IMPRECISION AND INDETERMINACY IN PROBABILITY JUDGMENT*

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Bayesians often confuse insistence that probability judgment ought to be indeterminate (which is incompatible with Bayesian ideals) with recognition of the presence of imprecision in the determination or measurement of personal probabilities (which is compatible with these ideals). The confusion is discussed and illustrated by remarks in a recent essay by R. C. Jeffrey.

1. Introduction. Some who deny that states of probability judgment ("credal states" as I shall call them) are numerically definite have sought to represent them in terms of a relation of comparative probability. Others use functions assigning upper and lower probabilities (or, alternatively, interval-valued probabilities) to hypotheses. Still others represent credal states by means of sets of probability functions defined for the relevant algebras of propositions or events.

I favor using sets of probability functions to represent credal states. (Levi 1974, 1980). One advantage of doing so is that this approach can be employed to make useful distinctions between credal states which formulations in terms of upper and lower or comparative probabilities cannot capture, while allowing for the expression of claims that these alternatives make. But, for many purposes, this point may be neglected. In spite of their differences, writers like A. P. Dempster (1967), M. Wolfenson and T. Fine (1982), P. Gärdenfors and N. -E. Sahlin (1982a), I. J. Good (1952, 1962), B. O. Koopman (1940), H. E. Kyburg (1961, 1974), C. A. B. Smith (1961), F. Schick (1958, 1984), and P. Williams (1976) all explore the possibilities of using upper and lower probability functions or the sets of probability distributions that such functions envelop in the systematic characterization of credal states.

Levi 1974 and 1980 point to an equivocation in the way sets of probability functions (or interval-valued functions) have been interpreted in the literature.

*Received May 1984; revised May 1984 and September 1984.

†Thanks are due to Richard Jeffrey, Henry E. Kyburg, Jr., Sidney Morgenbesser, Calvin Normore, Nils-Erik Sahlin, and especially Teddy Seidenfeld and the two referees for *Philosophy of Science* for helpful criticisms and comments.

Philosophy of Science, 52 (1985) pp. 390–409.

According to what I now call the "strict Bayesian" viewpoint, a rational agent is committed to recognizing a single probability function for use in computing expected utilities in any given context of deliberation. That probability function is supposed to represent his credal state in a quantitatively precise manner.

Strict Bayesians have often concerned themselves with devising procedures whereby one might, at least in principle, use choice behavior or other data to ascertain details of the agent's credal state. (See Savage 1971 for a discussion of this approach.) The problem is to measure degrees of belief or credal probability. And it is widely recognized that the techniques available to third parties and even to the agent himself may be too crude to yield precise numerical determinations of credal probabilities. We surely cannot expect more precision in this domain than we can in physics.

Thus, it is to be expected that we should not be in a position to identify an agent's credal judgments with utter precision, but only be able to make claims like the judgment that X's degree of belief that h is no less than r and no greater than $\bar{r}$.

Such claims are compatible with maintaining that the agent has a numerically definite, strictly Bayesian credal state; or, at any rate, that the agent is committed to having one whether he has lived up to his commitment or not. Moreover, it becomes possible to describe the set of hypotheses which fully specify the agent's strictly Bayesian credal state by a set of probability functions. Each function corresponds to a hypothesis as to the agent's strictly Bayesian credal state.

Exploiting a metaphor used in Good 1962, I have proposed to call the use of a set of probability functions to represent a range of hypotheses concerning the agent's credal state, each of which is consistent with the data, a "black box" construal. According to strict Bayesians, there is exactly one "permissible" distribution in the black box representing the system of probability judgments the agent is committed to accepting whether he knows it or not. The set of distributions represents a set of rival hypotheses about the unknown contents of the black box.[1]

By way of contrast, when a set of distributions is used under the permissibility construal to represent an agent's credal state, a fundamental tenet of strict Bayesianism is denied. The credal state in the agent's black box is not always representable by a single probability function. Like R. A. Fisher, J. Neyman, and H. E. Kyburg, advocates of the permissibility

[1]It is possible to entertain an *als ob* black box construal that embraces all the ramifications of black box construals for decision making but shies away from asserting that there is a fact of the matter as to which probability function is the correct representation of the credal state. For the purposes of this discussion, the difference between this view and the black box view is unimportant and shall, therefore, be ignored.

construal like myself hold that rational agents often do not and should not regard exactly one real-valued probability function to be permissible for use in assessing expected utilities. The credal state should be represented by a set of permissible probability functions. The agent may or may not know the contents of his credal state—that is, the contents of his black box. That is not being disputed. What is being emphasized is that the credal state in the black box may itself allow more than one distribution to be permissible in determining expected utilities. In that case, a set of probability functions can be used to characterize the credal state as a set of permissible probability distributions and not as a set of possibly true hypotheses concerning the unknown uniquely permissible distribution.

There is some recent evidence that this distinction between black box construals of sets of distributions and permissibility construals is not well understood. There is a tendency to understand the permissibility construal as if it were equivalent to the black box construal. This mistake is one that strict Bayesians are especially prone to make; for it supports the idea that those, like myself, who mean to reject strict Bayesian dogma are not really doing so but are, instead, merely taking note of a point which strict Bayesians themselves have often emphasized—to wit, that human agents cannot identify and ought not to be expected to be able to identify their strictly Bayesian credal probability judgments with full numerical precision. Acknowledging this point coheres well with adherence to strict Bayesian ideals. Hence, critiques of strict Bayesian doctrine that object to the numerical precision required by the ideal of strict Bayesianism are accommodated in the ample bosom of Mother Bayes.

