse a form of fictionalism; see Leng ([2005], [2010])). However, in what follows, I assume—together with other indispensabilist–explanationists—that we should require the truth of the explanans (hence of their mathematical components) in a good explanation.[9] Another typical realist assumption is also made here, namely, that the simpler and more unifying an explanation is, the better. These points

[8] Note, however, that this last inference is controversial. Helman ([1989]) and Chihara ([1990]), for instance, reject it. They develop philosophies of mathematics which construe the truth of the mathematical statements as not requiring the existence of mathematical 'objects'. See Shapiro ([1997]) for criticisms of this strategy.

[9] I suspect that the fictionalist conflates the explanatory role of the mathematical posits with their representational (modelling) role. Baker ([2009], pp. 625–7) offers a more elaborate defense of the conflation charge, and I'd direct the interested reader to his arguments.

should be kept in mind, as they will be relevant for assessing the new example of an explanation I propose later on.

A look at the recent literature reveals that both realists and nominalists agree on where the disagreement between the two metaphysical camps lies[10]: it is premise (ii). Both parties accept that mere indispensability might not be enough. The question agreed by both sides to be crucial is whether convincing examples can be found, in which, as Melia insisted, the best explanation of a physical phenomenon features mathematics in an essential way. Thus, the realists have set out to meet Melia's challenge, as clarified previously. Such examples have been proposed, and extensively discussed recently, especially by Colyvan ([2001], [2008]).[11]

Against this background, let me finally state my goals in this article. I introduce and discuss a number of examples, and I advance an analysis of the desiderata that these explanations must satisfy to be able to support a realist ontology. After I show why a new example is needed, I present such an example and argue that it satisfies the constraints I outline in the first part of the article.

2 Mathematical Explanations

A recent collection of interesting mathematical explanations is in Colyvan ([2001]). They are drawn from a wide range of scientific fields, from meteorology to special relativity and dynamic non-linear systems theory. Consider one from meteorology: given a certain moment of time, we want to explain why are there two antipodal points, P_0 and P_1, on the earth's surface with the same temperature and barometric pressure ([2001], p. 49)? As Colyvan argues, it is the Borsuk–Ulam theorem in algebraic topology that constitutes an essential part of an explanation as to why such points exist. Another more recent example appeals to the explanatory power of a mathematical construct called the phase space. On Colyvan and Lyon's account (which develops a suggestion of Malament [1982]), the introduction of this notion (together with the concept of a Poincare map) has a crucial role in explaining why 'high energy Henon–Heiles systems exhibit chaotic and predictable motion and why low energy ones exhibit regular and predictable motion' (Colyvan and Lyon

[10] (Baker [2005]) is an intervention in a debate in *Mind* between Colyvan ([2002]), on the realist side, and Melia ([2002]), on the anti-realist (or 'nominalist') side. Meanwhile, a growing number of other authors joined the discussion–Azzouni ([2004]), Leng ([2005], [2010]), Pincock ([2007]), Saatsi ([2007]), Mancosu ([2008]), Bangu ([2008]), Daly and Langford ([2009]), Batterman ([2010]), Colyvan and Bueno ([2011]), and so on–with divided sympathies. I don't have space here to get into any of the details of these positions, but the interested reader should be warned that there are subtle differences even among the advocates of the same orientation, either realist or antirealist.

[11] (Balaguer [1998]) contains interesting examples too. Baker's cicada example, in (Baker [2005]) is also central in this context, and I'll comment on it separately later on.

[2008], p. 56). Colyvan also discusses Minkowski's geometrical explanation of certain relativistic effects (such as Lorentz contraction) and points out that it is 'not obvious how such explanations could be obtained otherwise'—that is, other than by introducing mathematical assumptions ([2001], p. 50).

Some of the examples mentioned involve explanatory power indirectly, by drawing on the well-known connection between unification and scientific explanation (Friedman [1974], Kitcher [1981], Morrison [2000]). More specifically, the employment of certain mathematical concepts and structures often results in unifying a scientific theory, and hence contribute to enhancing its explanatory power. Arguing along these lines, Colyvan points out that it is precisely this feature that is 'hard to reproduce without mathematical entities' ([2001], p. 89).[12]

2.1 'Simplicity'

Although I generally agree with all this, let's observe that these examples come from scientific fields characterized by a high degree of mathematical complexity (applied algebraic topology, chaos theory, and so on). 'Complexity' is of course a vague notion, but it is unquestionable that these examples don't feature anything properly called elementary, familiar, or simple. Note, further, that by its very nature the challenge posed to the nominalist seems to require an appeal to this complexity. The examples presented to the nominalist must be non-elementary—*hard*—precisely because it must be hard to see how the explanatory power of the mathematized theories can be reproduced without mathematics. The idea is that the nominalist must be overwhelmed by the complexity of the example and declare that nominalization manoeuvres (such as the one described in Footnote 12) aren't in sight, and thus mathematics is indispensable to formulating the explanation. By presenting complex examples, the realist wins. But this victory is not as crushing as it could be, were the examples simpler. Faced with a very complex example, the nominalist might admit defeat; yet, he is entitled to claim that this defeat is only temporary. He replies that nominalization is still possible, but just difficult to achieve right away, as the theory is so complex—call this 'the complexity excuse' for failing to nominalize.

