Uniqueness of Simultaneity

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Abstract

We consider the problem of uniqueness of certain simultaneity structures in flat spacetime. *Absolute simultaneity* is specified to be a non-trivial equivalence relation which is invariant under the automorphism group `Aut` of spacetime. `Aut` is taken to be the identity-component of either the inhomogeneous Galilei group or the inhomogeneous Lorentz group. Uniqueness of standard simultaneity in the first, and absence of any absolute simultaneity in the second case are demonstrated and related to certain group theoretic properties. *Relative simultaneity* with respect to an additional structure X on spacetime is specified to be a non-trivial equivalence relation which is invariant under the subgroup in `Aut` that stabilises X. Uniqueness of standard Einstein simultaneity is proven in the Lorentzian case when X is an inertial frame. We end by discussing the relation to previous work of others.

Introduction

Simultaneity is a relational structure on or of spacetime which helps to globally organise events. It may or may not be thought of as intrinsic property *of* spacetime, depending on the limitations on the amount of structure put on the set of events. If it is definable solely by means of the structural elements assigned to spacetime one usually speak of *absolute* simultaneity. In this case a natural question is whether it is unique. If it is not definable in such a way, one has to add some further structure with the help of which we may then define some *relative* simultaneity, 'relative' to the added structure.

Adding sufficiently much structure will always allow to define *some* notion of relative simultaneity, even though such a definition will generally be far from unique. Uniqueness is no issue for practical applications, as many of the modern global navigational systems – like GPS or LORAN-C – demonstrate. For them to work it is sufficient that suitable and well defined methods for clock synchronisation exist. Typically, such methods will heavily depend on the contingent properties of the physical situation at hand, that is, abstractly speaking, on the properties of specific solutions to the dynamical equations of many body systems (earth, satellites etc.). Such solutions will clearly not respect any of the fundamental spacetime symmetries of the underlying theory.

There is hence no question that abundant simultaneity structures can be defined *on* spacetime. But this fact does not teach anything to the spacetime theorist. He or she is interested in the structure *of* spacetime and whether simultaneity may be regarded as (unique?) part of it. If spacetime does not provide a sufficiently rich structure by itself, the natural question would be 'how much' structure is missing. That is, whether we can add some *minimal* amount of structure with respect to which a (unique?) characterisation of simultaneity can be given. Here, 'minimal' could, for example, mean that the added structure should be invariant under as many of the original symmetries as possible. This will be the intuitive idea behind the approach followed here. The added structure should of course be physically interpretable and give rise to a physically 'reasonable' definition of simultaneity. For example, one may regard any definition as inadmissible in which a physically realizable world-line is allowed to intersect a set of mutually simultaneous events in more than one point. This will also be a criterion which we adopt in this paper.

Our discussion is close in spirit to that of Malament (1977) and Sarkar & Stachel (1999), and was in fact motivated by them. These authors are also concerned with certain uniqueness properties of Einstein's definition of synchronisation. Their background is the debate about the 'conventionality-of-simultaneity-thesis', an updated summary of which was given by Janis (1999). Here the issue of uniqueness comes in because one adopted strategy to refute this thesis is to first identify non-conventionality with uniqueness and then to prove the latter. Clearly, this identification can be challenged upon the basis that every proof of uniqueness rests upon some hypotheses which the simultaneity relation is supposed to satisfy and which may themselves be regarded as conventional. Arguments of this kind were brought forward in particular by Janis (1983) and Anderson et al. (1998, 123-126) and merely point towards a certain ambiguity in the possible meaning and range of the word 'convention'.[1] For this reason we concentrate on the question of uniqueness, which seems to be a much better behaved notion about which definite statements can be made once conditions for simultaneity relation are specified.

There are various difficulties with the uniqueness arguments by Mala-

[1] Janis (1983, 101) characterises a simultaneity structure as free of conventions, if it is 'singled out by facts about the physical universe'. But this is potentially also full of ambiguities. For example, are solutions to equations of motion which are actually realised in nature considered as 'facts about the physical universe'? If yes, and this is hard to deny, we can use them to define arbitrarily many (possibly practically very useful) simultaneities. To choose between those clearly requires a convention. On the other hand, on the largest observable scales our world is well described by a 'cosmological' solution in which the electromagnetic microwave background radiation singles out a preferred (irrotational) congruence of local observers whose orthogonal spatial hypersurfaces define a preferred simultaneity structure. Does the 'cosmological' character sufficiently distinguish this solution to let us conclude non-conventionality?

ment (1977) and Sarkar & Stachel (1999). Whereas the arguments given by Malament appear mathematically correct, we agree with Sarkar & Stachel that he puts *physically* unwarranted restrictions on the simultaneity relation.[2] On the other hand, the mathematical arguments given by Sarkar & Stachel are mathematically incomplete. Our last section is devoted to a more detailed discussion of these issues. In this paper we wish to present a fresh and systematic approach from first principles which, we believe, is free from the uncertainties just mentioned.