In the remainder of this paper, I shall try to bring out the difference between the black box and permissibility construals of sets of probability functions or, equivalently, between imprecise and indeterminate probabilities. I shall try to do so by relating the distinction to the ideas developed by R. C. Jeffrey in his 1965 and 1983 that seem to exhibit a subtle variant of the misunderstanding to which I am referring.

2. Commitment versus Performance. Jeffrey, I, and everyone else concede or should concede that the demands of strict Bayesian principles cannot be met by ordinary mortals in contexts of any complexity. Failures of memory and computational capacity, lack of self-awareness, and emotional stress can all present obstacles to compliance.

A strict Bayesian might respond in one of two ways:

(a) Human agents are in strictly Bayesian credal states unknown to themselves even when they are not able at the moment to recognize those states.

(b) A human agent is committed to being in some strictly Bayesian credal state without actually being in that state due, perhaps, to his failure to recognize what is entailed by living up to that commitment, or due to emotional factors inhibiting his fulfillment of those commitments.

In any case, these difficulties should not count as an objection to strict Bayesian ideals of rationality unless they count against all alternative ideals as well. No matter what alternative code one follows, contexts can and will arise where the tasks of computation and the strains and stresses of life will prevent the agent from conforming with the alleged canons of reason. There are no useful norms of rational belief, valuation, or action that humans can live up to under all circumstances.

Thus, I. J. Good's black box metaphor is applicable whether one is strictly Bayesian or not. An agent may not be able to identify the credal state to which he is committed even if he is of an open mind as to whether that credal state allows more than one probability function to be permissible. The difference between a critic of strict Bayesianism and a strict Bayesian in this respect concerns the contents of the black box. For the strict Bayesian, the contents to which the agent is committed (whether he is aware of it or not) consist of a single distribution; for his opponent, the contents may consist of several distributions.

Still strict Bayesians display a special fascination for black box interpretations of sets of probability distributions. The reason is that they are opposed to recognizing rationally acceptable credal states with more than one permissible probability function. When critics of Bayesianism like R. A. Fisher or J. Neyman complain of the arbitrariness or dogmatism involved in assigning numerically definite credal probabilities to hypotheses without grounding in knowledge of statistical probability, strict Bayesians deflect the complaint by reconstruing the objection as pointing to the agent's ignorance of his own state of subjective probability judgment. Indeterminate probabilities (which strict Bayesians cannot countenance) are transmuted into imprecise probabilities (which they can).

The reduction is an illegitimate one. To illustrate this, a well-known example taken from Ellsberg (1964) will be discussed.

3. The Ellsberg 'Paradox'. Suppose the agent faces an urn which he knows contains thirty red balls and sixty balls which are black and yellow in some unknown proportion. He is told that a ball is to be selected at random from the urn and is offered the following pair of bets with the payoffs specified:

	Red	Black	Yellow
I	$100	$0	$0
II	$0	$100	$0

After declaring how he would choose were these the sole options open to him, he is asked how he would choose were the following alternatives available to him rather than I and II:

III	\$100	\$0	\$100
IV	\$0	\$100	\$100

Under plausible assumptions, we might suppose that the agent assigns a credal probability of $^1/_3$ to the hypothesis that a red will be selected; and, indeed, authors like Fisher and Neyman would endorse such a judgment. The reason is that the agent can ground his credal probability judgment on knowledge of statistical probabilities or chances. But they would insist that the agent has no such warrant for making any definite credal probability assignment to the hypotheses that the ball will be black or that the ball will be yellow. All that can be said is that there is a $^2/_3$ probability that it will be one or the other.

Utilizing the permissibility construal of sets of distributions, I propose reconstruing their view as recommending that in such situations rational agents regard all credal distributions to be permissible that: (i) assign $^1/_3$ to red, $^2/_3$ to black and 0 to yellow; (ii) assign $^1/_3$ to red, 0 to black and $^2/_3$ to yellow; or (iii) assign values that are some weighted average of (i) and (ii).

How does this bear on the agent's choices? In each problem, determine the expected utilities (expected values) of the options using the payoffs as utilities and using some given permissible probability distribution as given by (i)–(iii). The result is a permissible expected-utility function defined for the feasible options which in this case are I and II. An option is "*E*-admissible" if it comes out best according to some permissible expected-utility function. A rational agent is required to restrict his choices to *E*-admissible options. Once this is done, other tests of admissibility may be applied to the *E*-admissible set to determine narrower sets of admissible options. (See chapters 6 and 7 of Levi 1980.)

In the first problem, both options are *E*-admissible. Hence, considerations of expected utility fail to decide between the feasible options. If one appeals to considerations of security (i.e., if one maximins), option I will be favored. But even if one does not maximize one's security level as a secondary criterion, when considerations of expected utility fail to decide, one is free to choose option I in the first problem.

In the second problem too, both options are *E*-admissible. Considerations of security argue in this case for choosing IV. But even if one is not a maximiner, one might reasonably choose IV. Hence, a rational agent might be prepared to choose I in the first problem were he to face it and choose IV in the second. In particular, there is no violation of the sure thing principle here; for the agent does not categorically prefer I over II, or IV over III (Levi 1982, pp. 405–8).