[12] In the current jargon, to purge a statement of its mathematical constituents is to 'nominalize' it (Field [1980]). The statement 'there are two bananas on the table' seems to make reference to a mathematical object, the number two. Indeed, one way to reformulate it is as 'the number of bananas on the table is two'. However, we can show that there is yet another way to reformulate it in first order logic dispensing with reference to numbers altogether: $\exists x \exists y (Fx \wedge Fy \wedge x \neq y \wedge \forall z (Fz \supset (z = x \vee z = y)))$, where F is the concept 'banana on the table'. The nominalist project is to treat every single scientific theory (and thus every scientific explanation) in this spirit.

Moreover, the complexity of an example, by making nominalization hard to achieve, generates two unwelcome effects. For one thing, it doesn't allow us to see how mathematics is actually explanatory; given the entanglement of highly sophisticated mathematical and physical assumptions, the role of mathematics becomes rather difficult to discern. Furthermore, because the nominalization is not available, a comparison of the two versions of the explanations (the mathematized and the nominalized one) becomes impossible. This seems to me a disadvantage for the realist: in so far as a nominalized version of the explanation is not entirely explicit, she can't credibly argue that the mathematized version of the explanation is better (the best).[13]

These points should reveal the virtues of some simpler examples. While it would indeed be much riskier for the realist to test whether the nominalist can nominalize a simple example; such an example naturally increases the nominalist's chances to succeed in dealing with it. Hence, a more promising approach for the realist would be to challenge the nominalists by presenting them with examples as elementary as possible, in which the appeal to some basic mathematical vocabulary significantly enhances our explanatory resources. An elementary example resisting nominalization will still amount to a plausibility argument for realism; although nothing can guarantee that all nominalization attempts will fail, such an example is more convincing in so far as the complexity excuse will no longer be available to the nominalist.

To sum up, this desideratum—call it 'Simplicity'—is reasonable because it makes the challenge to present a successful nominalization more pressing for the nominalist. The failure to succeed in nominalizing a simple example is surely more telling in favour of realism than the failure to nominalize a highly complex one. In addition, because it is easier to carry out a successful nominalization, then, when this is done, a comparison becomes possible between the explanatory virtues of the mathematized and the nominalized versions of an explanation.

Before I discuss an example that satisfies the simplicity desideratum (by Baker [2005]), let me sketch out two other desiderata that should regulate the use of the IBE strategy when employed to support mathematical realism.

If we recall Field's idea from the Introduction, we note that a realist eager to use it faces the following dichotomy. Either

(i) the explanandum/conclusion contains ineliminable (i.e. non-nominalizable) mathematical vocabulary, or

(ii) the mathematics is eliminable.

If (i), the realist's task becomes extremely difficult. Roughly speaking, the problem is this: The realist has to take the explanandum to be true (otherwise

[13] Not a serious disadvantage though; I agree with one of the referees for this article, who pointed out that until an attractive nominalization is available, the realist wins.

why bother explaining it?) and, in this case, the mathematical part of the explanandum (which, again, can't be eliminated) has to be true as well. But, if this is so, she just assumes realism before arguing for it, so she would beg the question against the nominalist. Consequently, the second desideratum is that the explanandum be nominalizable ('nominalize' for short).[14]

Branch (ii) is more promising for the realist. In this case, the mathematical component of the explanandum is there but only superficially, as it can be eliminated via a nominalization procedure. So, by taking the explanandum to be true, the realist doesn't beg the question anymore. But now the nominalist needs to be provided with a further reason to see how the mathematical explanans (which must occur as part of the premises to talk about mathematical explanation in the first place) can have any explanatory relevance for an explanandum that is in fact non-mathematical, that is, free of any traces of mathematical vocabulary. Thus, the realist needs a further argument to show that (at least one of) the explanans she uses to derive the explanandum contains indispensable mathematical components. If they (these premises–explanans) can be nominalized too, then what we get in the end is a nominalizable explanandum (the conclusion) and nominalizable explanans (the premises). Thus, to the nominalist's satisfaction, it turns out that mathematics was there only to capture (represent, describe, model, and so on) some essentially non-mathematical content (the premises–explanans and the conclusion–explanandum) in a more elegant fashion.

So, the challenge to the indispensabilist realist is to show that the mathematics in the explanans is indispensable, given that the conclusion is nominalizable. (I'll abbreviate this desideratum as 'indispensability'.) That the mathematics in the explanans is not eliminable needs to be shown each time one proposes a mathematical explanation, as a matter of principle. However, whether this can be done in each case can't be decided in advance, as it depends on the specific form of the explanans.

