REDUCTION, REPRESENTATION AND COMMENSURABILITY OF THEORIES*

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Theories in the usual sense, as characterized by a language and a set of theorems in that language ("statement view"), are related to theories in the structuralist sense, in turn characterized by a set of potential models and a subset thereof as models ("non-statement view", J. Sneed, W. Stegmüller). It is shown that reductions of theories in the structuralist sense (that is, functions on structures) give rise to so-called "representations" of theories in the statement sense and vice versa, where representations are understood as functions that map *sentences* of one theory into another theory. It is argued that commensurability between theories should be based on functions on *open* formulas and *open* terms so that reducibility does not necessarily imply commensurability. This is in accordance with a central claim by Stegmüller on the compatibility of reducibility and incommensurability that has recently been challenged by D. Pearce.

1. Introduction. It is a central claim within the structuralist approach to scientific theories as proposed by Sneed (1971) and further developed by Stegmüller (1973) that a theory T may be reducible to a theory T' even if T and T' are incommensurable in the sense of Kuhn. This thesis serves as an argument against the relativistic consequences to which Kuhn's approach may lead: T's being reducible to T' provides a rational reason to prefer T' to T, even if T and T' cannot be compared using the concept of commensurability. In the following it will be referred to as "Stegmüller's thesis" since it was Stegmüller who put most emphasis on it (see Stegmüller 1973, 1986).

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Unfortunately, the notions of commensurability and incommensurability, although widely used, are still far from having a clear meaning. However, in spite of the vagueness and incoherencies in many discussions of "(in)commensurability", it seems to be generally accepted that these terms have something to do with whether and, if so, how the vocabulary and the statements of a theory T can be understood or represented in another theory T'. This means that, even if not yet made fully precise, "commensurability" and "incommensurability" are linguistic concepts that are based on the statement-view of theories. "Reducibility", in contrast, although often understood as relating statements as well, can be successfully defined in a purely model-theoretic framework, as has been demonstrated in the structuralist school. This suggests: (1) investigating if this model-theoretic concept of reduction has some equivalent in the framework of the statement view; (2) looking to see if from this equivalent a satisfactory notion of commensurability can be obtained.

The main purpose of this paper is to correlate the structuralist concept of reduction with a concept based on the statement view that we call "representation of theories". It will be shown that each model-theoretic reduction can be associated with a representation of a certain kind and vice versa. More precisely, we will distinguish between weak and strong concepts of reduction and representation and show that the weak and the strong concepts mutually correspond. The weak concept of reduction is the concept favored by Sneed and Stegmüller, the strong one being a plausible sharpening of it. The weak concept of representation resembles Eberle's (1971) notion of a "representing function" in some respects, which in turn is closely related to Tarski's (1953) concept of the interpretability of theories. Roughly speaking, a weak representation of a theory in another is a mapping between the sentences of the theories that respects consequence and refutability and maps theorems to theorems. In contradistinction to Eberle and Tarski, however, we work with *partial* consequence as a restricted version of logical consequence, allowing for full logical consequence as a limiting case. A strong representation also respects non-consequence and maps non-theorems to non-theorems, and is thus related to what is sometimes discussed as "strong interpretability" (see Bonevac 1982). However, our notion of representation is not intended to capture immediately the idea of reduction as do the interpretability notions in the philosophy of science, but is understood as an independent concept whose relationship to a model-theoretically defined concept of reduction is to be investigated. In our study, representations are considered to belong to the statement view, reductions to the non-statement view of theories.

We chose the term "representation" for a linguistic mapping between theories in accordance with Eberle's (1971) terminology. Admittedly, "representation" has a different meaning in mathematics and also in some branches of philosophy of science (such as the theory of measurement). There it denotes a certain mapping between models of theories (see Suppes 1957, chap. 12) and thus comes close to what we call "reduction" (following the structuralists). However, other terms like "interpretation" or "translation" seemed inappropriate to us because of their unwanted connotations.

In the final section, we will extend our framework of representations of theories by defining commensurability functions between theories and thus the commensurability of theories through the existence of commensurability functions. Roughly speaking, the difference between representations and commensurability functions is that the latter also represent *open* formulas and *open* terms of one theory in another and not only *sentences* (= closed formulas). This is intended to capture the idea of a representation of *concepts* that is intuitively associated with "commensurability", going beyond the mere representation of *statements*. Since reductions only give rise to representations and not necessarily to commensurability functions, it turns out that the close relationship between reductions and representations is no argument against Stegmüller's thesis, if commensurability is understood in the way proposed.

Underlying our investigation is the thesis that relating concepts of the non-statement view to those of the statement view can give us a clearer understanding of the model-theoretic concepts, both with respect to their logical strength and their philosophical adequacy. This means that even though the model-theoretic view might be closer to actual scientific practice (Suppes), more capable of coping with the problem of theoretical terms (Sneed), and of greater value when reconstructing many of Kuhn's theses (Stegmüller), linguistic concepts cannot be discarded in favor of model-theoretic ones without loss. In many cases they are understood more easily and directly than the model-theoretic concepts. It seems that our intuitions concerning linguistic conceptualizations are much more immediate and reliable than those concerning model-theoretic conditions. This is particularly true with respect to a notion like that of commensurability. Whereas the logical understanding of scientific progress—including the distinction between normal and revolutionary science—is well dealt with from the global perspective of the non-statement view, whether or not two theories are commensurable is an issue requiring explicit reference to scientific terms. It is thus a local question more appropriately addressed within the statement view. This is in accordance with Feyerabend's and Kuhn's original rendering of the (in)commensurability problem. Although not explicitly considering formal languages, they pose it as a question concerning the preservation of meanings of scientific concepts and the deducibility of "old" laws in a "new" theory (Feyerabend 1962; Kuhn 1962, p. 101f.).

That there is a close relationship between statement and non-statement view, is obvious. Although Sneed and Stegmüller define "reduction" purely in terms of relations between sets of models, these sets must somehow be characterized linguistically when one wants to treat concrete examples. No formal system is necessary here: a mathematical characterization using set-theoretic predicates as in Suppes (1957, chap. 12) is sufficient. This is what Stegmüller calls the "quasi-linguistic mode of speech" (1986, p. 24). But even when characterized in such an informal way, structures are *eo ipso* related to formal languages. A structure is nothing but a collection of domains, relations, functions, and individuals, which can be described by a certain similarity type containing information about the arity of relations, etc. This similarity type is at the same time the similarity type of a formal language having corresponding predicate, function and individual constants in its non-logical vocabulary. This language is often first-order, though this does not imply that the considered set of structures is first-order definable. Therefore, given the abstract model-theoretic conception of theory reduction, it is not surprising that under certain conditions it has an equivalent on the linguistic level.

Our general claim is that the Sneed-Stegmüller conception of scientific theories would gain much argumentative clarity and intuitive plausibility if the connection between linguistic and model-theoretic concepts were given more attention. D. Pearce was the first to investigate this connection in relation to the structuralist view of science. In his recent discussion with Balzer and Stegmüller, Pearce (1982a, see also 1982b; and Tan 1986) argued that model-theoretically defined reducibility implies translatability and thus commensurability. Balzer (1985) defended Stegmüller's thesis by calling into question the adequacy of Pearce's accounts of translation and of incommensurability. Stegmüller (1986) further developed some of Balzer's ideas together with an improved definition of commensurability. The present study intends to make the correspondence between concepts of the statement and non-statement views of theories more obvious than does Pearce's work. Pearce's notion of "translation" suffers from several defects. First, it is itself relativized to a reduction in the structuralist sense that is supposed to be given and thus mixes up from the very beginning the two views of theories. (So one may ask if it should be called "translation" at all, as Balzer and Stegmüller remarked.) Contrary to that, representations in our sense are independently defined linguistic concepts and are thus more genuine explications of the idea of "translation". Secondly, Pearce's "translations" only concern *languages* and not *theories*. Nothing is required of how *theorems* of the reduced theory relate to the *theorems* of the reducing theory, a crucial aspect that seems to be overlooked by Balzer and Stegmüller. This is an additional point against the usefulness of this concept. Again, our representations also relate the theorems of the considered theories in a specific way. More precisely, we distinguish representations of languages from representations of theories, the latter being the concept in which we are really interested.

Our definition of commensurability is considerably different from Balzer's and Stegmüller's proposals in their replies to Pearce. In particular, we do not require commensurable concepts to be literally identical. Furthermore, whereas Balzer and Stegmüller work in Pearce's framework, which is based on reductions of theories in the model-theoretic sense, we do not use model-theoretic concepts in our account of "commensurability". We strictly confine ourselves to theories in the sense of the statement view and consider the meaning of the concepts of a scientific theory to be given internally by the theorems of the theory and not externally by their extensions in certain models.

As regards our definition of reduction, we follow Pearce, Balzer and Stegmüller in formulating this structuralist notion in a very restricted version, since the refinements concerning theoretical versus non-theoretical concepts, special laws and constraints do not immediately have to do with the questions investigated here.