Strict Bayesians cannot tolerate this kind of analysis. They may admit that honest men are often unprepared to assign definite credal probabilities to the hypotheses that a black or that a yellow will be drawn; but they insist that this reluctance is due to a lack of self-knowledge or to an incapacity to live up to their commitments. Still, to be rational, the agent should act as if he did have a strictly Bayesian credal state where only one probability distribution is permissible. This means that if he chooses I over II, he should choose III over IV.[2]

That is to say, even if the agent does not know the precise contents of his black box, he knows it contains a single permissible probability function. Hence, if he makes an estimate of that function, he should use that estimate in both decision problems as the uniquely permissible probability.

According to the permissibility interpretation, several distributions are permissible in both decision problems. Some favor one option and some favor another. There is not the slightest obligation on the rational agent to choose in a way that optimizes relative to the same probability distribution in the two distinct hypothetical contexts.

To be sure, in the quite distinct context where the agent is offered the two decision problems jointly so that he faces only one problem with four options (choosing I and III, choosing I and IV, choosing II and III, and choosing II and IV), all four options are *E*-admissible while both choosing I and IV and choosing II and III maximize security. But in the Ellsberg paradox, we are invited to consider the two decision problems in separate contexts where one does not face the two problems jointly. According to the permissibility construal, the treatment of these two problems comes out quite differently from the treatment of the case where both choices are offered jointly.

An entirely parallel observation may be made concerning indeterminate

[2]Gärdenfors and Sahlin (1982a) derive the recommendations allowed by the permissibility interpretation of credal states from a decision theory that exploits sets of probability distributions in the treatment of belief states. This feature implies that the Gärdenfors-Sahlin interpretation of these sets is not a black box construal. But the permissibility construal does not quite fit either; for the decision theory they employ differs from the approach congenial to the permissibility interpretation. I have rehearsed some difficulties I have with their approach in Levi 1982b. In Gärdenfors and Sahlin 1982b, they reply that their sets are to be interpreted in a way I had not anticipated they would intend—to wit, as sets of subjective distributions. But this construal is a black box interpretation! It coheres poorly with their treatment of the Ellsberg problem. As I read them, Gärdenfors and Sahlin are proposing to modify the Bayesian dogma in a spirit similar to the attitude I favor. The disagreements I have with them concern the best way to proceed.

utility judgment and E-admissibility in the case of the well-known Allais paradox. (See Levi 1982, pp. 241–43.)

Thus, when belief states are represented as sets of probability distributions under the permissibility interpretation, there are implications ensuing for rational choice which strict Bayesians cannot countenance even when they want to put a "human face" on things.

Before linking these ideas up with the theory of Jeffrey (1965), it should be emphasized that those who insist on the reasonableness of indeterminacy in probability judgment under the permissibility interpretation mean to claim that even superhumans ought not always to have credal states that are strictly Bayesian.

In this respect, advocates of the reasonableness of indeterminacy follow in the footsteps of C. S. Peirce, R. A. Fisher, J. Neyman, E. S. Pearson, and H. E. Kyburg, all of whom were or are well known anti-Bayesians. Except for Kyburg, these authors tend to suggest that sometimes it is rational to make no probability judgment. Following Kyburg, I prefer to say that it is sometimes rational to make no determinate probability judgment and, indeed, to make maximally indeterminate judgments. Here I am supposing, as all these authors have, that refusal to make a determinate probability judgment does not derive from a lack of clarity about one's credal state. To the contrary, it may derive from a very clear and cool judgment that on the basis of the available evidence, making a numerically determinate judgment would be unwarranted and arbitrary.

I think it fair to say that anyone who abandons strict Bayesianism has abandoned Bayesianism in its dominant guise. To understand Bayesianism in so generous a way as to encompass Peirce, Fisher, Neyman, Pearson, and Kyburg in the fold is to foster confusion and to show disrespect both for the integrity of Bayesian ideas and for the ideas of these anti-Bayesian authors.[3]

4. Technicalities. Because Jeffrey's own special way of confounding the distinction between imprecise and indeterminate probabilities is related to some of the technical details of Jeffrey 1965, it will be useful to reformulate the conception of indeterminate probability (and utility and ex-

[3]These critics of strict Bayesianism share their rejection of the requirement that credal probability judgments be numerically determinate. They differ from one another, however, in their responses to this point. And my own views differ from theirs in several critical respects. My ambition has been, in part, to be able to furnish a framework within which the diverse Bayesian and anti-Bayesian viewpoints may be seen as disagreeing about specific assumptions, relative to a background of shared assumptions. Within this framework, I have sought (in Levi 1980) to elaborate a position of my own. In this paper, I am concerned to emphasize my dissent from the dogmas of strict Bayesianism and am happy to associate my dissent with the anti-Bayesian authors I have cited. In other respects, which I have discussed elsewhere, my views exhibit a strong Bayesian character.

pected utility) in a way rendering it more directly comparable to Jeffrey's theory, even though some of the technicalities would not be required if we did not have the comparison with Jeffrey's theory in mind. In this section, I shall introduce the technicalities and turn to Jeffrey's ideas in the sections that follow.

Suppose agent X has a body of knowledge represented by a deductively closed set of propositions K. A K-proposition is representable by a set of propositions equivalent given K. K itself may then be represented by the K-proposition T_k.

Attention shall be focused on a Boolean algebra of K-propositions $W(U) = W$ generated by taking all Boolean combinations in a finite set U of K-propositions such that each is consistent with K and such that K entails the truth of at least and at most one element of U.