Importantly, note that 'indispensability' is only a sufficient condition, not a necessary one. It has to be balanced against the fourth (and essential) desideratum—namely, that the mathematics involved in the premises be genuinely explanatory (desideratum 'explanation' for short). I'll discuss this constraint

[14] This is essentially the point I made in my ([2008]). The problem is even more pressing for the mathematical explanations of mathematical statements. Such explanations can't count as supporting mathematical realism, as Leng correctly notes:

> one might wonder why it is mathematical explanations of physical phenomena that get priority. For if there are [. . .] some genuine mathematical explanations [of mathematical facts] then these explanations must also have true explanans. The reason that this argument can't be so is that, in the context of an argument for realism about mathematics, it is question begging. For we also assume here that genuine explanations must have a true explanandum, and when the explanandum is mathematical, its truth will also be in question. ([2005], p. 174)

later, in the context of the new example; however, before we move on, let me clarify why 'indispensability' is not necessary. Indispensability realists say that mathematics is indispensable in the sense that it is needed in achieving certain goals. In this case, the goal is maximal explanatory power (other goals, such as descriptive accuracy are often mentioned). So, even if a nominalist manages to eliminate the mathematics from the explanans of a scientific explanation, his case against the realist is not yet completed. He still has to show that the nominalized version of the explanation is better—more attractive—than the mathematized version. So, if desideratum 'explanation' is satisfied and the mathematics in the explanans is explanatory, but by nominalizing the explanans the theory loses explanatory power, then achieving a nominalization of the explanans is a hollow victory. The realist is entitled to claim that the mathematized version of the explanation is superior, hence she is in the possession of 'the best explanation'.

Back to the discussion of the examples in the literature, Baker ([2005]) deserves special mention here. His article argues that mathematical assumptions feature essentially in the explanation of the fact that the life cycle of North American magi-cicadas is a prime number. From the perspective I've articulated here, what is important about this example (its intrinsic merits aside) is that its complexity is rather minimal. (In essence, it is two lemmas drawn from number theory that constitute the gist of the explanation. See Baker [2005], p. 232)). Although desideratum 'simplicity' is met, the cicada example is not without problems, as I argued in my previous article ([2008]) (for the larger context of this argument, see (Mancosu [2008], Section 3])). The key difficulty is that this example seems to fail to satisfy desideratum 'nominalize'. Tellingly, Baker ([2009], p. 619) revisits this issue and discusses the nature of the explanandum—'the cicadas' life-cycle is a prime number'—in the end acknowledging that it can't be nominalized! However, even assuming that this difficulty with the cicada example gets sorted out eventually, I should emphasize that a new example featuring the same low degree of complexity is necessary.[15] This is so not only because the cicada example has already been attacked on other grounds as well,[16] but also because the plausibility of the explanationist

[15] To clarify: (Baker [2009]) also contains an argument to the effect that despite this feature of the example, the charge of circularity I raised in my ([2008]), can be avoided. Therefore, until some more careful assessments of Baker's response are available, I should say that what I count against the cicada example here is only the suspicions that it might fail to satisfy constraint 'nominalize'. So, while I personally am not convinced of Baker's response to my criticism, I'm ready to admit that it might work. Yet, even if it does, a new example, free of such suspicions, should be welcome by the realists.

[16] See also (Leng [2005]) and (Saatsi [2007]). Daly and Langford ([2009]) point out that the fourth desideratum (to use the terminology introduced here) is not satisfied. (Baker [2009]) addresses these criticisms, but (Rizza [2011]) raises a new one.

version of the indispensability argument is better supported by expanding the number and variety of cognate examples.[17]

In what follows, I propose such a new argument example.[18] After I describe it, I point out the serious difficulties faced by the nominalist when attempting to offer a non-mathematical treatment of it. I shall pay special attention to showing that all four desiderata mentioned previously are satisfied in a very natural way.

3 An Average Story: The Banana Game

Consider the following game, played by two people, call them A and B. They start by making two large crates, labelled 'X' and 'Y'. Crate X contains two identical smaller crates, labelled x_1 and x_2, each containing bananas. Similarly, crate Y contains two smaller crates, labelled y_1 and y_2, each containing bananas as well. A possible distribution of them would be: five bananas in x_1, seven in x_2, one in y_1, and nine in y_2. Such a distribution will be described as [(5, 7); (1, 9)]. Of course, nominalists can't express this information in this way, as they lack numbers.[19] But they are able to do pairings (one-to-one correspondences) and apply predicates such as '_ contains more bananas than _' or '_ contains as many bananas as _' correctly when presented with any two crates. (They just take all bananas out from the two crates they want to compare and then form the pairs). Thus, the nominalists have no difficulty noticing that, for example, crate x_1 contains more bananas than crate y_1 and fewer bananas than y_2, or that y_2 contains more bananas than any other small crate. They also have access to the fact about the two large crates that 'crate X contains more bananas than crate Y in total'. (Obviously, they would have noticed this if they had availed themselves of numbers and, by counting the bananas, they had gotten twelve bananas for crate X and ten for Y.)

The rules of the game (call it 'Game') are as follows:

(a) The players know what is in each large crate, namely, that X and Y contain small crates; they also know what is in each small crate.

(b) One player (the first player, say A) is asked to choose either crate X or crate Y. The other player (B) will be left with the other large crate.

[17] Baker himself invites such developments: 'it is clearly less than ideal to rest the argument for the existence of abstract mathematical objects on a single case study from science. Thus one line of further inquiry on the platonist side is to look for more good examples of mathematical explanation in science' [2009, p. 631].