The plan of this paper is as follows: in section 2 we present some preliminaries concerning notation and concepts used. In section 3 representations of first-order languages and related concepts are defined and basic properties are established. Section 4 extends these notions to representations of theories. In section 5 theories in the structuralist sense and reductions between them are defined. Section 6 then shows how a given reduction between theories in the structuralist sense can be used to define a corresponding representation between theories in the sense of the statement view and vice versa. In section 7 we discuss the strength and adequacy of our results in relation to Stegmüller's thesis and propose a definition of commensurability.

2. Formal Preliminaries. The Notion of Partial Consequence. Given a relation $R \subseteq M \times N$, let $\mathrm{Dom}(R) = \{x \in M : (\exists y \in N)\ xRy\}$ and $\mathrm{Ran}(R) = \{y \in N : (\exists x \in M)\ xRy\}$, and for $M_1 \subseteq M$ let $R(M_1) = \{y \in N : (\exists x \in M_1)\ xRy\}$. R^{-1} denotes the converse of R. Note that for $M_1 \subseteq M$, $R(M_1)$ is defined even if M_1 is not a subset of $\mathrm{Dom}(R)$; more precisely, $R(M_1) = R(M_1 \cap \mathrm{Dom}(R))$. This notation also applies to functions, which are considered to be special cases of relations. We do not distinguish between sets and classes and will often refer to "sets" of structures where classes are meant.

We investigate first-order languages L and L', which differ only in their non-logical vocabulary. Intuitively, L is to be understood as the language of the reduced ("old") theory T and L' as the language of the reducing ("new") theory T', although all our formal notions and theorems are independent of this motivation. To simplify notation, we adopt the convention that unprimed syntactical variables relate to L and primed syntactical variables to L'. For example, if Σ denotes a set of sentences in L, Σ' denotes a set of sentences in L', and so on. Thus we will explain our notation only for L; by this convention everything extends to L' *mutatis mutandis*.

Except for a few places, where it is explicitly mentioned, the only logical sign of L and L' to which we will refer is negation $\neg$. (This is because negation is the only logical sign of the object-language for which a preservation property will be explicitly required.) All other logical signs ($\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$, &) are used metalinguistically. Sent(L) denotes the set of sentences (formulas without free variables) of L, Σ stands for arbitrary subsets of Sent(L), and σ (with and without indices) for elements of Sent(L) (that is, for sentences). Str(L) denotes the set of structures of the similarity type determined by the non-logical constants of L, S stands for subsets of Str(L), x and y for elements of Str(L) (that is, for structures). As usual, $x \models \sigma$ means that σ holds in x, and $x \models \Sigma$ that each element of Σ holds in x; in addition, $S \models \sigma$ means that σ holds in each element of S, and $S \models \Sigma$ that each element of Σ holds in each element of S. $\Sigma \models \sigma$ means that σ holds in all structures in which each element of Σ holds (that is, σ is a logical consequence of Σ).

We can now introduce the notion of partial consequence with respect to a set of structures S: $\Sigma \models_S \sigma$ means that σ holds in all structures *of* S in which each element of Σ holds, that is,

$$\Sigma \models_S \sigma \Leftrightarrow (\forall x \in S)(x \models \Sigma \Rightarrow x \models \sigma). \tag{2.1}$$

Obviously, logical consequence is a limiting case of partial consequence, where $S = \text{Str}(L)$. It follows immediately from (2.1) that for any S, $S^* \subseteq \text{Str}(L)$,

$$\Sigma \models_S \sigma \;\&\; S^* \subseteq S \Rightarrow \Sigma \models_{S^*} \sigma.$$

$\sigma_1 \dashv\models_S \sigma_2$ means that both $\{\sigma_1\} \models_S \sigma_2$ and $\{\sigma_2\} \models_S \sigma_1$.

$\text{Th}_L(S)$ designates the set of all sentences of L which hold in each element of S, and, conversely, $\text{Mod}_L(\Sigma)$ the set of all structures in which each element of Σ holds. If the reference to L or L' is obvious (for example, by the absence or presence of a prime in syntactical variables), we omit the subscript. (2.1) can now be reformulated as

$$\Sigma \models_S \sigma \Leftrightarrow \text{Mod}(\Sigma) \cap S \models \sigma.$$

$\equiv_L$, expressing elementary equivalence, is then defined as

$$x \equiv_L y \Leftrightarrow (\forall \sigma)(x \models \sigma \Leftrightarrow y \models \sigma),$$

that is,

$$x \equiv_L y \Leftrightarrow \mathrm{Th}_L(\{x\}) = \mathrm{Th}_L(\{y\})$$

(where, again, subscripts are omitted if no confusion can arise). Let $S_\equiv$ be the closure of S under $\equiv$, that is, $S_\equiv = \{y : (\exists x \in S) x \equiv y\}$.

Now suppose an arbitrary function $f : \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ is given. We define a restricted notion $\equiv_f$ of elementary equivalence with respect to f as

$$x' \equiv_f y' \Leftrightarrow (\forall \sigma)(x' \vDash f(\sigma) \Leftrightarrow y' \vDash f(\sigma)).$$

x' and y' are elementarily equivalent with respect to f if they cannot be distinguished by images under f. Let $S'_{\equiv_f}$ denote the closure of S' under $\equiv_f$. The following lemma follows immediately from the definitions.

LEMMA 1. For all Σ, Σ' and σ,

(i) $\Sigma \vDash_S \sigma \Leftrightarrow \Sigma \vDash_{S_\equiv} \sigma$.
(ii) $f(\Sigma) \vDash_{S'} f(\sigma) \Leftrightarrow f(\Sigma) \vDash_{S'_{\equiv_f}} f(\sigma)$
(instead of $f(\Sigma)$ we could also write $\Sigma' \cap \mathrm{Ran}(f)$). □

Each function $f : \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ induces a relation $R_f \subseteq \mathrm{Str}(L') \times \mathrm{Str}(L)$ which is defined as

$$x' R_f x \Leftrightarrow (\forall \sigma)(x' \vDash f(\sigma) \Leftrightarrow x \vDash \sigma), \tag{2.2}$$

where $x \in \mathrm{Str}(L)$ and $x' \in \mathrm{Str}(L')$. R_f is called the associated relation between structures. The following is obvious:

LEMMA 2. (i) R_f is invariant with respect to $\equiv$ and $\equiv_f$ in the sense that for all x, y, x' and y':

$$x \equiv y \Rightarrow (x' R_f x \Leftrightarrow x' R_f y)$$

$$x' \equiv_f y' \Rightarrow (x' R_f x \Leftrightarrow y' R_f x),$$

(ii) R_f is unique on the right modulo $\equiv$ and on the left modulo $\equiv_f$ in the sense that for all x, y, x', y':

$$(x' R_f x \;\&\; x' R_f y) \Rightarrow x \equiv y$$

$$(x' R_f x \;\&\; y' R_f x) \Rightarrow x' \equiv_f y'.$$ □

Thus R_f would become a partial one–one function if its arguments were equivalence classes under $\equiv$ and $\equiv_f$, instead of single structures.

3. Language Representations. Basic Lemmas. We define representations between languages quite generally, considering the possibility of partial consequence with respect to certain sets S and S' of structures instead of full logical consequence. The usefulness of this approach will become obvious when we formulate an equivalent of reduction in terms of representations. By taking S and S' to be Str(L) and Str(L'), respectively, we obtain a non-relativized concept of representation as a limiting case.

A *weak representation* of L in L' with respect to S and S' is a function f: Sent(L) $\rightarrow$ Sent(L') for which, for all Σ and σ,

$$\Sigma \vDash_S \sigma \Rightarrow f(\Sigma) \vDash_{S'} f(\sigma) \tag{3.1}$$

and

$$f(\neg\sigma) \Dashv\vDash_{S'} \neg f(\sigma). \tag{3.2}$$

If (3.2), but instead of (3.1) its converse

$$f(\Sigma) \vDash_{S'} f(\sigma) \Rightarrow \Sigma \vDash_S \sigma \tag{3.3}$$

hold, we will speak of a *conversely weak representation*. If all three of (3.1), (3.2) and (3.3) hold, f is called a *strong representation* of L in L' with respect to S and S'. The concept of a conversely weak representation is mainly of technical relevance. It has been defined to enable us to distinguish assertions for which proof only (3.3) is needed from those that require both (3.1) and its converse (3.3). An example for a weak representation is given in section 6 after Theorem 1.

Apart from being based on partial consequence instead of full logical consequence, our notions of representations differ from Eberle's (1971) notion of a representing function and Tarski's (1953) notion of weak interpretability in at least two respects:

(i) The syntactical form of sentences need not be respected by f, that is, a conjunction need not necessarily be transformed into a conjunction, a disjunction not necessarily into a disjunction, etc. From (3.1) and (3.2) it follows only that propositional connectives are respected *modulo (partial) consequence*. There is no reason in the present context to demand that, for example, conjunctions of the "old" theory become conjunctions of the "new" theory.