Let us point out right at the beginning that our requirements on simultaneity differ slightly from the ones used by Malament and Sarkar & Stachel: they require the simultaneity-defining equivalence relation to be invariant under all *causal* automorphisms (explained in section 5), whereas we only require invariance under spacetime automorphisms (to be defined below), which form a proper (i.e. strictly smaller) subgroup of the former. This implies that a priori our invariance-requirement is weaker and will therefore generally allow for *more* invariant equivalence relations. Uniqueness results in our setting should therefore be considered as stronger. But our actual motivation for sticking with spacetime automorphisms is simply that those causal automorphisms which are not spacetime automorphisms, like constant scale transformations, are no *physical* symmetries.[3] For this reason we will also not include space- and time-reflections in the group of spacetime automorphisms. The latter were also excluded by Sarkar & Stachel but not by Malament. Since our presentation aims to be self-contained, it will also contain a fair amount of background material.

1 Flat Spacetime and its Automorphisms

Throughout we deal with flat spacetime which we denote by $\mathcal{M}$. It consists of a manifold together with certain geometric structures. The manifold is assumed to be diffeomorphic to $\mathbb{R}^4$ with its natural differentiable structure. We think of $\mathbb{R}^4$ as being endowed with a basis which is fixed once and for all, unless explicitly stated otherwise. Linear transformations are then interpreted as diffeomorphisms (active point transformations), not as changes of bases.

An effective method to (implicitly) specify geometric structures is via the choice of an *automorphism group* `Aut`, which is a subgroup of the group of bijections of $\mathbb{R}^4$. In most cases, like in ours, it will turn out to be a finite dimensional Lie group which acts by diffeomorphisms, but a priori this need not necessarily be so. Once the choice of `Aut` is made, a geometric structure is

[2]For fairness one should say that Malament had a different motivation, namely to prove that the standard simultaneity relation of special relativity is uniquely definable in terms of the relation of causal connectibility. Since the latter is invariant under a strictly larger group than the group of physical symmetries of spacetime, he had indeed good reasons to put the stronger invariance requirement. Note that this makes existence less and uniqueness – provided existence holds – more likely.

[3]We argue on the level of modern classical and quantum field theories in flat space, thereby *ignoring* General Relativity.

said to exist on, or be a property of, spacetime $\mathcal{M}$ iff[4] this structure is invariant under `Aut`.[5] Invariant structures are sometimes called 'absolute' (like absolute simultaneity), but one should keep in mind that this notion of *absolute* depends on, and is hence relative to, the choice of `Aut`. That choice should really be considered as a *physical* input.[6]

In this paper `Aut` is either the inhomogeneous Galilei or the inhomogeneous Lorentz group. Let us briefly recall the essential requirements which lead to these groups.

- Elements of `Aut` should be *bijections* of $\mathbb{R}^4$; that is, they should be maps which are injective (same as 'into') and surjective (same as 'onto'). Hence each transformation has an inverse and no point (event) 'gets lost' in a transformation. Note that a priori we do not require transformations to be continuous or even smooth, which – if you think about it – would be hard to justify physically. In fact, smoothness will be implied by this and the next condition.

- We assume we are given 'forceless point-particles', that is, elementary point objects whose inertial trajectories define an affine structure on $\mathcal{M}$ with respect to which the trajectories become 'straight lines'. `Aut` is now required to preserve this affine structure, i.e., transformations in `Aut` must map straight lines to straight lines.