Obviously such an algebra of K-propositions will not suffice to characterize all hypotheses X might consider at some time or other relative to K. But we shall never find a finest partition U^{**} (i.e., a space of possible worlds). In any context of inquiry, we focus on some range of alternative hypotheses of interest on the assumption that should we enlarge our horizons we could consistently extend the appraisals we have already made concerning the elements of W to a larger algebra in which W is embedded.[4]

The next ideas to be introduced are the notions of a credal state and a value structure. A credal state corresponds to a state of belief or subjective probability judgment. A value structure corresponds to a state of utility appraisal.

A credal state B for W is a convex set of normalized and finitely additive probability functions over W. Again, for the sake of simplicity, I shall consider only those cases where each probability measure in B assigns positive probability to every element of U and, as a consequence, for each permissible probability function in B, the conditional probability on some proposition on some K-proposition in W consistent with K is uniquely determined by the unconditional permissible probability function.

A value structure $V(U)$ for the elements of U is a nonempty, convex set of real valued utility functions closed under positive affine transformations where the domain of definition of the utility functions is the set

[4]With this understood, my restriction of U to a finite number of elements is intended to avoid special complications arising when U goes infinite. I discussed some of these in Levi 1980, but more profound discussions are available in Seidenfeld and Schervish 1983, and in the papers by J. B. Kadane, T. Seidenfeld, and M. Schervish referenced there. The interesting issues concerning infinity and probability discussed in these essays are tangential to the points I mean to emphasize here.

U. Any pair of utility functions in $V(U)$ that are not positive affine transformations of one another are nonequivalent permissible valuations of elements of U.

Given a permissible probability function Q in B and a permissible utility (or permissible valuation) function v in $V(U)$, we can define a permissible desirability function *des* over the elements of $W - \{F_k\}$ where F_k is the inconsistent K-proposition. Any G in $W - \{F_k\}$ is a disjunction of some nonempty subset of elements $H_1, H_2, \ldots, H_k$ in U.

$$des(G) = \Sigma Q(H_i)v(H_i)/\Sigma Q(H_i).$$

It follows, of course, that for each H_i in U, $des(H_i) = v(H_i)$.

Thus, the credal state and the value structure determine a desirability structure (expected value structure) $D(W)$ that is a convex set of permissible desirability functions closed under positive affine transformations.

The desirability structure determines a preference structure. Given any permissible desirability function in $D(W)$, there is a permissible ranking of the K-propositions in W determined by it. The set of such permissible rankings is the preference structure $P(W)$.

For each G and G' in W, G is categorically strictly (weakly, indifferently) preferred to G' if and only if G is strictly (weakly, indifferently) preferred to G' according to every permissible ranking in $P(W)$. Categorical preference induces a quasi-ordering on the elements of W.

Suppose agent X faces options representable by K-propositions belonging to some subset O of W. Given the desirability and preference structures for W, a desirability structure $D(O)$ is defined for the feasible options in O. A preference structure $P(O)$ is, therefore, defined as well. If G is in O, choosing the options represented by G is choosing true the proposition G and all consequences of K and G. Hence, if there are options representable by G and GvG' where these are not equivalent given K, these options are exclusive. The agent cannot choose both of them. We are supposing, however, that he can choose at least and at most one option in O.

G is E-admissible relative to O, B, and $V(U)$ if and only if there is a permissible probability Q in B and a permissible utility v in $V(U)$ such that the pair (Q,v) define an element of $P(O)$ that ranks the option G as optimal among all elements in O. (Notice that G' is not E-admissible if some feasible G is categorically strictly preferred to it. However, the converse is not, in general, true.)

We have now returned to the point reached informally in our earlier discussion of E-admissibility. A rational agent X ought to restrict his choice from among the feasible options to the E-admissible ones. As already mentioned, other tests for admissibility may then be applied to reduce the admissible options still further.

When sets of probability functions over W and utility functions over U are used to define E-admissibility in the manner indicated and rational agents are, according to the proposed decision theory, to restrict choice to E-admissible options, the sets of probability distributions and the sets of utility functions represent states of belief and states of valuation under the permissibility interpretation.

Strict Bayesians insist that credal states recognize only one Q-function to be permissible. They also require that states of valuation recognize only a single set of equivalent utility functions to be the permissible set—i.e., the set should be unique up to a positive affine transformation.

If strict Bayesian strictures are observed, the set of permissible desirability functions will also be unique up to a positive affine transformation and $P(W)$ will allow exactly one preference ranking to be permissible. All E-admissible options will be E-optimal—i.e., best according to the categorical or uniquely permissible preference ranking.

Armed with these technicalities, we are in a position to address Jeffrey's ideas.