(ii) We only talk about sentences, not about formulas with free variables. We do not necessarily expect a uniform mapping between formulas of L and L' to be definable. In particular, nothing is required of quantifiers. This is crucial for the relationship between the existence of representations and commensurability and will be discussed in the final section.

It is easy to see that (3.1) and (3.2) imply that if $f(\Sigma)$ is satisfiable in S' then Σ is satisfiable in S. Similarly, (3.3) and (3.2) imply that if Σ is satisfiable in S then $f(\Sigma)$ is satisfiable in S'. Furthermore, it is obvious that a weak representation of L in L' with respect to S and S' is a weak representation with respect to any S^* and S'^* for which $S^* \supseteq S$ and $S'^* \subseteq S'$. A conversely weak representation of L in L' with respect to S and S' is a conversely weak representation with respect to any S^* and S' for which $S^* \subseteq S$ (it follows from (3.2) that we cannot arbitrarily choose a set of structures $S'^* \supseteq S'$).

We now turn to study properties of the associated relation R_f between structures defined by (2.4) and give necessary and sufficient conditions for f's being a representation of a certain kind.

LEMMA 3. If f is a weak representation of L in L' with respect to S and S', then

(i) $(\forall x' \in S'_{\equiv_f})(R_f(\{x'\}) = \mathrm{Mod}(\{\sigma : x' \models f(\sigma)\}) \cap S_{\equiv} \neq \emptyset)$,
(ii) $S'_{\equiv_f} \subseteq \mathrm{Dom}(R_f)$,
(iii) $R_f(S') = R_f(S'_{\equiv_f}) \subseteq S_{\equiv}$.

Conversely, given any $f: \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ and $S' \subseteq \mathrm{Dom}(R_f)$, then f is a weak representation of L in L' with respect to $R_f(S')$ and S'; in fact, $R_f(S')$ is the least set S closed under $\equiv$ such that f is a weak representation of L in L' with respect to S and S'.

Proof. (i) Suppose $x' \in S'_{\equiv_f}$. If $x' R_f x$, then $(\forall\sigma)(x' \models f(\sigma) \Rightarrow x \models \sigma)$ by the definition of R_f, which is the same as $x \models \{\sigma : x' \models f(\sigma)\}$. Hence, $R_f(\{x'\}) \subseteq \mathrm{Mod}(\{\sigma : x' \models f(\sigma)\})$. If $x \models \{\sigma : x' \models f(\sigma)\}$, then $(\forall\sigma)(x' \models f(\sigma) \Rightarrow x \models \sigma)$. So in particular, $(\forall\sigma)(x' \models f(\neg\sigma) \Rightarrow x \models \neg\sigma)$. Since $x' \in S'_{\equiv_f}$, this implies by (3.2) and Lemma 1 (i), that $(\forall\sigma)(x' \models \neg f(\sigma) \Rightarrow x \models \neg\sigma)$, and so $(\forall\sigma)(x \models \sigma \Rightarrow x' \models f(\sigma))$, from which the converse inclusion $\mathrm{Mod}(\{\sigma : x' \models f(\sigma)\}) \subseteq R_f(\{x'\})$ follows immediately. Next we show that $\mathrm{Mod}(\{\sigma : x' \models f(\sigma)\})$, that is $R_f(\{x'\})$, is nonempty and contained in $S_{\equiv}$. From Lemma 2 (ii) we know that all the elements of $R_f(\{x'\})$ are elementarily equivalent. Thus it suffices to show that $\{\sigma : x' \models f(\sigma)\}$ is satisfiable in S. But if $\{\sigma : x' \models f(\sigma)\}$ is not satisfiable in S, then $f(\{\sigma : x' \models f(\sigma)\})$ is not satisfiable in S', contradicting the fact that x' satisfies $f(\{\sigma : x' \models f(\sigma)\})$ in $S'_{\equiv_f}$, and consequently that some $y' \equiv_f x'$ satisfies it in S'.

(ii) and (iii) are immediate consequences of (i).

Now suppose $S' \subseteq \mathrm{Dom}(R_f)$. Then for any $x' \in S'$ there is an $x \in R_f(S')$ such that for all σ, $x' \models f(\sigma)$ iff $x \models \sigma$. Thus $x' \models f(\Sigma)$ implies $x \models \Sigma$, which in turn, supposing $\Sigma \models_{R_f(S')} \sigma$, implies that $x \models \sigma$.

Therefore $x' \models f(\sigma)$, showing that $f(\Sigma) \models_{S'} f(\sigma)$ and thus (3.1). Similarly, $x' \models f(\neg\sigma)$ iff $x \models \neg\sigma$ iff not $x \models \sigma$ iff not $x' \models f(\sigma)$ iff $x' \models \neg f(\sigma)$. This proves (3.2). That $R_f(S')$ is least, follows from (iii). □

Lemma 3 implies that a weak representation f of L in L' with respect to S and S' is one with respect to $R_f(S')$ and S' (and thus also one with respect to *any* $S^* \supseteq R_f(S')$ and S'). Because the parameter S can always be replaced by $R_f(S')$, it is, unlike S', rather unspecific for f.

LEMMA 4. If f is a conversely weak representation of L in L' with respect to S and S', then

(i) $(\forall x \in S_{\equiv})(R_f^{-1}(\{x\}) = \text{Mod}(\{f(\sigma) : x \models \sigma\}) \cap S'_{\equiv_f} \neq \emptyset)$,
(ii) $S_{\equiv} \subseteq \text{Ran}(R_f)$,
(iii) $R_f^{-1}(S) = R_f^{-1}(S_{\equiv}) \subseteq S'_{\equiv_f}$.

Conversely, given any f: Sent(L) → Sent(L') and $S \subseteq \text{Ran}(R_f)$, then f is a conversely weak representation of L in L' with respect to S and $R_f^{-1}(S)$; in fact, $R_f^{-1}(S)$ is the least set S' closed under $\equiv_f$ such that f is a conversely weak representation of L in L' with respect to S and S'.

Proof. (i) Suppose $x \in S_{\equiv}$. Since $\{\sigma : x \models \sigma\}$ is satisfiable in $S_{\equiv}$ and thus in S, $f(\{\sigma : x \models \sigma\})$ is satisfiable in S'. Thus $\text{Mod}(\{f(\sigma : x \models \sigma\}) \cap S'_{\equiv_f}$ is nonempty. If $x' \models \{f(\sigma) : x \models \sigma\}$ and $x' \in S'_{\equiv_f}$, then $(\forall\sigma)(x \models \sigma \Rightarrow x' \models f(\sigma))$. Thus in particular, $(\forall\sigma)(x \models \neg\sigma \Rightarrow x' \models f(\neg\sigma))$, therefore by (3.2), $(\forall\sigma)(x \models \neg\sigma \Rightarrow x' \models \neg f(\sigma))$, and hence $(\forall\sigma)(x' \models f(\sigma) \Rightarrow x \models \sigma)$. Thus we have proved $\emptyset \neq \text{Mod}(\{f(\sigma) : x \models \sigma\}) \cap S'_{\equiv_f} \subseteq R_f^{-1}(\{x\})$. Now suppose $x' R_f x$. Then $(\forall\sigma)(x \models \sigma \Rightarrow x' \models f(\sigma))$ by the definition of R_f, that is, $x \models \{f(\sigma) : x \models \sigma\}$. Hence $R_f^{-1}(\{x\}) \subseteq \text{Mod}(\{f(\sigma) : x \models \sigma\})$. That $R_f^{-1}(\{x\}) \subseteq S'_{\equiv_f}$ follows from the fact that $R_f^{-1}(\{x\})$ contains an element of $S'_{\equiv_f}$ and that, by Lemma 2 (ii), all the elements of $R_f^{-1}(\{x\})$ are elementarily equivalent with respect to f.

(ii) and (iii) are immediate consequences of (i).

Now suppose $S \subseteq \text{Ran}(R_f)$. Then for any $x \in S$, there is $x' \in R_f^{-1}(S)$ such that for all σ, $x \models \sigma$ iff $x' \models f(\sigma)$. Thus $x \models \Sigma$ implies $x' \models f(\Sigma)$ which in turn, supposing $f(\Sigma) \models_{R_f^{-1}(S)} f(\sigma)$, implies that $x' \models f(\sigma)$. Therefore $x \models \sigma$, showing that $\Sigma \models_S \sigma$ and thus (3.3). (3.2) is then proved as in the previous lemma. That $R_f^{-1}(S)$ is least, follows from (iii). □

Lemma 4 implies that a conversely weak representation f of L in L' with respect to S and S' is one with respect to S and $R_f^{-1}(S)$ (and thus also one with respect to S and any S'^* such that $S' \supseteq S'^* \supseteq R_f^{-1}(S)$). Because the parameter S' can always be replaced by $R_f^{-1}(S)$, it is, unlike S, rather unspecific for f.