It is the main result of real affine geometry that bijections of $\mathbb{R}^n$ $(n \geq 2)$ which map straight lines to straight lines must be affine maps: $x \mapsto Ax + a$, where A is an invertible $n \times n$-matrix and $a \in \mathbb{R}^n$. A proof of this fact is given in sections 2.6.3-4 of (Berger 1987).[7] Hence `Aut` must be a subgroup of the real affine group in 4-dimensions, called $\texttt{Aff}(4, \mathbb{R})$, which is given by the semi-direct product $\mathbb{R}^4 \rtimes \texttt{Gl}(4, \mathbb{R})$ of the group of translations ($\mathbb{R}^4$) with the group of general linear transformations ($\texttt{Gl}(4, \mathbb{R})$). For (a', L') and (a, L) in $\texttt{Aff}(4, \mathbb{R})$ their multiplication law reads:

$$(a', L')(a, L) = (a' + L'a, L'L). \tag{1}$$

[4] Throughout we use 'iff' as abbreviation for 'if and only if'.

[5] This is the central idea of the 'Erlanger Programm' of Felix Klein (1893): to characterise a geometry (in a generalised sense) by its automorphism group (Klein calls it 'Hauptgruppe'). Relations belong to that geometry (are 'objective' according to Weyl (1949, Chap. III.13)) iff they are invariant under `Aut`. This statement stands independent of the logical question of whether any such 'objective' relation is actually *derivable* or *definable* within a given axiomatic setting (Weyl 1949, 73). This issue has recently been raised again by Rynasiewicz (2000) in the context of the 'conventionality of simultaneity' debate.

[6] Eventually it depends on the fundamental dynamical laws of quantum field theory (without gravitation), which denies the notion of empty space even locally. What we call the automorphisms of spacetime is the stabiliser of the ground state ('vacuum') within the group of dynamical symmetries of the theory. Note that this point of view eventually also denies that there exists a fundamental distinction between kinematical and dynamical symmetries.

[7] To complete Berger's (1987) argument one needs to supply a proof of his proposition 2.6.4, which states that there are no non-trivial automorphisms of the real numbers. This well known fact can be shown in an elementary fashion.

- We assume that all spacetime-translations are part of $\mathtt{Aut}$ ('homogeneity of spacetime'). Hence $\mathtt{Aut}$ is of the form $\mathbb{R}^4 \rtimes \mathtt{Aut}^*$, where $\mathtt{Aut}^* \subseteq \mathtt{Gl}(4,\mathbb{R})$. Of $\mathtt{Aut}^*$ it is further assumed that it contains the spatial rotations ('isotropy of space') as matrices of the form:[8]

$$R(D) = \begin{pmatrix} 1 & \vec{0}^\top \\ \vec{0} & D \end{pmatrix}, \tag{2}$$

  where $D \in \mathtt{SO}(3)$. Note that we did not include space- and time-reflections.

- We assume the relativity principle to hold, which says that velocity transformations $B(\vec{v})$ (called boosts) are part of $\mathtt{Aut}^*$. The boosts are assumed to be continuously and faithfully labelled by $\vec{v} \in V \subseteq \mathbb{R}^3$, where V is connected. Finally, let $R(D)$ be as in (2), then we assume the following equivariance condition[9], which should be regarded as part of the requirement of 'isotropy of space':

$$R(D)B(\vec{v})[R(D)]^{-1} = B(D\vec{v}) \,. \tag{3}$$

Given these conditions marked with •, one can rigorously show that $\mathtt{Aut}$ is either the inhomogeneous Galilei or the inhomogeneous Lorentz group for some yet undetermined velocity parameter c. The identification of c with the velocity of light is a logically independent step which need not concern us here. The idea to just use the relativity principle and not the invariance of the velocity of light in order to arrive at (something close to) the Lorentz group was first consistently spelled out by Frank & Rothe (1911). The way sketched here is mathematically more complete and partly based on the work of Berzi & Gorini (1969).

2 Simultaneity

Simultaneity, S, is a *relation* on $\mathcal{M}$, that is, a subset of $\mathcal{M} \times \mathcal{M}$. If (p,q) belongs to this subset we write $S(p,q)$, which stands for the statement: 'the point (event) p on $\mathcal{M}$ is in relation (later called 'simultaneous') to the point q'. More precisely, we require S to be an *equivalence relation*, which means that it ought to satisfy the following three conditions:

$$S(p,p)\ \forall p \in \mathcal{M} \qquad \text{(reflexivity)}\,, \tag{4}$$

$$S(p,q) \Rightarrow S(q,p)\ \forall p,q \in \mathcal{M} \qquad \text{(symmetry)}\,, \tag{5}$$

$$S(p,q) \text{ and } S(q,r) \Rightarrow S(p,r)\ \forall p,q,r \in \mathcal{M} \qquad \text{(transitivity)}\,. \tag{6}$$

[8] Vectors in $\mathbb{R}^3$ carry an arrow overhead and are considered as 1×3-matrices. The superscript $\top$ denotes matrix-conjugation. 4×4-matrices are written in time $\oplus$ space - form.