5. Jeffrey's Quantization. In a recent essay (Jeffrey 1984), R. C. Jeffrey writes that the Bolker-Jeffrey system, which he expounded in Jeffrey 1965 and reiterated in the second revised edition (Jeffrey 1983) gives

> one a clear version of Bayesianism in which belief states—even superhumanly definite ones—are naturally identified with infinite sets of probability functions, so that degrees of belief in particular propositions will normally be determined up to an appropriate quantization—i.e., they will be interval valued (so to speak). Put it in terms of the *the thesis of primacy of practical reason,* i.e., a certain sort of pragmatism, according to which belief states that correspond to indentical preference rankings are in fact one and the same. (Jeffrey 1984, p. 138)

Jeffrey himself does not insist on the thesis of the primacy of practical reason. However, he asserts that it is an intelligible claim consonant with Bayesianism. He then writes:

> Applied to the Bolker-Jeffrey theory of preference, the thesis of the primacy of practical reason yields the characterization of belief states as sets of probability functions (Jeffrey, 1965, 6.6). Isaac Levi (1974) adopts what looks to me like the same characterization but labels it 'unBayesian'. (Jeffrey 1984, p. 139)

Jeffrey's "quantization" undoubtedly generates a set of probability distributions for a suitable algebra. But this does not mean that the set of probability distributions so obtained represents (is "naturally identified" with) a belief or credal state even when the theory enunciated in Jeffrey 1965 is supplemented by a thesis of the primacy of practical reason. In the subsequent discussion, I shall show that Jeffrey's theory cannot be interpreted successfully as using sets of probability distributions to represent credal states—counter to the claim he makes in the passages just cited. In particular, I shall argue the following:

Thesis 1: Jeffrey's sets of distributions derived from "quantizations" cannot be given a permissibility interpretation without contradicting the "superhuman" conditions to which Jeffrey alludes in the passages just cited and which form the cornerstone of the discussion in Jeffrey 1965. Hence, Jeffrey cannot be using these sets to represent credal states in the sense I explicitly intended in my 1974 and 1980 without drastic revision of the doctrine of Jeffrey 1965. Adding a principle (whatever it might be) that yields a permissibility interpretation must also lead to contradiction.

Thesis 2: Jeffrey's sets of distributions can be given a black box interpretation; but then the set of probability distributions cannot be "naturally identified" with (i.e., represent) the belief state. Rather, each member of the set of distributions is to be regarded as a black box hypothesis asserting that the unknown strictly Bayesian belief state of the agent is represented by that member of the set. Such a reading makes his position beg the normative issue I have raised about the contents of the black box.

Thesis 3: One may deny that there is a belief state separate from a state of desire to represent and, hence, deny that the set of probability functions determined by Jeffrey's quantization represents a separate belief state. On this view, there is a state of belief-desire representable by each member of a given set of pairs of functions. One member (the first by our convention) is a probability function defined over the given algebra. The second is a utility function or, alternatively, a desirability function. If A is the set of such pairs, each pair (Q,v) in A is equivalent to any other as a representation of the belief-desire state. Of course, one can formally construct the set LA. But, I shall argue, it is nonsense to say that LA represents or is naturally identified with a belief or credal state. There is no credal state but only a belief-desire state.

I do not know which of the readings described above captures Jeffrey's intent when he claims that my proposal can be derived from his with the aid of a thesis of the primacy of practical reason. Only he can tell us whether he meant any one of them or had some other idea up his sleeve. But as I shall hope to show, the prospects are dim for his being able to make good his claims. We have, I fear, another illustration of a Bayesian wolfman seeking to disguise himself in non-Bayesian clothing.

6. Thesis 1: The Permissibility Construal. Suppose U consisted of just the three K-propositions H_1, H_2, and H_3. Consider any pair (Q,v) consisting of a probability function over $W(U)$ and a utility function over U that assigns positive probability to every element of U. Finally suppose that the utility of H_1 is greater than that of H_2 which in turn is greater than that of H_3.

From this slender information we can conclude that the desirability function determined by the pair (Q,v) ranks the seven K-propositions in $W(U) - \{F_k\}$ in one of the following three ways:

(a)	(b)	(c)
H_1	H_1	H_1
$H_1 \cup H_2$	$H_1 \cup H_2$	$H_1 \cup H_2$
H_2	H_2, $H_1 \cup H_3$, T_k	$H_1 \cup H_3$
T_k	$H_2 \cup H_3$	T_k
$H_1 \cup H_3$	H_3	H_2
$H_2 \cup H_3$		$H_2 \cup H_3$
H_3		H_3

Since we are supposing that all utility functions over U that are positive affine transformations of one another are equivalent, we may for convenience stipulate that $v(H_1) = 1$ and $v(H_3) = 0$. Then $v(H_3)$ will equal some value s between 0 and 1.

In (a), $des(H_1 \cup H_3)$ is less than s. In (b) it is equal to s (which must equal $des(T_k)$), and in (c) it must be greater than s.

$$des(H_1 \cup H_3) = \frac{[Q(H_1)\, v\,(H_1) + Q(H_3)\, v\,(H_3)]}{Q(H_1) + Q(H_3)}$$

$$= \frac{Q(H_1)}{Q(H_1) + Q(H_3)}$$

Consequently we are in a position to derive information about probability-utility pairs representing an agent's preference ranking over $W(U) - \{F_k\}$ from full information about the ranking itself.

Thus, if we know that the only permissible preference ranking in an agent's preference structure $P(W)$ is (a), we may say that the probability assignments to all elements of U are positive and $v(H_2)$ is greater than $Q(H_1)/[Q(H_1) + Q(H_3)]$.

For any probability function Q assigning positive probability to all of the H_i's, there will be at least one function v which assigns H_1, H_3 the value 0 and H_2 a value s so that the ranking (a) is generated. Furthermore, the same ranking must be generated when the probability function is combined with any positive affine transformation of v.

It is also true that for a given value function (utility function) v, there will be several probability functions which combine with it to generate the ranking (a). For example, if $v(H_2) = 0.5$, then the Q-function assigning 0.49 to H_1 and 0.5 to H_3 can combine with the value function to generate (a). So can the value function assigning 0.245 and 0.25, the function assigning 0.163 and 0.167, etc.