[18] Its conceptual background is in economics. Similar games are discussed in von Neumann and Morgenstern's *Theory of Games and Economic Behavior* ([2007]; first published in 1944).

[19] As a referee pointed out, can't the nominalists express the content (for example, 5 bananas in x_1, 7 in x_2) in the familiar way using first-order logic and identity, as explained in Footnote 12? They surely can; but they can't use the signs '5' or '7' to express this content in the same way the realist uses them, as standing for some objects.

(c) Once A decides on a large crate, the choice of a small crate inside it is purely chancy. However, the probabilities of choosing x_1 or x_2 are equal, and the same goes for y_1 and y_2.

(d) Once a small crate is picked, A collects all the bananas in it; B does the same.

(e) The goal of the game is to collect the most bananas. So, player A should make the initial choice of the large crate with this goal in mind.

(f) The choice of the large crates X or Y is made only once, at the beginning of the game. If A has chosen to withdraw bananas from large crate X (say), then he'll keep doing this until the end of the game.

(g) After a small crate is emptied, it is refilled with as many bananas as there were in it initially.

More concretely, the 'Game' is played as follows: Let us say that player A goes for crate X, so B is left with crate Y. Now, let's say A happens to pick small crate x_1 from large crate X, then he will collect all bananas in it. B, the second player, is left with large crate Y, and he randomly picks a small crate inside it. Let's say he picks y_1, then he collects what's in it, and the first run ends. Now crates x_1 and y_1 will be refilled with the same number of bananas they contained initially, and a new turn begins, with the first player picking from crate X. After many turns—say, a whole day of playing—the two players' piles are compared by using the predicate '_ contains more bananas than _'. Let's also assume that, quirky characters as they are, the two players enjoy the Game and play it for weeks. At the end of one day, after many turns, the two players compare their piles, and the winner is recorded. The next day, they play again, with player A making his/her choice of a large crate, followed by many turns. The winner is recorded again, and so on.

What is the outcome of the Game? Before start of play, the two large crates don't seem very different. The total amount of bananas each of them contains is not markedly disproportionate (twelve versus ten), so the thought that each large crate will win some runs would sound reasonable; in other words, it might seem that any large crate can be chosen and only luck will decide. Yet, as the game unfolds, and the days pass, the players observe an interesting regularity, or pattern: there is a noticeable discrepancy in how many times one and the same large crate wins, when compared with the other. Thus, one of the crates (as it happens, crate X) wins overwhelmingly often (and, naturally, the other one, crate Y, loses almost all the time).

Consider, furthermore, a new game (call it Game*) within the following set-up. There are 0 bananas in x_1, 16 in x_2, 12 in y_1, 12 in y_2, and 6 in y_3. The rules are the same as for the first Game, except for rule (c) that fixes the probabilities:

$$Pr^*(\text{pick } x_1) = \frac{1}{4}$$
$$Pr^*(\text{pick } x_2) = \frac{3}{4}$$
$$Pr^*(\text{pick } y_1) = \frac{1}{6}$$
$$Pr^*(\text{pick } y_2) = \frac{2}{6}$$
$$Pr^*(\text{pick } y_3) = \frac{3}{6}$$

Note that although the nominalist can't use numbers, specific probabilities (such as $\frac{1}{2}$ or $\frac{3}{4}$) are still accessible to him. They can be constructed as follows: Assume there is some randomization device yielding equi-probable outcomes. For the first Game, a fair coin will do. A convention may be set up: if the coin lands heads-up, then the player who collects from crate X will collect from x_1; if it lands tails-up, he'll collect bananas from x_2. A similar convention would work just as well for crate Y and player B. Same reasoning can be used for Game*. To understand the probability of $\frac{1}{4}$ (say) nominalistically, we introduce something like a physically symmetric four-sided die, whose faces (call them s_1, s_2, s_3, s_4) are equi-probable. Thus, we convene that x_2 would be selected when the four-sided die lands any of faces s_1 to s_3 up, and we select x_1 when the four-sided die lands face s_4 up.

On playing Game*, the players notice, again, that one of the crates wins most of the time (crate X). Again, the wins are not evenly distributed, as one might think on a superficial examination of the set-up: one crate, X, is (almost) always the winner. In fact, this result is puzzling when compared with what happened in the Game: while the winning crate in Game contained more bananas than the losing crate (twelve vesus ten), in Game*, the winning crate contains fewer bananas than the losing one (sixteen versus thirty).

Now, the players notice that the two games resemble one another in this interesting respect: *in both games, one crate wins overwhelmingly often*. Is this a mere accident, they wonder, or a fact that can be explained? Call the italicized sentence 'the explanandum'. Thus, the task set for the nominalist and the realist is to identify what can account for it, or in other words, to explain what makes the two games alike in this regard, i.e. unidirectional. This is, I claim, something that the realist can handle relatively easily, whereas the nominalist faces serious difficulties.