Remark. The proofs of Lemma 3 and Lemma 4 are not completely parallel. The proof of Lemma 3 (i) contains the demonstration that $\mathrm{Mod}(\{\sigma : x' \vDash f(\sigma)\}) \subseteq S_{\equiv}$. The corresponding assertion for Lemma 4 (i), namely that $\mathrm{Mod}(\{f(\sigma) : x \vDash \sigma\}) \subseteq S'_{\equiv_f}$. does not necessarily hold, since $\{f(\sigma) : x \vDash \sigma\}$ may have models that are not elementarily equivalent with respect to f, unless we extend (3.2) by dropping the restriction to S' and take full logical consequence instead. We have only that $R_f^{-1}(\{x\}) = \mathrm{Mod}(\{f(\sigma) : x \vDash \sigma\} \cup \{\neg f(\sigma) : x \vDash \neg\sigma\}) \subseteq S'_{\equiv_f}$.

LEMMA 5. If f is a strong representation of L in L' with respect to S and S', all assertions of Lemmas 3 and 4 hold and in addition

(i) $R_f(S'_{\equiv_f}) = S_{\equiv}$,
(ii) $R_f^{-1}(S_{\equiv}) = S'_{\equiv_f}$.

Conversely, given any $f: \mathrm{Sent}(L) \to \mathrm{Sent}(L')$, $S \subseteq \mathrm{Ran}(R_f)$ and $S' \subseteq \mathrm{Dom}(R_f)$ such that (i) and (ii) hold, then f is a strong representation of L in L' with respect to S and S'.

Proof. Immediately from Lemma 3 (iii) and Lemma 4 (iii). □

There follow some results about f relative to any subrelation of R_f. The main result is the following.

LEMMA 6. Given $f: \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ and $R \subseteq \mathrm{Str}(L') \times \mathrm{Str}(L)$ such that $R \subseteq R_f$; then f is a strong representation of L in L' with respect to $S = \mathrm{Ran}(R)$ and $S' = \mathrm{Dom}(R)$, a weak representation with respect to any $S \supseteq \mathrm{Ran}(R)$ and $S' \subseteq \mathrm{Dom}(R)$, and a conversely weak representation with respect to any $S \subseteq \mathrm{Ran}(R)$ and $S' = \mathrm{Dom}(R)$.

Proof. Let S be $\mathrm{Ran}(R)$ and S' be $\mathrm{Dom}(R)$. Then by Lemma 2 (ii) we have $R_f(S'_{\equiv_f}) = S_{\equiv}$ and $R_f^{-1}(S_{\equiv}) = S'_{\equiv_f}$, from which the assertion follows by Lemma 5. □

In particular, each $f: \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ is a strong representation of L in L' with respect to $S = \mathrm{Ran}(R_f)$ and $S' = \mathrm{Dom}(R_f)$, a weak representation with respect to any $S \supseteq \mathrm{Ran}(R_f)$ and $S' \subseteq \mathrm{Dom}(R_f)$, and a conversely weak representation with respect to any $S \subseteq \mathrm{Ran}(R_f)$ and $S' = \mathrm{Dom}(R_f)$.

Next we can characterize $\mathrm{Dom}(R_f)$ and $\mathrm{Ran}(R_f)$ by certain maximality conditions.

LEMMA 7. Take $f: \mathrm{Sent}(L) \to \mathrm{Sent}(L')$. Then $\mathrm{Dom}(R_f)$ is the maximal set of structures S' for which there is S such that f is a weak representation of L in L' with respect to S and S'. Correspondingly, $\mathrm{Ran}(R_f)$ is the maximal set of structures S for which there is S' such that f is a conversely weak representation of L in L' with respect to S and S'.

Proof. That $\mathrm{Dom}(R_f)$ and $\mathrm{Ran}(R_f)$ are maximal was shown in Lemmas 3 (ii) and 4 (ii). That sets S for the weak representation and S' for the conversely weak representation exist, with the required properties, follows immediately from Lemma 6. □

Finally, we state a result by Feferman, whose importance for the present context was first pointed out by Pearce (see his 1982a, 1982b). We present it not in its full generality but only as far as is necessary for our purposes. For languages L and L', whose non-logical vocabularies are disjoint, a relation $R \subseteq \mathrm{Str}(L') \times \mathrm{Str}(L)$ is called projectively definable if there is a common extension L^* of L and L' by additional non-logical constants and a set Γ of sentences of L^*, such that

$$x'Rx \Leftrightarrow (\exists z^*)[x',x,z^*] \models \Gamma.$$

Here z^* stands for a tuple of elements for which $[x',x,z^*]$, that is, the structure composed of the elements of x', x and z^*, is a structure for L^*.

LEMMA 8. Let $R \subseteq \mathrm{Str}(L') \times \mathrm{Str}(L)$ be given. Suppose the non-logical vocabularies of L' and L are disjoint and R is projectively definable. Suppose furthermore that R is unique on the right modulo $\equiv$, that is, for all x, y, x':

$$(x'Rx \,\&\, x'Ry) \Rightarrow x \equiv y.$$

Then there is a function $f: \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ such that $R \subseteq R_f$.

Proof. See Feferman (1974). □

This assertion is actually a straightforward consequence of the interpolation lemma for first-order logic. Feferman proved it in the general setting of abstract model theory by supposing only that L and L' fulfill an abstract interpolation property, not that they are first-order languages.

4. Representations of Theories. Based on the notion of representations of languages we can define representations of theories. Theories are understood as pairs $\langle L,\Theta\rangle$ consisting of a language L and a set of sentences Θ of L, called the *theorems* of the theory. This is the standard notion of "theory", as used, for example, in mathematical logic, called by the structuralists the statement view. (The non-statement view will be considered in the next section.)

Representations of theories will be defined as representations of the corresponding languages with certain additional properties. Let $T = \langle L, \Theta \rangle$ and $T' = \langle L', \Theta' \rangle$ be theories where $\Theta \subseteq \mathrm{Sent}(L)$ and $\Theta' \subseteq \mathrm{Sent}(L')$. A function $f\colon \mathrm{Sent}(L) \to \mathrm{Sent}(L')$ is called a *weak representation* of T in T' with respect to S and S' iff f is a weak representation of L in L' with respect to S and S' and

$$f(\Theta) \subseteq \Theta'. \tag{4.1}$$

f is a *conversely weak representation* of T in T' with respect to S and S' iff f is a conversely weak representation of L in L' with respect to S and S' and

$$f(\Theta) \supseteq \Theta' \cap \mathrm{Ran}(f). \tag{4.2}$$

f is a *strong representation* of T in T' with respect to S and S' iff f is a strong representation of L in L' with respect to S and S' and

$$f(\Theta) = \Theta' \cap \mathrm{Ran}(f). \tag{4.3}$$

Again, conversely weak representations of theories have a mainly technical significance.

Note that we have required nothing of the sets of theorems Θ and Θ', not even that they be deductively closed. Thus we use "theorem" in a very wide sense. In principle it is possible to consider the axioms of a theory to be "theorems" in our sense (as, for example, in the example given in section 6 after Theorem 1). In many cases, however, it would then be difficult to establish a reduction between theories, since according to (4.1) axioms of T would have to be transformed into axioms of T', and not only into consequences of axioms.

(4.1) says that f maps theorems of the "old" theory to theorems of the "new" theory, and (4.3) that no non-theorem of the "old" theory is mapped onto a theorem of the "new" theory. These seem to be quite natural reconstructions of the idea of a "translation" between theories that not only concerns their languages but also their theorems. It might be added that our condition (4.1) has some similarity with Eberle's (1971) notion of the preservation of the consequences of a theory and with Tarski's (1953) notion of the interpretability of theories (see also Bonevac 1982).

5. Reductions of Theories. Next we consider theories with respect to the non-statement view and its notion of reduction. As mentioned in the introduction, we follow Pearce (1982a), Balzer (1985) and Stegmüller (1986, chap. 10) in working with notions of theory and reduction that are strongly simplified versions of the original concepts as developed in detail in, for example, Sneed (1971); Mayr (1976); and Stegmüller (1986, chap. 4). In particular, the notions of theoretical and non-theoretical concepts are kept distinctly apart.

A theory T_m is now taken to be a pair $\langle M_p,M \rangle$, where M_p is a set of structures and M a subset of M_p. The subscript "*m*" expresses the fact that we are dealing with a theory which, in the non-statement sense, is defined by sets of *models* and not by languages and sets of theorems as were the theories T in the previous section. It is assumed that all structures in M_p are of the same similarity type. Thus there is a first-order language L of this similarity type such that $M_p \subseteq \mathrm{Str}(L)$. The elements of M_p are also called "potential models" and represent those structures which are considered to be the potential range of application of the theory. The choice of M_p instead of the whole of Str(L) represents the fact that in physical theories one usually disregards certain structures from the very beginning, for example, those in which relations and functions fail to have the appropriate mathematical properties. M_p is to be distinguished from the set I of intended applications of a theory which is also considered by the structuralists, but not in our restricted framework. This set further delimits the range of a theory by pragmatically and paradigmatically selecting certain physical structures. M represents the set of models of the theory, that is, those structures to which the theory is correctly applicable. Note, however, that in the structuralist framework, one abstracts from the way M is specified. For example, M need not be defined as the set of models of a certain set of first-order sentences. (For examples see Stegmüller 1986.)