[9] Condition (3) is usually not stated explicitly, but tacitly assumed in statements to the effect that one may w.l.o.g. (sic) restrict attention to boosts in a preferred direction (Berzi & Gorini 1969, 1519), and that rotations about the $\vec{v}$-axis necessarily (sic) commute with $B(\vec{v})$ (Torretti 1996, 80). On the other hand, Berzi & Gorini and Torretti explicitly make use of (3) but with R being a spatial reflection which reverses the boost direction (Torretti 1996, 79), which unnecessarily involves reflection transformations (which we exclude), whereas the same can be achieved by choosing for R a π-rotation about an axis $\perp$ to the boost direction.

It is hard to see how one could do without the first two conditions, but transitivity is certainly not needed in order to talk about the simultaneity of *pairs* of events. For example, it already allows to synchronise each member of a set of clocks with a preferred 'master-clock', which is indeed sufficient for certain practical purposes. However, transitivity *is* needed in order to consistently talk about *mutually* simultaneous events in sets of more than two.

2.1 Equivalence Relations

Let us recall a few general properties of equivalence relations which we will frequently use. An equivalence relation S on a set $\mathcal{M}$ is the same thing as a partition of $\mathcal{M}$. Recall that a 'partition' is defined to be a covering by non-empty, mutually disjoint sets. In the present context such sets are called equivalence classes. The equivalence class in which p lies is called p's equivalence class or $[p]$, and given by

$$[p] := \{q \mid S(p,q)\}\,. \tag{7}$$

This definition makes sense since $[p]$ and $[q]$ are either disjoint or identical. Before showing this, we first note that reflexivity implies $p \in [p]$. Hence no $[p]$ is empty and each p lies in some equivalence class. Now, if $S(p,q)$ then $[p] = [q]$ since symmetry and transitivity immediately imply that $S(p,r)$ iff $S(q,r)$. Moreover, in the same way we see that $r \in [p] \cap [q]$ implies $S(p,q)$ and consequently $[p] = [q]$, which proves the claim. Conversely, a partition $\mathcal{M} = \bigcup_i U_i$ defines an equivalence relation through $S(p,q) \Leftrightarrow p$ and q lie in the same U_i. The conditions of reflexivity, symmetry, and transitivity are easily checked. Hence we have shown that an equivalence relation on $\mathcal{M}$ is the same as a partition of $\mathcal{M}$.

Two particularly boring equivalence relations are: 1) $[p] = [q]\ \forall p,q \in \mathcal{M}$ (just one equivalence class), and 2) $[p] \neq [q]\ \forall p,q \in \mathcal{M}$ where $p \neq q$ (each point is a different class). We call an equivalence relation non-trivial if it is different from these two.

2.2 Invariant Equivalence Relations

Suppose $\mathcal{M}$ carries an action of a Group G: $(g,p) \mapsto g \cdot p$. We say that the equivalence relation S is invariant under this action iff

$$S(p,q) \Leftrightarrow S(g\cdot p, g\cdot q)\,, \quad \forall g \in G\,, \forall p,q \in \mathcal{M}\,. \tag{8}$$

Expressed in terms of the equivalence classes (7) this is the same as[10]

$$[g\cdot p] = g\cdot [p]\,, \quad \forall g \in G\,, \forall p \in \mathcal{M}\,. \tag{9}$$

Proof. That (8) implies (9) is seen as follows:

$$[g\cdot p] \quad = \quad \{q \mid S(g\cdot p, q)\}$$

[10] For $U \subseteq \mathcal{M}$ or $H \subseteq G$ we write: $g\cdot U := \{g\cdot p \mid p \in U\}$ and $H\cdot p := \{g\cdot p \mid g \in H\}$.

$$
\begin{aligned}
&= \{q \mid S(p, g^{-1} \cdot q)\} \\
&= \{g \cdot r \mid S(p, r)\} \\
&= g \cdot \{r \mid S(p, r)\} = g \cdot [p]\,.
\end{aligned}
$$

Conversely, if $S(p,q)$ then $q \in [p]$ and (9) implies $g\cdot q \in [g\cdot p]$ so that $S(g\cdot p, g\cdot q)$. This proves the equivalence of (8) and (9).