This circumstance raises the following question: Can we embed the preference ranking over $W(U) - \{F_k\}$ into a preference ranking over a larger set of propositions $W(U^*) - \{F_k\}$ in a way that reduces the number of pairs of probability and value functions allowed to represent the preference ranking (a).

The Bolker-Jeffrey theory offers an answer to this question. It is possible to identify an algebra $W(U^*)$ that contains $W(U)$ as a subalgebra and that has the following properties: If a preference ranking over $W(U^*) - \{F_k\}$ is given that preserves as a subranking the preference ranking (a) over $W(U) - \{F_k\}$, it will provide as much information as can be gleaned from preference rankings without supplementation from other sources concerning the probability and valuation functions that can generate desirability functions over $W(U^*) - \{F_k\}$ representing the given (a)-preserving ranking.[5]

Suppose we are given a preference ranking over $W(U^*) - \{F_k\}$ that preserves the preference ranking over $W(U) - \{F_k\}$, and that is related to the latter ranking in the manner indicated above. Given the "superhumanly definite" information conveyed in this ranking, it is possible to identify a set A of pairs (Q,v) of probability and value functions defined for $W(U)$ and extendable by means of (i), (ii), and (iii) of footnote 5 to

[5]One way to reach $W(U^*)$ from $W(U)$ is to generate a series of algebras $W(U^1)$, $W(U^2)$, . . ., $W(U^n)$, . . ., as follows: For each n, partition T_k into $2n$ K-propositions $G_1, G_2, \ldots, G_{2n}$ where, for fixed n, the G_j's meet the following conditions:

(i) Each G_j is consistent with each H_i (given K).
(ii) For every i, j, and j', $Q(H_i \,\&\, G_j) = Q(H_i \,\&\, G_{j'})$.
(iii) For every i, j, and j', $v(H_i \,\&\, G_j) = v(H_i \,\&\, G_{j'})$.

Let the boolean algebra generated from the refinement of U into U^n consisting of all H_i & G_j's be $W(U^n)$. Starting with a pair (Q,v) that generates the ranking over $W(U) - \{F_k\}$, conditions (i), (ii), and (iii) instruct us how to extend the definitions for the pair to $W(U^n) - \{F_k\}$ so that the new preference ranking preserves (a) as a subranking. This new preference ranking satisfies the existence, closure and G conditions in Jeffrey 1965, sec. 7.1, as well as the splitting condition, provided that condition is allowed to apply at most n times. This proviso is removed by shifting to $W(U^*)$ that is the union of the $W(U^n)$'s.

$W(U^*)$ such that each pair generates the given preference ranking over $W(U^*) - \{F_k\}$. This set is characterized by the transformations given in Jeffrey 1965, sec. 6.1 by means of conditions (6-1), (6-2), and (6-3).

Consequently, if we are primarily interested in the credal state and valuational structure for $W(U)$, we could exploit the information about the preference ranking that the agent identifies as uniquely permissible over $W(U^*) - \{F_k\}$ to obtain the desired answers.

Once we have the set A of pairs of probability and value functions, we can, of course, identify the set LA of left elements of such pairs and the set RA of right elements of such pairs. If I understand Jeffrey correctly, he claims that the set LA of probability functions obtained from A, where A is the set of all pairs that generate a specific preference ranking over $W(U^*) - \{F_k\}$, corresponds to the credal state that, on my proposal, would be recommended in such a situation.

This is demonstrably false.

If LA were to represent a credal state under the permissibility interpretation, each element of LA would have to be a permissible probability function. Moreover, each element of RA would have to be a permissible value function. And that implies that the desirability structure consists of every desirability function obtained by computation from a permissible Q-function in LA and a permissible v-function in RA.

That is to say, the set of permissible pairs of probability and value functions for use in computing desirabilities (i.e., expectations) would have to be the Cartesian product $LA \times RA$ of the sets LA and RA. But this Cartesian product will, in general, be a much larger set of pairs than the set A of pairs that generate a unique preference ranking over $W(U^*) - \{F_k\}$.

This point may be seen without appealing to the full technical apparatus of Jeffrey's theory. Consider the mini-algebra $W(U)$ and the preference ranking. Let $v(H_1) = 1$, $v(H_2) = 0.5$, and let $v(H_3) = 0$. Let $Q(H_1) = 0.2$, $Q(H_2) = 0.5$, and $Q(H_3) = 0.3$. This pair generates the ranking (a).

Contrast this with $v'(H_1) = 1$, $v'(H_2) = 0.3$, and $v'(H_3) = 0$ while $Q'(H_1) = 0.1$, $Q'(H_2) = 0.5$, and $Q'(H_3) = 0.4$. (Q',v') also generates the ranking (a).

However, if Q and Q' are both permissible probability functions in the agent's credal state and v and v' are permissible in his value structure, the preference ranking generated by (Q,v') ought also to be permissible. But this means that the preference ranking (c) would have to count as permissible in the agent's preference structure—counter to the assumption that (a) alone is permissible.

The phenomenon just illustrated operates also when we exploit the full resources of the ranking over $W(U^*) - \{F_k\}$.

Of course, if we start with an agent who does not satisfy strict Bayesian requirements and does recognize several preference rankings as permissible in his preference structure, there is nothing amiss. But in Jeffrey 1965 and in the second edition of that book, Jeffrey quite clearly restricts his attention to cases where only one preference ranking is permissible. Thus, the idea of using the set of probability functions obtained by his "quantization"—that is, the set *LA*—is inconsistent with the assumption that the agent recognizes only one ranking as permissible.