Given two theories $T_m = \langle M_p,M \rangle$ and $T_m' = \langle M_p',M' \rangle$, let L and L' be first-order languages corresponding to the similarity types of M_p and M_p', respectively. Then we define a *weak reduction* of T_m to T_m' as a relation $R \subseteq M_p' \times M_p$ such that

$$\mathrm{Ran}(R) = M_p \tag{5.1}$$

$$(\forall x' \in \mathrm{Dom}(R))(\forall x, y \in M_p)((x'Rx \;\&\; x'Ry) \Rightarrow x \equiv_L y) \tag{5.2}$$

$$R(M') \subseteq M. \tag{5.3}$$

Condition (5.2) expresses the fact that R is a function if one considers equivalence classes with respect to elementary equivalence instead of single structures in M_p. In the structuralist literature one usually finds the stronger condition that R is a function from a subset of M_p' onto M_p, but the weaker condition (5.2) seems quite natural. (5.1) expresses the idea that in order to reduce T_m to T_m', there must for each potential application of T_m be at least one corresponding potential application of T_m' (perhaps more, since T_m' may be more differentiated than T_m and, from the viewpoint of T_m', one may look at one potential application of T_m in different ways, that is, conceive them as different potential applications). Condition (5.3) says that R transforms each model of T_m' in the domain of R into a model of T_m.

We speak of a *conversely weak reduction* of T_m to T_m' if (5.1), (5.2) and the converse of (5.3),

$$M \subseteq R(M') \tag{5.4}$$

hold. (5.4) says that with each model of the "old" theory a corresponding model of the "new" theory is associated via R. A *strong reduction* of T_m to T_m' is defined as a reduction that is both weak and conversely weak, that is, for which (5.1), (5.2) and

$$R(M') = M \tag{5.5}$$

hold.

Our distinction between weak and strong reduction does not agree with similar distinctions in Sneed (1971) and Stegmüller (1986). The concept of reduction favored by Sneed and Stegmüller corresponds to our notion of a weak reduction. Our notions of a conversely weak and of a strong reduction partly correspond to notions discussed by Mayr (1976) and found by him to be more adequate. However, we do not want to continue any detailed discussion as to the most appropriate notion of reduction. We hope that we are at least considering some important candidates. For an overview of the proposals made in the structuralist literature, see Rott (1987). An example of a weak reduction is mentioned in section 6 after Theorem 2.

6. Relating Reductions and Representations. The Central Theorems. The main result of this section will be that a weak, conversely weak or strong representation of a theory T in a theory T' with respect to sets of structures S and S' induces a weak, conversely weak or strong reduction, respectively, of a theory T_m to a theory T_m' where the theories in the non-statement sense T_m and T_m' correspond to the theories in the statement sense T and T' in a canonical way. Conversely, under certain assumptions each weak, conversely weak or strong reduction of a theory T_m to a theory T_m' induces a weak, conversely weak or strong representation, respectively, of a theory T in a theory T' with respect to sets of structures S and S', where T, S and T', S' are obtained from the respective theories T_m and T_m' in a canonical way. As regards Stegmüller's thesis, this converse direction is even more important. Our central theorems follow straightforwardly from two lemmas that do not even require the framework of representations of languages but that are related to representations of theories in an obvious way.

LEMMA 9. Given a function f: Sent(L) → Sent(L') and a set $S' \subseteq$ Str(L'). Then:

(i) $f(\Sigma) \subseteq \Sigma' \Rightarrow R_f(\text{Mod}(\Sigma' \cap \text{Ran}(f)) \cap S') \subseteq \text{Mod}(\Sigma) \cap R_f(S')$.
(ii) $f(\Sigma) \supseteq \Sigma' \cap \text{Ran}(f) \Rightarrow R_f(\text{Mod}(\Sigma' \cap \text{Ran}(f)) \cap S') \supseteq \text{Mod}(\Sigma) \cap R_f(S')$.
(iii) $f(\Sigma) = \Sigma' \cap \text{Ran}(f) \Rightarrow R_f(\text{Mod}(\Sigma' \cap \text{Ran}(f)) \cap S') = \text{Mod}(\Sigma) \cap R_f(S')$.

Proof. (i): If $x' \in \text{Mod}(\Sigma' \cap \text{Ran}(f)) \cap S'$, then $x' \vDash \Sigma' \cap \text{Ran}(f)$, and so, by the assumption, $x' \vDash f(\Sigma)$. Hence by definition of R_f for each x such that $x'R_fx$, $x \vDash \Sigma$, that is, $x \in \text{Mod}(\Sigma) \cap R_f(S')$.

(ii) If $x \in R_f(S')$, then there is a corresponding $x' \in S'$ such that $x'R_fx$. If furthermore $x \in \text{Mod}(\Sigma)$, that is, $x \vDash \Sigma$, then by definition of R_f, $x' \vDash f(\Sigma)$. Thus by assumption, $x' \vDash \Sigma' \cap \text{Ran}(f)$, which means that $x \in \text{Mod}(\Sigma' \cap \text{Ran}(f)) \cap S'$.
(iii) follows from (i) and (ii). □

Under certain conditions concerning Σ, Σ' and S' even the converses of the clauses of this lemma can be proved. However, such results will not be used later on.

LEMMA 10. Given f: Sent(L) → Sent(L') and $R \subseteq$ Str(L') x Str(L) such that $R \subseteq R_f$, let M and M' be arbitrary sets of structures of L and L', respectively. Then:

(i) $R(M') \subseteq M \Rightarrow f(\text{Th}(M)) \subseteq \text{Th}(M' \cap \text{Dom}(R))$.
(ii) $R(M') \supseteq M \Rightarrow f(\text{Th}(M)) \supseteq \text{Th}(M' \cap \text{Dom}(R)) \cap \text{Ran}(f)$.
(iii) $R(M') = M \Rightarrow f(\text{Th}(M)) = \text{Th}(M' \cap \text{Dom}(R)) \cap \text{Ran}(f)$.

Proof. (i) if $\sigma \in \text{Th}(M)$, then $M \vDash \sigma$. Thus, assuming that $R(M') \subseteq M$, $R(M') \vDash \sigma$. Hence, since $R \subseteq R_f$, $M' \cap \text{Dom}(R) \vDash f(\sigma)$. (Note that $M' \vDash f(\sigma)$ does not necessarily hold since M' need not be completely contained in Dom(R).) This means that $f(\sigma) \in \text{Th}(M' \cap \text{Dom}(R))$.

(ii) If $f(\sigma) \in \text{Th}(M' \cap \text{Dom}(R))$, then $M' \cap \text{Dom}(R) \vDash f(\sigma)$. Thus, since $R \subseteq R_f$, $R(M') \vDash \sigma$, by the definition of R_f. Hence, assuming that $M \subseteq R(M')$, $M \vDash \sigma$. Thus $\sigma \in \text{Th}(M)$.
(iii) is an immediate consequence of (i) and (ii). □

Now we can prove our two main theorems.

THEOREM 1. Given theories $T = \langle L,\Theta \rangle$ and $T' = \langle L',\Theta' \rangle$ and a function f: Sent(L) → Sent(L'), let T_m and T_m' denote theories (in the non-statement sense) $\langle M_p,M \rangle$ and $\langle M_p',M' \rangle$, respectively, where M_p and M_p' are as stated in the clauses below and where M and M' are defined depending on M_p and M_p' as

$$M = \mathrm{Mod}(\Theta) \cap M_p \qquad M' = \mathrm{Mod}(\Theta' \cap \mathrm{Ran}(f)) \cap M_p'.$$

(i) Suppose f is a weak representation of T in T' with respect to S and S'. Let

$$M_p' \supseteq S' \qquad M_p = R_f(S') \qquad R = R_f \cap (S' \times M_p).$$

Then R is a weak reduction of T_m to T_m', where $M_p \subseteq S_{\equiv}$.

(ii) Suppose f is a conversely weak representation of T in T' with respect to S and S'. Let

$$M_p' \supseteq R_f^{-1}(S) \qquad M_p = S_{\equiv} \qquad R = R_f \cap (R_f^{-1}(S) \times M_p).$$

Then R is a conversely weak reduction of T_m to T_m', where $\mathrm{Dom}(R) \subseteq S'_{\equiv_f}$.