As already mentioned in the introduction, we regard a $G = \mathtt{Aut}$–invariant equivalence relation as a physical property of spacetime. Our central requirement on 'absolute simultaneity' then reads as follows:

Requirement 1 Absolute Simultaneity *is a non-trivial* $\mathtt{Aut}$*-invariant equivalence relation on* $\mathcal{M}$ *each equivalence class of which intersects any physically realizable trajectory in at most one point.*

If no such absolute structure exists, we need to add some further structural elements X to $\mathcal{M}$. X could be a subset of $\mathcal{M}$, like a single wordline which models an individual *observer*, as in Malament (1977), or a whole 3-dimensional family of such observers which define a *reference frame*, as in Sarkar & Stachel (1999). Straight lines or families of straight lines correspond to inertial observers and inertial reference frames respectively. Now, let $\mathtt{Aut}_X$ be the subgroup of $\mathtt{Aut}$ that preserves (stabilises) X. For example, if X is a subset of $\mathcal{M}$, this means that $\mathtt{Aut}_X$ should map points of X to points of X (pointwise X need not be fixed), and if X is a partition of $\mathcal{M}$ by subsets, like a foliation by straight lines, it means that $\mathtt{Aut}_X$ should preserve this partition, i.e., map lines to lines. A relation is then said to exist on $\mathcal{M}$ relative to X, or be a property of (M, X), iff it is invariant under $\mathtt{Aut}_X$.[11] In other words, the relation is required to break none of the residual spacetime symmetries which still exist relative to the structure X. Our central requirement on 'relative simultaneity' then reads as follows:

Requirement 2 Simultaneity relative to X *is a non-trivial* $\mathtt{Aut}_X$*-invariant equivalence relation on* $\mathcal{M}$ *each equivalence class of which intersects any physically realizable trajectory in at most one point.*

We see that in order to classify simultaneity-structures we essentially need to classify G-invariant equivalence relations, where G is $\mathtt{Aut}$ or some subgroup thereof. This will be done to some extent in the following sections. There we will make extensive use of the following simple observations: Let G_p denote the stabiliser subgroup of $p \in \mathcal{M}$ in G, that is, $G_p := \{g \in G \mid g \cdot p = p\}$. If $S(p,q)$ then (8) immediately implies $S(p, g\cdot q)$ for all $g \in G_p$. Hence the whole G_p-orbit of q, denoted by $G_p \cdot q$, lies in $[p]$. Moreover, suppose $S(p,q)$ and that for some g, with $p' = g \cdot p$ and $q' = g \cdot q$, we have

$$
G_p \cdot q \cap G_{p'} \cdot q' \neq \emptyset\,, \tag{10}
$$

then $[p] = [p']$. The proof is simple: since the orbits $G_p \cdot q$ and $G_{p'} \cdot q'$ lie in $[p]$ and $[p']$ respectively, $[p]$ and $[p']$ intersect and must hence be equal.

[11] Again this notion of relative existence of geometric relations may be found in Klein's 'Erlanger Programm' (Klein 1893, §2).

2.2.1 Existence

One may ask for general criteria for when G-invariant equivalence relations exist. For example, assume G's action on $\mathcal{M}$ to be 2-point-transitive, which means that for any set of four mutually distinct points p_1, p_2, q_1, q_2 there exists a $g \in G$ such that $g \cdot p_1 = q_1$ and $g \cdot p_2 = q_2$. This is equivalent to saying that the stabiliser subgroups G_p act transitively. Then, obviously, the only invariant equivalence relation is the trivial one where $[p] = \mathcal{M}$ for all p. On the other hand, if G's action is not transitive but still non-trivial, we can, for example, just set $[p] := G \cdot p$ to define a non-trivial G-invariant equivalence relation. Hence the mathematically most interesting situations arise when G acts transitively but not 2-point-transitively. This is precisely the situation we are dealing with. Due to the spacetime translations, $\mathtt{Aut}$ clearly acts transitively on $\mathcal{M}$, but the stabiliser subgroup of, say, the origin, $\mathtt{Aut}^*$, does not. Its orbits consist of 3-dimensional submanifolds which are planes in the Galilean and hyperbola or light-cones in the Lorentzian case.