Only if Jeffrey weakens his requirements so as to entertain allowing rational agents to regard several rankings as permissible could his theory be made to generate credal states in my sense; but then the sets of probability distributions he would derive would not look like those obtained by means of quantization in his 1965. And in any case, there is no way that Jeffrey can add a thesis of the primacy of practical reason (whatever that means) to his theory to derive characterizations of credal states as sets of probability measures under the permissibility construal unless the thesis he adds is inconsistent with his theory.

So much for thesis 1 and the permissibility construal of Jeffrey's quantizations.

7. Thesis 2: The Black Box Construal. According to the black box construal of *LA*, we might regard the information concerning the preference ranking over $W(U^*) - \{F_k\}$ as data to be used to make conjectures about the agent's credal state and value structure. Each pair (Q,v) in *A* could be construed as a conjunction of two claims: one to the effect that the agent's credal state countenances *Q* as uniquely permissible and one to the effect that all and only those value functions that are positive affine transformations of *v* are permissible. When *v* and *v'* are positive affine transformations of one another, the pairs (Q,v) and (Q,v') represent equivalent hypotheses. Otherwise pairs are rival, mutually incompatible conjectures about the agent's state of belief and desire.

On this black box construal, the set *LA* is a set of hypotheses consistent with the data concerning the agent's strictly Bayesian credal state.

This reading of the set *LA* is compatible with Jeffrey's theory of 1965 where agents are endowed with uniquely permissible preference rankings. But it should be apparent that the credal states and value structures are assumed to be strictly Bayesian. Indeed, there is no way compatible with the data about the ranking to attribute a credal state to the agent even conjecturally that is not strictly Bayesian.

But if this black box construal of the quantization *LA* represents Jeffrey's intent, it is, at best, misleading to say that *LA* is "naturally identified" with or represents the agent's belief or credal state. Each member of the set is entertained as a possibly correct representation of the credal state; but the set does not represent that state.

8. Thesis 3: Conventionalism. Suppose it is thought (I am not clear what Jeffrey thinks about this) that the preference ranking over the domain $W(U^*) - \{F_k\}$ exhausts all the relevant information that one could in principle obtain concerning the agent's state of belief and desire.

Given this assumption (whether Jeffrey makes it, I do not know), one might begin to think of the black box construal just discussed as endorsing a form of realism that ought, perhaps, to be abandoned. Rather than regarding (Q,v) and (Q',v') as rival conjectures (subject to the qualifications mentioned previously), suppose we were to adopt a "conventionalist" approach and regard all members of A as equivalent.

This conventionalism resembles the sort of conventionalism in geometry that Putnam (1975) introduces as a reconstruction of Reichenbach's view. According to Reichenbachian conventionalism, two pairs (M,P) and (M',P') are taken to be equivalent geometries-cum-physics when they make the same predictions about "possible trajectories." In a similar spirit, someoone might claim that the pairs (Q,v) and (Q',v') are equivalent characterizations of belief-desire states. In the case of general relativity, it would be nonsense to say that the geometry of space-time is "naturally identified" with or represented by the set of metrics that are left elements of pairs of metrics and laws. And it would be absurd to say that each metric is equivalent to every other metric. Only pairs consisting of metrics and laws are equivalent. If one fixes the right-hand component of such a pair, then it becomes possible to identify a metric that represents the geometry of space-time—relative to the right-hand component stipulated. But that metric is a member of the set of left-hand components and should not be confused with the set itself.

On the conventionalist reading of Jeffrey's theory we are now exploring, similar remarks ought to be made. It no longer makes sense to speak of the agent's credal state in absolute terms. Strictly speaking, what is being characterized is the belief-desire state. It is nonsense to say that the set of probability functions in the quantization represents or is identified with the agent's belief state. It is equally nonsensical to say that one such probability function in LA is equivalent to any other.

Of course, if a specific function v from RA is selected, the preference ranking, according to Jeffrey's theory, licenses a unique probability function that, on the conventionalist interpretation, represents the credal state in a relative way. But that function is not the set of all functions in the quantization LA and should not be confused with it.

As already pointed out, it is possible—even for a conventionalist of the sort being considered—to use something that looks like a representation of a credal state. Given the preference ranking (which constitutes the datum) and given a stipulated value function from RA, one may speak of the probability measure thus determined as representing the agent's credal state. It might be the case (though I doubt it) that one could identify a "natural" procedure in some interesting sense for bringing this off, just as is allegedly done in physics.

But even if this is granted for the sake of the argument, the representation obtained is a representation of the credal state by a single probability measure—not by the of measures in *LA* that constitute Jeffrey's quantization. Whether one wants to call this form of conventionalism "strict Bayesianism" or not is of no concern to me. The respect in which it differs from the black box construal discussed under (2) concerns metaphysical and epistemological issues pertaining to the theory of measurement, which have small bearing on the critique of strict Bayesianism I am advocating.

9. Fragmentary Preference. Is there any way in which the ideas of Jeffrey 1965 can be made to square with the anti-Bayesian or, perhaps, anti-strict Bayesian ideas advanced in Levi 1974 and 1980?