(iii) Suppose f is a strong representation or T in T' with respect to S and S'. Let

$$M_p' \supseteq S'_{\equiv_f} \qquad M_p = \mathrm{S}_{\equiv} \qquad R = R_f \cap (S'_{\equiv_f} \times M_p).$$

Then R is a strong reduction of T_m to T_m', where $\mathrm{Dom}(R) = S'_{\equiv_f}$ and $M_p = S_{\equiv}$.

Proof. In all three cases, $M \subseteq M_p$ and $M' \subseteq M_p'$, so T_m and T_m' are theories in the non-statement sense; furthermore, (5.2) holds by Lemma 2 (ii).

(i) It is easy to see that $\mathrm{Ran}(R) = M_p$, so (5.1) is verified. Since by definition we have

$$\begin{aligned} R(M') &= R(\mathrm{Mod}(\Theta' \cap \mathrm{Ran}(f)) \cap M_p') \\ &= R_f(\mathrm{Mod}(\Theta' \cap \mathrm{Ran}(f)) \cap S') \end{aligned}$$

and

$$M = \mathrm{Mod}(\Theta) \cap R_f(S'),$$

and since $f(\Theta) \subseteq \Theta'$ by assumption, we obtain by Lemma 9 (i) $R(M') \subseteq M$, that is, (5.3). $M_p \subseteq S_{\equiv}$ holds by Lemma 3 (iii).

(ii) By Lemma 4 (ii) we have that $S \subseteq \mathrm{Ran}(R_f)$. Thus $R_f(R_f^{-1}(S)) = S_{\equiv} = M_p$, which implies that $\mathrm{Ran}(R) = M_p$, that is, (5.1) is verified. Furthermore we have

$$\begin{aligned} R(M') &= R(\mathrm{Mod}(\Theta' \cap \mathrm{Ran}(f)) \cap M_p') \\ &= R_f(\mathrm{Mod}(\Theta' \cap \mathrm{Ran}(f)) \cap R_f^{-1}(S)) \end{aligned}$$

and

$$M = \text{Mod}(\Theta) \cap S_{\equiv} = \text{Mod}(\Theta) \cap R_f(R_f^{-1}(S)).$$

Since $f(\Theta) \supseteq \Theta' \cap \text{Ran}(f)$ by assumption, we obtain by Lemma 9 (ii) $R(M') \supseteq M$, that is, (5.4). $\text{Dom}(R) \subseteq S'_{\equiv_f}$ holds by Lemma 4 (iii).

(iii) This follows from (i) and (ii) by use of Lemma 5 and Lemma 9 (iii). □

Theorem 1 says that representations of the various kinds induce corresponding reductions. The parameters of the induced reduction, in particular Dom(R) and M_p = Ran(R)(M_p' is an arbitrary superset of Dom(R)), are canonically related to the parameters of the given representation f of L in L' with respect to S and S'. In case (i) S' defines Dom(R), and $R_f(S')$ (which, as remarked after Lemma 3, is more specific for f than S itself), defines M_p. In case (ii), $S_{\equiv}$ defines M_p, and $R_f^{-1}(S)$ (which, as remarked after Lemma 4, is more specific for f than S' itself), defines Dom(R). In case (iii), Dom(R) and M_p are directly defined by the parameters S and S' of the representation, namely, as $S'_{\equiv_f}$ and $S_{\equiv}$.

It remains to show that M and M' were chosen in a canonical way. For M this is obvious. We did not choose Mod(Θ) but its intersection with M_p in order to guarantee that $M \subseteq M_p$. As for M', one may ask why $\text{Mod}(\Theta') \cap M_p'$ was not taken instead of $\text{Mod}(\Theta' \cap \text{Ran}(f)) \cap M_p'$. The reason is that $\Theta' \backslash \text{Ran}(f)$ is completely unspecific as to the representation f. Any (weak, conversely weak, strong) representation of $\langle L,\Theta\rangle$ in $\langle L',\Theta'\rangle$ with respect to S and S' is also one of $\langle L,\Theta\rangle$ in $\langle L',\Theta' \cap \text{Ran}(f)\rangle$. The subset of theorems of T' outside Ran(f) can be changed arbitrarily without changing the character of the representation.

A simple mathematical example for the application of Theorem 1, which is related to elementary examples used in the theory of measurement, may be given as follows. Let L and L' be first-order languages with identity where L has the binary predicate letter "$<$" and L' the binary predicate letters "$<'$" and "I" as non-logical constants. Let f associate with each sentence σ of L the sentence σ' of L' which results from σ by replacing "$<$" by "$<'$" and the identity sign "$=$" by "I". Let S' be the set of those structures $\langle A,<'_s,I_s\rangle$ of L' with domain A and relations $<'_s$ and I_s corresponding to the relation symbols "$<'$" and "I" in L', such that I_s is an equivalence relation and $<'_s$ is invariant with respect to I_s, that is,

$$a <'_s b \ \&\ bI_s c \Rightarrow a <'_s c, \quad \text{and}$$

$$a <'_s b \ \&\ aI_s c \Rightarrow c <'_s b.$$

Then f is a weak representation of L in L' with respect to Str(L) and S'. Obviously, $R_f(S') = \text{Str}(L)$, since for each $x = \langle A,<_s\rangle \in \text{Str}(L)$, we have $x'R_f x$ for $x' = \langle A,<_s,I_s\rangle \in S'$, where I_s is the set of all pairs (a,a) for $a \in A$. Let Θ contain formulas formally expressing in L that "<" is asymmmetric, transitive and connected (that is, $u = w \vee u < w \vee w < u$, for u and w being individual variables and $\vee$ being the disjunction sign of the object-language), and Θ' be the corresponding formulas expressing in L′ that "<′" is asymmetric, transitive and connected with respect to I (that is, $uIw \vee u <' w \vee w <' u$). Obviously, $f(\Theta) = \Theta' = \Theta' \cap \text{Ran}(f)$, so that f is a weak representation of the theory $\langle L,\Theta\rangle$ in the theory $\langle L,\Theta'\rangle$ with respect to Str(L) and S'. Then according to Theorem 1 (i), R_f is a weak reduction of $T_m = \langle \text{Str}(L),\text{Mod}(\Theta)\rangle$ to $\langle S',\text{Mod}(\Theta') \cap S'\rangle$. Using the terminology of Suppes and Zinnes (1963, pp. 23–26), Mod(Θ) is the class of all *series* and, due to the choice of S', Mod(Θ') $\cap$ S' the class of all *quasi-series*. Therefore we have obtained a model-theoretic reduction of the theory of series to that of quasi-series.

THEOREM 2. Given theories in the non-statement sense $T_m = \langle M_p, M\rangle$ and $T_m' = \langle M_p',M'\rangle$, and a relation $R \subseteq M_p' \times M_p$; define theories in the statement sense $T = \langle L,\Theta\rangle$ and $T' = \langle L',\Theta'\rangle$ as follows. Let L and L' be first-order languages of the similarity types of M_p and M_p', respectively, such that the non-logical vocabularies of L and L' are disjoint. Let Θ and Θ' be defined as

$$\Theta = \text{Th}(M) \qquad \Theta' = \text{Th}(M' \cap \text{Dom}(R)).$$

Suppose R is projectively definable.

(i) If R is a weak reduction of T_m to T_m', then there is a weak representation of T in T' with respect to any S and S' such that $S \supseteq M_p$ and $S' \subseteq \text{Dom}(R)$.
(ii) If R is a conversely weak reduction of T_m to T_m', then there is a conversely weak representation of T in T' with respect to any S such that $S \subseteq M_p$ and to $S' = \text{Dom}(R)$.
(iii) If R is a strong reduction of T_m to T_m', then there is a strong representation of T in T' with respect to $S = M_p$ and $S' = \text{Dom}(R)$.

Proof. It is obvious that T and T' are theories in the statement sense. Since we have assumed that the non-logical vocabularies of L and L' are disjoint and R is projectively definable, and because (5.2) is supposed to hold for all three kinds of reductions, we can apply Lemma 8, obtaining an $f\colon \text{Sent}(L) \to \text{Sent}(L')$ such that $R \subseteq R_f$. Lemma 6 then implies that we have representations of the language L in L' with respect to S and S' of the appropriate kinds (remember that by (5.1), $\text{Ran}(R) = M_p$). That they are representations of the theory T

in T' of the corresponding kinds follows immediately from the clauses of Lemma 10. □

Theorem 2 says that under certain conditions, especially projective definability, representations can be obtained from reductions. That the choice of Θ, Θ', S and S' is canonical can be seen as follows. The choice of Θ is completely natural. To take Θ' as Th($M' \cap$ Dom(R)) instead of Th(M') is justified since the notions of reduction of T_m to T_m' are completely unspecific as to $M_p'\backslash$Dom(R) and $M'\backslash$Dom(R). A (weak, conversely weak, strong) reduction of T_m to T_m' is at the same time one of $\langle M_p,M\rangle$ to $\langle M_p' \cap \text{Dom}(R), M' \cap \text{Dom}(R)\rangle$. What falls outside the intersection with Dom(R) can be chosen in an arbitrary way. Note that Θ and Θ', as defined in Theorem 2, are deductively closed in the usual sense although this is not required by our notion of a theory. S and S' can always be chosen as M_p and Dom(R), that is, as those sets of structures that are actually related by R, and may therefore be considered to contain the essential structures to be compared by the reduction.