Let's begin with the realist. She will proceed by introducing the elementary mathematical notion of an expectation value, that is, a number associated with picks from each large crate. More precisely, if we symbolize the expectation value associated with picks from X and Y by $[X]$ and $[Y]$, respectively, in Game we have[20]:

$$[X] = Pr(\text{pick } x_1) \times (\#\text{ of bananas in } x_1) + Pr(\text{pick } x_2) \times (\#\text{ of bananas in } x_2)$$

$$[Y] = Pr(\text{pick } y_1) \times (\#\text{ of bananas in } y_1) + Pr(\text{pick } y_2) \times (\#\text{ of bananas in } y_2)$$

Because all probabilities are equal to $\frac{1}{2}$, given the numbers of bananas in each small crate, we calculate $[X] = 6$ and $[Y] = 5$.

Moreover, the realist observes that there is a way to know what happens in the long run. The player who selected crate X will gather approximately $([X] \times n)$ bananas after n runs, and the one who chose Y will collect $([Y] \times n)$ bananas. This claim is justified by applying the well-known mathematical theorem called the '(weak) law of large numbers' or wLLN for short. Consider $V_1, V_2, \ldots, V_n$, a sequence of independent and identically distributed random variables with finite expected values $\mu = [V_1] = [V_2] = \ldots = [V_n]$ and finite variance. Let $\overline{V_n} = \frac{1}{n}(V_1 + V_2 + V_3 + \ldots + V_n)$ be the arithmetic mean (or 'sample average') of these variables. The (weak) law of large numbers states that the arithmetic mean converges in probability to the expected value. (More precisely, for any positive number ε, $P(|\overline{V_n} - \mu| < \varepsilon) \rightarrow 1$, as n approaches infinity.) In essence, the theorem says that as n increases, the sample average $\overline{V_n}$ gets closer and closer to the expected value, μ.

The application of this theorem to our game is immediate. Consider crate X. At every turn, a pick from crate X can yield either small crate x_1 (i.e. five bananas) or x_2 (i.e. seven bananas), each of them with probability $\frac{1}{2}$. Thus, a pick from crate X can be regarded as yielding values for a random variable (i.e. two values, either five or seven) with equal probability $\frac{1}{2}$. Let A_1 be this random variable, associated with the first pick from X. Because A_1 takes numerical values (either five or seven) with probability $\frac{1}{2}$, the expected value is $[A_1] = 6$. The same holds for the second, third, etc. pick from X, so let us call these random variables $A_2, A_3, \ldots, A_n$. The expected values are $[A_2] = [A_3] = \ldots = 6$. In this case then, $\mu = 6$. Now, consider the arithmetic mean $\overline{A_n} = \frac{1}{n}(A_1 + A_2 + A_3 + \ldots + A_n)$. We know from wLLN that for sufficiently large n (that is, after many turns), we have that $\overline{A_n}$ gets arbitrarily close to μ. Or, equivalently, the quantity of interest (i.e. the quantity of bananas in the first player's pile) is $(A_1 + A_2 + A_3 + \ldots + A_n)$, and it approaches $n\mu$. More concretely, after (say) $n = 10{,}000$ turns, the first player's pile will contain approximately 60,000 bananas, whereas the second player's will contain

[20] How does a realist know that such a function exist? As it turns out, the existence of a function $[\cdot]$ having the needed mathematical properties can be proved axiomatically (from three conditions that hold for our games). A proof of the existence of a utility function (what function $[\cdot]$ actually is) was given by von Neumann and Morgenstern); for a more modern presentation, see theorem 8.4 in Fishburn ([1970], pp. 112–5). I thank Professor Teddy Seidenfeld for drawing my attention to this literature.

approximately 50,000. The difference in the amount of collected bananas increases with n and, given that $[X] > [Y]$, even if some statistical anomalies occur now and then, it is clear that the pile associated with X will be recognized as amassing more bananas than the one corresponding to Y (almost) all the time.

The crucial point is that the result in the first Game (one crate winning almost always) tends to occur because of an inequality of expectation values: the value corresponding to crate X is higher than the one corresponding to Y. Essentially, the same reasoning can be transferred to the other game, Game*. In this game, we have that the expectation value associated with one crate (X) is higher than the expectation value of the other (Y): compare $[X^*] = 12$ with $[Y^*] = 9$. Hence, if one wants to know what is common to both games, and thus what accounts for the explanandum, the realist offers this: in both games, we have an inequality of expectation values. A common feature of the games was identified, and this is what explains why the two games evolve the same way in the long run. This feature has been shown, in a rigorous fashion, to be responsible for the observed unidirectionality, that is, the explanandum. This explanation is given in terms of a simple mathematical notion ('expectation value'), so we are entitled to count this explanation as a mathematical one.

To ensure that this example complies with 'nominalize', the issue in need of clarification at this point is whether the explanandum is an unquestionable physical fact, expressible in nominalistic language. The method to deal with this problem has been presented already, and now I'll add a few, hopefully useful, details. Let's focus on Game. As we saw, using the pairing method, the players notice that the pile collected from crate X contains more bananas than the pile originating in crate Y. They record the result of the first day of playing: next to crate X, they scratch a mark on the ground—say ϒ. As the Game progresses, after many days, the marks recording the wins for the two crates accumulate (say the win mark for Y is ⅄). Now, after many rounds (days), they compare the two collections of physical marks using, again, the nominalistically acceptable procedure of pairing them. They note that there are overwhelmingly more ϒs than ⅄s, and this gives the outcome of Game: one crate (X) wins. The players note a striking imbalance, as the wins don't switch from one crate to the other: the vast majority of wins corresponds to only one crate.