As long as the Bolker-Jeffrey theory is made to require that an ideally rational agent allow exactly one ranking to be permissible in the suitably rich domain being investigated, the answer must be no. On the other hand, if Jeffrey were to concede that preference structures over *K*-propositions in the domain under scrutiny may allow several distinct preference rankings to be permissible in the sense that leads to the criterion of *E*-admissibility and, hence, that grounds the analysis of the Ellsberg problem given previously, he can allow consistently for indeterminate probability judgment.

I have not found the slightest indication in Jeffrey 1965 or elsewhere to indicate that Jeffrey has deviated from the strict Bayesian fold.

There are, to be sure, references to "fragmentary" preference rankings in Jeffrey 1984, pp. 139–40; adumbrated in Jeffrey 1973, p. 156. Perhaps these remarks are a sign that Jeffrey has converted from the doctrine put forth in Jeffrey 1965. I do not think so. I conjecture that Jeffrey intends such fragmentary preference rankings to be used at least as if they represent partial information about the strictly Bayesian complete preference ranking that the agent, if rational, is committed to having in his black box.[6]

[6]Jeffrey has denied being a "black boxer" in personal correspondence; but I still think he displays all the symptoms of that position. For the purposes of this discussion, it makes little difference whether he is an *als ob* black boxer or an *eigentlich* black boxer. Three considerations serve to support my interpretation of Jeffrey's view as expressed in the written record. Only he can say, of course, whether these symptoms of strict Bayesian black boxism represent his current view:

(1) Jeffrey does not appear to require that the set of probability distributions in his "probasition" be convex. There is no reason to require convexity under the black box

10. Concluding Remarks. The idea that probability judgments may be imprecise is scarcely novel. Both strict Bayesians and their critics can agree about this. Even if magnitudes have precise values, measurement aimed at determining these values is always liable to imprecisions of various sorts. This point lies at the heart of black box construals of sets of credal probability functions that, as I have been belaboring, are not the sorts of construals I have been deploying in my critique of strict Bayesian doctrine.

The issue concerns indeterminate probability—not imprecise probability. Unfortunately many strict Bayesians have found it conceptually difficult to recognize the distinction and have followed in the footsteps of F. P. Ramsey who focused attention on the problem of measuring credal probability. Jeffrey's 1965 book is best understood, in my judgment, as an important contribution to this measurement theoretic tradition. My complaint regarding Jeffrey's attitude in his 1984 is that he does not appear to think there can be any issues to discuss if they are not related to the measurement theoretic approach. His strategy seems to be to reinterpret the complaints of critics who maintain that rational agents may refuse to make determinate probability judgments. By conflating imprecision with indeterminacy, the thrust of the criticism is allegedly disarmed.

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interpretation. However, under the permissibility interpretation, convexity should be imposed (Levi 1980, sec. 9.5.)

(2) In Jeffrey 1984, p. 142, he admits that probasitions may have only roughly defined boundaries. The vagueness of the thresholds of upper and lower probability is plausible when we are thinking of measurement theoretic contexts where the black box construal is appropriate. However, when speaking of the credal state to which the agent is committed, problems of imprecision of measurement are being ignored. Just as credal states which are strict Bayesian recognize exactly one permissible distribution, so too credal states allowing many permissible distributions are representable by well-defined sets.

(3) Jeffrey suggests that one might introduce higher-order distributions over the space of probability distributions of which a probasition is a subset in order to characterize more adequately the vagueness of thresholds. This suggests that he is treating the probability distributions in the probasition as if they were hypotheses concerning the unknown distribution in the black box appropriate to use in computing expectations.

Jeffrey's comments on Good's views (Good 1952, p. 114) on higher-order probabilities are somewhat puzzling. He attributes to Good an objection to assigning a single higher-order distribution over the space of lower-order distributions which I cannot find expressed anywhere in Good's essay. To the contrary, in Good 1962, as in his 1952, not only is Good prepared to use higher-order distributions but he is perfectly prepared to use sharp higher-order distributions to integrate the lower-order distributions and come up with a new lower distribution. Good sees nothing objectionable in this. Nonetheless, on independent grounds, he thinks that it is implausible that higher-order probabilities should be precise. The objection Jeffrey attributes to Good is found in Savage 1954, p. 58, with a footnote in the second edition, which appeared in 1972, referring sympathetically to convex set representations of belief states. Jeffrey seems prepared to retain the second-order distribution, just as Good is, but is anxious to avoid the integration or averaging that Savage found objectionable. The possibility of doing this is mentioned in Levi 1980, p. 186n. See also my exchange with Gärdenfors and Sahlin, referenced in footnote 2, above.

When strict Bayesians are not begging questions against their critics in this way, they often respond to challenges by citing the considerable systematic virtues and comprehensive character of Bayesian ideas in accounting for rational choice and the relation between such choice, rational belief, and rational valuation. They challenge skeptics to furnish alternatives as viable as strict Bayesian doctrine in these respects.

Levi 1974 and 1980 seek to meet this challenge. They furnish an alternative far more comprehensive and systematic than strict Bayesianism. From the perspective of this approach, strict Bayesian decision theory, principles of probability judgment and the like are interesting special cases that apply under narrowly circumscribed circumstances. The same is true of decision principles like maximin.

The pretensions of these proposals to greater generality than strict Bayesian doctrine are, of course, themselves open to challenge and criticism; and their adequacy is clearly a debatable question.

Important issues are obscured, however, by suggesting that those who favor such proposals are Bayesian pussycats after all—especially when the basis of the claim is the conflation of indeterminate and imprecise probability.

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