The standard example of a reduction in the model-theoretic sense is the reduction of classical rigid body mechanics to classical particle mechanics as treated in Adams (1959) and Sneed (1971, chap. 7). It is based on the idea that a rigid body may be considered a set of particles with the property that the distance between any two particles does not change over time. Following Sneed, a weak reduction relation R between the two theories can be defined. Pearce (1982b) was able to show that R fulfills certain conditions that imply the assumptions of Theorem 2. Thus a weak representation of corresponding theories in the statement sense can be obtained in this case.

7. Commensurability and Stegmüller's Thesis. We have shown that under certain conditions, representations and reductions of theories can be transformed into each other. As regards Stegmüller's thesis ("reducibility is compatible with incommensurability"), the transition from reductions to representations, as expressed by Theorem 2, is of specific importance. More precisely, since the concept of reduction favored by Stegmüller corresponds to what we call a weak reduction, Theorem 2 (i) is the crucial assertion which says, in particular, that from a weak reduction R of a theory T_m in the non-statement sense to a theory T_m' there can be obtained a weak representation of a corresponding theory T in the statement sense to a theory T' with respect to the sets of structures M_p and Dom(R). This would immediately refute Stegmüller's thesis if first, the conditions under which Theorem 2 holds were always fulfilled for weak reductions, and if secondly, a weak representation (with respect to the indicated sets of structures) always made theories commensurable.

We do not want to tackle the first question in detail here. Theorem 2 mainly depends on Lemma 8, which assumes that R is projectively definable, and whose proof essentially uses the interpolation theorem, which is valid in the framework of first-order logic that we have assumed throughout this paper. In his reply to Pearce (1982a), Balzer (1985) argued that all relevant theories use at least second-order logic, for which the interpolation theorem is not available. However, even if Balzer is right in his claim that most or all of the important theories are of second or higher order (better arguments for this thesis than his own can be found in Shapiro 1985), this does not necessarily refute the applicability of Lemma 8. What is required there is only that the *reduction relation* R be projectively definable in a first-order language, not that all sets making up the theories considered be definable in that way. It may well be that the reduction relation R between theories $T_m = \langle M_p, M \rangle$ and $T_m' = \langle M_p', M' \rangle$ is projectively definable in a first-order language (and therefore also M_p, being equal to Ran(R)), even though M_p', M and M' are sets that are definable only in second-order logic. Pearce (1982b) gives arguments that in central cases R *is* first-order definable. In the previous section we have at no place assumed that certain sets of structures can be characterized by sets of first-order sentences. The disjointness of the non-logical vocabularies of L and L' that is also required in Lemma 8 presents no problem to us, since we may achieve it by re-labeling constants. However, it *is* a problem for Balzer and Stegmüller who base their commensurability concept on the identity of symbols (see below).

As regards the second point, we must give a definition of commensurability and show how it is related to the notions of representation. We shall now give such a definition, indeed, many, interrelated definitions. Our proposals are based on two ideas both of which make commensurability differ from representability:

(i) Commensurability concerns scientific *concepts* and not only *statements,* "concept" here understood as comprising functions as well (we avoid speaking of "scientific terms" because of the possible confusion with the logical usage of "term" as opposed to "formula"). In logical terminology this means that predicate constants or, more generally, *open* formulas, and function constants or, more generally, *open* terms of one theory must be given a meaning from the viewpoint of another theory. For example, we not only want to say that an atomic sentence σ_1 of T which has the form $P(t_1)$ is to be understood as σ_1' in T', σ_2 of the form $P(t_2)$ as σ_2' and σ_3 of the form $P(t_3)$ as σ_3', but more specifically something like the following: To $P(v)$ as an open formula of T with individual variable v there should correspond an open formula $A'(v')$ of T' with individual variable v' in such a way that the sentences $P(t_1)$, $P(t_2)$ and $P(t_3)$ are related to sentences $A'(t_1')$, $A'(t_2')$ and $A'(t_3')$ for certain closed terms t_1', t_2' and t_3'. That is, applications of one and the same P to different arguments in T should be related to applications of *one and the same* formula $A(v)$ to different arguments in T'. In general, correspondence of sentences should be a consequence of correspondence of (perhaps open) formulas and (perhaps open) terms.

(ii) Commensurability may be partial. It should make sense to speak, for example, of $\{P_1, \ldots, P_n\}$-commensurability of theories for certain distinguished predicate letters $P_1, \ldots, P_n$, expressing that the *specific* constants $P_1, \ldots, P_n$ but not necessarily *all* constants of one theory can be understood from the viewpoint of another. *Full* commensurability with respect to *all* non-logical constants of a theory is just a limiting case.

The idea of commensurability as a correspondence of *open* formulas and terms ("term-for-term correspondence" in the terminology of the philosophy of science) also seems to underlie Balzer's (1985) and Stegmüller's (1986) proposals. However, they arrive at definitions of commensurability and incommensurability that are different from ours. In particular, Balzer's and Stegmüller's definitions of commensurability are relativized to reductions, and their definitions of incommensurability contain a quantification over all reduction functions between the theories considered. Thus they presuppose the structuralist concept of reduction whereas we define commensurability and incommensurability in a framework based purely on the statement view, not presupposing any concept of reduction to be given. Furthermore, Balzer's and Stegmüller's basic idea is that commensurable theories must contain *literally* equal constants that have the same extensions. Such a procedure has the awkward consequence that if, for example, T' results from T by re-lettering the non-logical constants, then T' and T are incommensurable, although they are in a plausible sense (namely, modulo re-naming of constants) the same theories. (This has also been pointed out by Pearce (1986) in his recent rejoinder to Balzer and Stegmüller.) Finally, Balzer's and Stegmüller's proposals contain some technical deficiencies (for example, they contain no condition on how to handle substitutions in open formulas or terms like (7.1) and (7.3) below).

In the following definitions, we continue to work within our framework of partial consequence, which we developed in order to have a notion of representation that exactly corresponds to the model-theoretic notion of reduction. However, if one reads the following as an attempt to define "commensurability" independently of its relationship to model-theoretic reductions, one may well skip the reference to S and S' in reading, or, in what comes to the same thing, consider S to be $\mathrm{Str}(L)$ and S' to be $\mathrm{Str}(L')$.

Our central concept will be that of a commensurability function *F from* a theory T *to* a theory T' which intuitively means that F makes concepts of T understandable in T' by giving them an analogue in T'. Symmetric notions of the commensurability of theories T *and* T' can then easily be defined. It seems to us, however, that one usually has an asymmetric notion in mind even if one speaks of the (in)commensurability of T *and* T'. For example, when one discusses the question of whether Newtonian physics and the general theory of relativity are commensurable or not, one asks whether the central concepts of the Newtonian theory can be understood in the modern theory, not the converse question, or the two together.

Let X be a set of non-logical constants of L, that is, of predicate, function and individual constants of L. Let Term(L) and Fml(L) be the sets of terms and formulas, respectively of L, and let Term(L,X), Fml(L,X) and Sent(L,X) be the sets of terms, formulas and sentences, respectively, of L that contain no constants beyond those in X. Of course, L must be a language with identity if it contains function constants. Let Var(L) be the set of individual variables of L. We use ϕ, v and t, with and without indices, as syntactical variables for elements of Fml(L), Var(L) and Term(L), respectively. Let $\alpha(v/t)$ denote the result of substituting t for v in an expression α, provided t is free for v in α, and otherwise α. As before, primed characters denote corresponding objects of L'. Let F be a function

$$F : \mathrm{Fml}(L,X) \cup \mathrm{Term}(L,X) \cup \mathrm{Var}(L) \rightarrow \mathrm{Fml}(L') \cup \mathrm{Term}(L') \cup \mathrm{Var}(L')$$

such that

$$F(\mathrm{Fml}(L,X)) \subseteq \mathrm{Fml}(L')$$
$$F(\mathrm{Term}(L,X)) \subseteq \mathrm{Term}(L')$$
$$F(\mathrm{Var}(L)) \subseteq \mathrm{Var}(L').$$

Let sets of structures $S \subseteq \mathrm{Str}(L)$ and $S' \subseteq \mathrm{Str}(L')$ be specified. Let $\models_S$ and $\models_{S'}$ now be understood as relating formulas rather than only sentences in the obvious way. For all $\phi \in \mathrm{Fml}(L,X)$ and t, t_1, $t_2 \in \mathrm{Term}(L,X)$ let the following conditions be fulfilled:

$$F(\phi(v/t)) \dashv\models_{S'} F(\phi)(F(v)/F(t)) \tag{7.1}$$

$$F(t_1 = t_2) \dashv\models_{S'} F(t_1) = F(t_2) \tag{7.2}$$

$$\models_{S'} F(t(v/t_1)) = F(t)(F(v)/F(t_1)). \tag{7.3}$$

Then, if for all $\Phi \subseteq \mathrm{Fml}(L,X)$ and $\phi \in \mathrm{Fml}(L,X)$,

$$\Phi \models_S \phi \Rightarrow F(\Phi) \models_{S'} F(\phi) \tag{7.4}$$

and

$$F(\neg\phi) \dashv\vDash_{S'} \neg F(\phi) \tag{7.5}$$

hold, F is called a *weak X-commensurability function* from L to L' with respect to S and S'. If instead of (7.4) its converse

$$F(\Phi) \vDash_{S'} F(\phi) \Rightarrow \Phi \vDash_S \phi \tag{7.6}$$

holds, we speak of a *conversely weak X-commensurability function,* and if all three of (7.4), (7.5) and (7.6) hold, of a *strong X-commensurability function* from L to L' with respect to S and S'.