In essence, the same nominalistically acceptable procedures are used for Game*. As we saw, the selection of small crates is made by tossing a fair four-sided device and a six-sided one. In this game, the players also note a conspicuous imbalance. Again, this game goes in one direction too: the wins don't shift from time to time, but one crate collects them all (almost). The outcome of Game* can be expressed in nominalistic terms, like the outcome of Game. Given that the outcome of Game* is the same as the outcome of Game (both are clearly unidirectional one-crate winners), the players wonder if this is a mere accident or can it be explained? But, recall that this is the explanandum: in both games, one crate almost certainly wins. This is a fact accessible to a nominalist, insofar as the outcome of each game is, as we saw, nominalistically expressible.

3.1 Some clarifications

Two contentious aspects of this example must be discussed before we evaluate the nominalist resources to deal with it. First, one might be bothered by the fact that the outcome of the games is not guaranteed to obtain; as said, crate *X* wins 'most of the time' and not 'all the time' ('overwhelmingly often' and not 'always'). Indeed, it's possible that crate *Y* wins more rounds, and thus balances out the wins of crate *X*; hence, the players won't be stricken by the imbalance and won't take the outcome of each of the games to be 'one crate wins overwhelmingly often'. Thus, at this point, an important empirical assumption under which this example works needs to be made fully explicit: namely, that the games are played in a world in which the (weak) law of large numbers holds. The example stands or falls with this assumption; if the wins are more evenly distributed, then such an example is out of the question, as the explanandum is not true. But, as far as we know, this assumption is true: we do live in such a world, so the outcomes and the explanandum, as described here, do obtain. Virtually all physical processes we have ever observed strongly corroborate this assumption.[21] This is precisely why the condition imposed on these games is that they are played for a large number of rounds.

The second problematic issue is whether the nominalist can accept this example in the first place because it is couched in terms of 'games' and these are presumably abstract objects. This is a fair point, but one should keep in mind that we talk of games here in purely operational terms, as a succession of procedures and manipulations of otherwise unremarkable physical objects (crates, bananas, scratches on the ground, and so on). Moreover, one should also realize that if such over-stringent nominalist constraints are imposed, virtually every example is blocked, including the cicadas one (on the face of things, it requires at least quantification over species, which are, one might argue, abstract entities too). Here, although quantification over games might be necessary, if we want to express the explanandum in first-order logic, the operational way in which it is meant to be understood doesn't existentially commit us to anything beyond mere physical objects.

[21] Coin tossing is perhaps the simplest example of wLLN holding. In fact, an approximately equal number of heads and tails begin to appear after a rather small number of tosses (hundreds, or even tens).

Essentially, in this example, unlike the cicadas one and others, no mathematical object or property appears in the formulation of the explanandum.[22]

3.2 Hopes and troubles for the nominalist

So, what are the nominalist resources to deal with this example? If the nominalist neglects utterly irrelevant features of the crates (such as them having different colours, different textures, and so on), he will notice that the following conditions hold true of Game:

(a) There are more bananas in one crate (the one that eventually wins) than in the other one.

(b) There are as many small crates in one large crate as there are in the other one.

(c) The probabilities of selecting small crates from the large crates are equal.

Note that each of these three conditions is nominalistically acceptable. The nominalist's hope is that the conjunction of these conditions reveals what is distinctive about Game. So, he advances the following 'qualitative' explanation (call it **Q**) for the outcome of Game:

> Assume (a), (b), and (c). Therefore, one crate (*X*) wins overwhelmingly often.

But there is a problem. How does the nominalist show that this conditional statement ('If (a), (b), (c), then one crate wins overwhelmingly often') holds? Or, equivalently, that these three conditions are sufficient to derive the consequent? More precisely, the question is why does he list only these three conditions? Why not add a fourth one, for instance, that 'one large crate must not contain a small crate that contains fewer bananas than any other small crate'? This condition holds true of the set-up of Game, and might be relevant for the outcome. Why isn't such a condition explanatory, like the previous three?

Thus, there is apparently no principled reason to exclude a fourth, fifth, and so on, condition from formulating an explanation. A correct and complete formulation of **Q** (hence a rigorous proof of it) seems to be beyond the

[22] This point has been added in response to a referee's comment. Another referee pointed out that an additional worry could be that talk of 'manipulations' and 'procedures' is also talk of abstract objects, the reason being that we deal with types of procedures and types of manipulations. The referee, however, also suggested that my reply to the first referee's worry–roughly, that we're going to have to talk about some such abstract objects in presenting any examples (and unless the nominalist can construe such talk in some nominalistically acceptable way or other, then he is unable to describe empirical phenomena, let alone explaining them, nominalistically)–applies to this new worry as well. I agree.

nominalist's conceptual resources. Hence, he has no right to claim that the antecedents of **Q** are the explanans for the explanandum in question (the outcome of Game).