The conditions (7.1), (7.2) and (7.3) express the idea that scientific concepts, that is, predicate, function and individual constants, must be mapped to formulas, terms and individual constants, and not only sentences to sentences. (7.4), (7.5) and (7.6) are just the conditions we had before for representations, but now for formulas instead of sentences. If one is only interested in what we call a *weak* X-commensurability function and wants to work without partial consequence, then one may use the compactness of the first-order consequence relation $\vDash$ and replace (7.4) by

$$F(\phi_1 \wedge \phi_2) \dashv\vDash F(\phi_1) \wedge F(\phi_2)$$

where "$\wedge$" is the conjunction sign of the object language. Furthermore, it would then suffice to require (7.1) for atomic formulas ϕ and add

$$F(\bigwedge v\phi) \dashv\vDash \bigwedge; F(v)F(\phi)$$

as a condition for the universal quantifier "$\bigwedge$" of the object language. This would yield a definition of "commensurability function" which simply requires that F distributes over substitution in atomic formulas and over the logical constants. Thus, apart from the relativization to X, it comes very close to Tarski's (1953) notion of weak interpretability.

The concept of a theory in the statement-sense need not be changed since theorems that are open formulas may be identified with their universal closures, and therefore sets of theorems can as before be considered to be sets of sentences. Thus given theories $T = \langle L,\Theta\rangle$ and $T' = \langle L',\Theta'\rangle$, we call F a *weak [conversely weak, strong] X-commensurability function from T to T′ with respect to S and S′*, whenever F is a weak [conversely weak, strong] X-commensurability function from L to L' with respect to S and S' and $F(\Theta) \subseteq \Theta'$ [$F(\Theta) \supseteq \Theta' \cap \mathrm{Ran}(F)$, $F(\Theta) = \Theta' \cap \mathrm{Ran}(F)$].

Remember that $F(\Theta)$ contains all sentences $F(\sigma)$ for $\sigma \in \Theta \cap \mathrm{Dom}(F)$, that is, "$F(\Theta)$" is a meaningful expression even if Θ is not fully contained in the domain of F. In particular, $F(\Theta)$ is empty if each sentence in Θ contains at least one non-logical constant that is not in X, that is, if $\Theta \cap$

$\mathrm{Fml}(L,X) = \varnothing$. In such a case, F is a weak X-commensurability function by trivial reasons. This is not counterintuitive. If all theorems of T mix constants from X with constants outside X, no separate assertion is made in T about the elements of X, so what is said in T about the elements of X alone (namely nothing) can trivially be understood from the viewpoint of T'. What is *implicitly* said in T about X, in the sense that constants outside X are involved, can be understood in T' only through a Y-commensurability function for some $Y \supseteq X$ that contains these additional constants. It is quite plausible that, for example, if the theorems of T characterize the predicate constant P_1 only together with the constants P_2 and P_3, then a $\{P_1,P_2,P_3\}$-commensurability function is required to give P_1 (and at the same time P_2 and P_3) an analogue in T'. By similar reasons, if an X-commensurability function F_1 and a Y-commensurability function F_2 are given, then an $X \cup Y$-commensurability function need not necessarily exist. It may well be that F_1 transforms those theorems of T that only involve constants from X and F_2 those theorems that only involve constants from Y without there being an F that transforms those theorems that involve constants from both X and Y. If one wants to avoid such consequences, one has to work with "full" commensurability functions only, that is, with X-commensurability functions where X contains the whole non-logical vocabulary of L. We do not see any possibility of defining a concept of commensurability that on the one hand is *restricted* to a specific non-logical constant P but on the other hand respects the full meaning of P in a theory T even if in T this meaning of P is only implicitly determined together with the meanings of other constants.

When we describe it as the task of an X-commensurability function to represent in T' the meaning that a constant from X has in T, "meaning" is understood as something that is specified by the *theorems* of T, that is, by certain laws that hold of this constant. In other words, we rely on the statement view of theories when dealing with commensurability (see section 1 above). This makes our proposal strongly differ from Balzer's and Stegmüller's who refer to extensions in models as the meanings of non-logical constants and require equality of extensions in related models as the central criterion of commensurability (see section 1 above). Our rendering of "meaning" as something that is purely *internal* to a theory seems to us to be narrower to the notion of commensurability as used, for example, in discussions in the context of Feyerabend's and Kuhn's writings (see Feyerabend 1962, p. 74ff.; Kuhn 1962, pp. 102 and 128f.). There the question was always whether or not certain conceptual frameworks can be related to each other, and not whether extensions of concepts (which are external to the theories in which the concepts are embedded) are the same. (Feyerabend and Kuhn would probably even deny that we can speak of extensions of scientific concepts independent of theories.) The fact that Balzer and Stegmüller base their notion of commensurability on the equality of extensions of constants in different theories seems to us to be due to the fact that they consider commensurability only in the context of model-theoretic reduction.

In order to relate representations to commensurability functions, we define the following: We say that a weak [conversely weak, strong] X-commensurability function F from T to T' with respect to S and S' is an extension of a weak [conversely weak, strong] representation of T in T' with respect to S and S', if $F|_{\mathrm{Sent}(L)} = f|_{\mathrm{Sent}(L,X)}$, where the vertical bar expresses the restriction of the domains of F and f to the indicated sets. If X contains the whole non-logical vocabulary of L, this means the same as $F|_{\mathrm{Sent}(L)} = f$. It is obvious that a weak [conversely weak, strong] representation f of T in T' with respect to S and S' cannot necessarily be extended to a weak [conversely weak, strong] commensurability function F from T to T' with respect to S and S', since, if f is given, nothing is known about open formulas and open terms. This holds especially if f results by Theorem 2 from a reduction. If the assumptions of Lemma 8 are fulfilled, a reduction gives rise to a representation, but not necessarily to an X-commensurability function, however X may be chosen. Stegmüller's thesis that reduction does not imply commensurability is fully confirmed by the given reasoning, provided one accepts our definition of "X-commensurability function" as an appropriate approach to the notion of "commensurability". This does not, of course, preclude that by *strengthening* the conditions for reductions (for example, by adding certain specific model-theoretic requirements), commensurability functions may be obtained from reductions (see van Benthem and Pearce 1984).

To complete our definitions, we say what commensurability as distinguished from commensurability functions should mean. For simplicity, we omit the specifications "weak", "conversely weak", and "strong" and also skip the reference to S and S'. It is clear how by use of these additional specifications different concepts of commensurability can be obtained. We define: *T is X-commensurable in T'* if there is an X-commensurability function from T to T'. *T is fully commensurable in T'* if T is X-commensurable in T', where X comprises all the non-logical constants of L. Similarly, a symmetric concept of commensurability can be defined: If X and X' are sets of non-logical constants of L and L', respectively, then *T and T'* are called *(X,X')-commensurable,* if T is X-commensurable in T' and T' is X'-commensurable in T. *T and T' are fully commensurable* if T is fully commensurable in T' and T' is fully commensurable in T. One can obtain weaker notions of X-commensurability if one only requires that there be an X-commensurability function from T to a *consistent extension of T'*. According to this approach, T and T' are already commensurable with respect to certain concepts of T, if the theorems of T' do *not exclude* that these concepts have analogues in the framework of T', although T' may still be too weak to characterize these analogues sufficiently. (This proposal seems to correspond to the view expressed in Feyerabend 1962, p. 74–76.)

Various concepts of *non-commensurability* can be obtained by negating the corresponding notions of commensurability. However, non-commensurability need not mean *incommensurability*. Intuitively, incommensurability means non-commensurability in the presence of a certain relatedness of theories. Theories that have "nothing to do with each other" are non-commensurable, but not incommensurable. To give this relatedness of theories a precise rendering is still a desideratum in the philosophy of science.

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