Yet things turn out to be even worse for the nominalist. In addition to not being able to find out what the antecedents–explanans in **Q** should be, he faces another difficulty. Suppose that we accept, for the sake of the argument, that an incomplete list of these antecedents–explanans in **Q** has been compiled. Hence, such an incomplete **Q** will count as a (quasi-)explanation of the result of Game. It is now crucial to note that this (quasi-)explanation necessarily fails as a possible (quasi-)explanation of the outcome of Game*. This is so because even if the list of the antecedents–explanans can be eventually completed, none of the explanans on the list so far (that is, conditions (a), (b), and (c)) holds for Game*! In Game*, there are fewer bananas in the large crate that will eventually win (X), there are fewer small crates in the large crate that will eventually win (X), and the probabilities are not equal. It is thus important to stress that even if the nominalist comes up with a (quasi-)explanation for the outcome of Game* (call it **Q***),[23] the nominalist has now two distinct (quasi-) explanation: a (quasi-)explanation, **Q**, for the outcome of Game and another (quasi-)explanation, **Q***, for the outcome of Game*. And **Q*** is necessarily different from **Q** regardless of how (or whether) the nominalist manages to complete the list of antecedents–explanans. (It is different because **Q** and **Q*** don't—and can't!—share some of the antecedent–explanans.)

3.3 New hopes?

Another route the nominalist can take is to devise an operational procedure by which he can find out that one crate (X) will almost always be the winner in Game. The procedure, carried out in nominalistic terms, is as follows. For Game, he begins by simply listing a possible distribution of outcomes for the withdrawals of bananas from X and Y.

Table 1 should be interpreted as follows. In round one, player A, who takes out bananas from crate X, happens to pick small crate x_1 and thus he collects five bananas. Player B happens to pick y_1 and thus collects only one banana. In round two, A picks x_1 and collects five bananas once again; this time, the coin lands such that B picks y_2 and collects nine bananas. In round three, A adds seven bananas to his pile, whereas B adds only one, so on and so forth.

Two remarks about Table 1 are in order. Firstly (and obviously), it is just illustrative; this is only one of the many possible ways in which Game might unfold. The amount of bananas each player collects in each round depends on

[23] **Q*** might be a conditional like this: 'If (a*), (b*), and (c*), then one crate (X) is the winner', where (a*) = there are fewer bananas in total in crate X, (b*) = there are fewer small crates in crate X, and (c*) = the probabilities are not equal.

Table 1. Possible distribution

Round	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Player *A*																		
Crate *X*	5	5	7	7	7	7	5	5	7	5	5	5	7	7	7	5	7	5
Player *B*																		
Crate *Y*	1	9	1	9	9	1	9	9	1	1	9	1	9	1	9	1	1	9

Table 2. Rearrangement

Round	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Player *A*																		
Crate *X*	**5**	**7**	5	7	5	7	5	7	5	7	5	7	5	7	5	7	5	7
Player *B*																		
Crate *Y*	**1**	**9**	1	9	1	9	1	9	1	9	1	9	1	9	1	9	1	9

which small crate they collect from, which is in turn determined probabilistically by how the coin lands. Secondly, and essentially, in so far as the probabilities are fixed, we know for sure that no matter how the rounds go, in the long run, there will be an approximately equal number of rounds when each of these values (5 and 7, and 1 and 9, respectively) will appear in the two rows. Thus, what the nominalist can do is just rearrange the values he tabulated above (Table 2).

The next step then consists in dividing Table 2 in cells of four values, the upper ones consisting in a (5, 7) pair, and the lower ones being a (1, 9) pair (the first such 'cell' is indicated by the bold fonts). Given the probabilities, this division in cells should be exhaustive. Now, just by simply applying the predicate '_contains more than_' to the two upper values and the two lower values in each repeating cell, the nominalist can infer that A, the player who chose X, will have more bananas in his crate in the end, as each cell indicates this. (Using numbers: the advantage of *A* increases with two bananas, twelve versus ten, with each cell.)

I'll call such a procedure a 're-arrangement'. It is available to the nominalist regardless of what the values in the cells are, and what the probabilities are. There is no doubt that a rearrangement allows the nominalist to predict the result of Game. But one might want to maintain that he can also explain why Game almost certainly ends in this way. If I happen to expect that some wins will go to one crate, some to the other, once you show me how the rearrangement works, I'll understand that only one will win. So, the existence of a rearrangement seems to show that a nominalistic explanation of the result in Game is possible. But recall that the task set for the nominalist was to find a nominalistic explanation of the result that one crate wins overwhelmingly often in both games (recall that this is our explanandum). So, the next step is to devise a rearrangement for Game*. To be sure, such a procedure can be found, but I will not present it in detail.[24]

Hence, the question now is what is the nominalistic account of the explanandum? It must be this: the nominalist explains why is it the case that 'in both games, one crate wins overwhelmingly often' by pointing out that this can be expressed as the conjunction of 'in Game, one crate wins overwhelmingly often' and 'in Game*, one crate wins overwhelmingly often', and by offering nominalistic explanations in the form of a rearrangement for each conjunct.

3.4 New troubles

We can now draw the contrast between the nominalist and the realist situation more forcefully. An analogy with the common-cause type of explanation might be illuminating here.[25] (I stress that this is only an analogy, as nothing has been said or implied about these explanations as being 'causal'—on the contrary.) Suppose two friends, Joe and Moe, arrive separately at the bus station at 3 p.m. Also, suppose that they are followed by two detectives who scrupulously record every move they make. The first detective has gathered a lot of information, and is about to write down an account as to why Joe and Moe arrived at the bus station at 3 p.m. The second detective followed Joe and Moe closely too and gathered the same information as the first detective, but this second detective, benefiting from some listening devices, also intercepted a phone conversation between Joe and Moe the day before, in which they actually agreed to arrange their schedules for the next day such that they meet at 3 p.m. at the bus station.

Now consider the event 'Joe and Moe both arrived at the bus station at 3 p.m.'. If asked to account for this event, the first detective explains it by giving the full details of Joe's trip from his place to the bus station. He will also provide the details of Moe's trip from work to the bus station. It is no surprise for him that Joe went to the bus station (say, he knows he goes to visit his mother); he can even explain why he was there at 3 p.m. instead of 4 p.m. (the bus taking him to his mother leaves at 3:05 p.m., and so on). He has a similar story about Moe as well (say, he goes to visit his father, and the bus taking him there leaves at 3:07 p.m.). Knowing all these details, the detective actually

[24] The reader can try to identify it herself. Hint: the repeating cell has the length of twelve boxes.
[25] Reflections on (Owens [1992]) have been helpful in designing it.

expects that the two will bump into each other at 3 p.m. at the town's small bus station. In brief, he has an explanation of the meeting event. If asked about the meeting event, he describes it as a conjunction (of 'Joe arrived at the bus station at 3 p.m.' and 'Moe arrived at the bus station at 3 p.m.'), and he has an explanation of each conjunct. Yet, it is intuitively clear that the second detective has a better explanation of the event, in so far as he was able to identify an additional relevant element—the phone call. Although for the first detective, who missed it, the meeting is still explained, for the second, the meeting event is better explained.

To see how this analogy is relevant for the banana game, we should begin by observing that the realist was able to identify the factor responsible for the unidirectional tendency observed in Game, expressed as an abstract mathematical (structural) feature of the game: Game is such that one crate has a higher expected value than the other. Moreover, the same factor is responsible for the tendency exhibited in Game*. Hence, the realist is in the possession of the 'common factor' that accounts for the fact that in both games, one crate wins overwhelmingly often (the explanandum). For the realist, the mathematical apparatus of expectation functions enabled him to isolate this common unifying element, just as the listening devices helped the second detective intercept the phone call.

Does the nominalist have the resources to identify such a common unifying factor? The discussion of the tentative nominalist explanations **Q** and **Q***made it clear that they are of no help in identifying such a common factor. Hence, the only hope is that the explanations in terms of rearrangements might yield such an element. A nominalist might claim that a rearrangement like the one in Table 2 explains the result in Game. Because the same procedure would explain the result in Game*, the nominalist too seems to be in the possession of a common element responsible for the result in each game. Yet, the realist is entitled to ask the nominalist to specify why the rearrangement used in Game is the same as the rearrangement used in Game*. The question is pertinent because, after all, one who sets up Tables 1 and 2 for Game doesn't set up identical tables for Game*. So, in what respect are the two rearrangements the same? What do the two rearrangements have in common?

Thus, the realist presses the nominalist to acknowledge that what the two procedures have in common is a certain structure, which in this case is a mathematical one; more precisely, to acknowledge that it is no accident that these procedures are available for the two games, and that they work for both. The realist has what I take to be an excellent answer to the question above: the games share the same abstract mathematical structure, they both instantiate an inequality of expectation values. The realist's challenge to the nominalist can also be poignantly expressed in counterfactual terms: it is easy to see that if one changed the mathematical structural relations (more precisely: the expectation value relations) among the crates, neither of the two procedures would work anymore! It is this abstract mathematical factor that 'lies underneath' the efficacy of the two rearrangements. In so far as the nominalist can't recognize this factor, he must accept that the realist explanation is superior.

4 Conclusion

The bananas example fares well with respect to all four desiderata. Consider desideratum 'simplicity' first: The example is simpler than other examples, and it is at least as simple as Baker's cicadas. But, unlike the cicada example, this new example does not raise any suspicions with regard to the second desideratum 'nominalize', as no mathematical concept enters the formulation of the explanandum. With regard to the third desideratum ('indispensability'), we must ask whether the mathematics of expectation values used here is indispensable. Recall that the nominalist can come up with an account of the explanandum, so the mathematics seems dispensable. Yet, when the fourth desideratum ('explanation') is taken into account, realism remains the more attractive position. As we saw, the nominalist's conjunctive explanation of the result scores lower than the realist's precisely on the explanatory power scale. Hence, I believe it is fair to say that the realist's explanation is the best explanation available.

Finally, let me stress that I offer this new type of example in a constructive spirit. I propose it not so much as a replacement of the ones discussed so far in the literature, but rather as providing necessary additional ammunition for the realist camp.

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