A general criterion for the existence of G-invariant equivalence relations, or equivalently, G-invariant partitions, does indeed exist. Before we state it, recall that a subgroup K of G is called 'maximal' iff there is no proper subgroup H of G which properly contains K. We have

Theorem 1 *Let G act transitively on $\mathcal{M}$. There exists a non-trivial G-invariant equivalence relation on (equivalently: partition of) $\mathcal{M}$ iff the stabiliser subgroups G_p are not maximal.*

Note that maximality either applies to all or none of the stabiliser subgroups, since for a transitively acting G they are all conjugate: $G_p = g \cdot G_q \cdot g^{-1}$ if $p = g \cdot q$. A proof of Theorem 1 may be found as proof of Theorem 1.12 in Jacobson (1974). We will not make essential use of this theorem because we prefer to give direct arguments. But it is still useful to know since it highlights a group theoretic property (maximality of stabiliser subgroups) that distinguishes the inhomogeneous Galilei from the inhomogeneous Lorentz group and which pinpoints the mathematical origin of their different behaviour regarding the existence of absolute simultaneity-structures.

3 Galilean Relativity

We speak of *Galilean relativity* if $\mathtt{Aut}$ is the inhomogeneous (including translations), proper (no space reflections), orthochronous (no time reflection) Galilei group, which we denote by $\mathtt{IGal}$. According to the general results given above we only need to specify its homogeneous part $\mathtt{Aut}^*$. It is given by the homogeneous Galilei group $\mathtt{Gal}$, which is the semi-direct product of spatial rotations ($R \in \mathtt{SO}(3)$) and boosts ($\vec{v} \in \mathbb{R}^3$). Hence we have

$$\mathtt{IGal} = \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \mathtt{SO}(3)) \,, \tag{11}$$

where the first $\rtimes$ on the right side comes from (1) and the second corresponds similarly to the law $(\vec{v}', R')(\vec{v}, R) = (\vec{v}' + R'\vec{v}, R'R)$. It is implemented by letting

$Aut^* \subset \mathtt{Gl}(4,\mathbb{R})$ be the subgroup of 4×4 – matrices of the form:

$$\begin{pmatrix} 1 & \vec{0}^\top \\ \vec{v} & R \end{pmatrix} . \tag{12}$$

Note that the first semi-direct product in (11) is such that the action of $\mathtt{Aut}^*$ on $\mathbb{R}^4$ is not irreducible: it leaves invariant the subgroup $\mathbb{R}^3$ of spatial translations (due to the zero-vector in the upper right corner of (12)). Consequently, $\mathbb{R}^3 \rtimes \mathtt{Aut}^*$ is a subgroup of $\mathtt{Aut}$. Note also that the same is not true for time translations. With respect to Theorem 1 this implies that $\mathtt{Aut}^*$ – the stabiliser subgroup of the origin in $\mathbb{R}^4 \cong \mathcal{M}$ – is not a maximal subgroup of $\mathtt{Aut}$, since we can still adjoin the group $\mathbb{R}^3$ of spatial translations. Hence Theorem 1 guarantees the existence of a non-trivial $\mathtt{Aut}$-invariant equivalence relation. But this will be proven directly below.

$\mathtt{IGal}$ is parameterised by ten real numbers: three for R, three for a boost-vector $\vec{v}$, three for a spatial translation vector $\vec{a}$ and one for a time-translation b. A general element $g \in \mathtt{IGal}$ can then be uniquely labelled by $(R,\vec{v},\vec{a},b)$. The law for multiplication and inversion then simply read[12]

$$g'' = (R'R\,,\,\vec{v}'+R'\vec{v}\,,\,\vec{a}'+R'\vec{a}+b'\vec{v}\,,\,b'+b)\,, \tag{14}$$
$$g^{-1} = (R^{-1}\,,\,-R^{-1}\vec{v}\,,\,-R^{-1}(\vec{a}-b\vec{v})\,,\,-b)\,. \tag{15}$$

Writing p in $\mathcal{M}$ as $(t,\vec{x})$, the action of g on p reads:

$$\vec{x}\mapsto\vec{x}' = R\vec{x}+\vec{v}t+\vec{a}\,, \tag{16}$$
$$t\mapsto t' = t+b\,. \tag{17}$$

The subgroup $\mathtt{Euc}\subset\mathtt{IGal}$ of Euclidean motions consists of spatial rotations and translations. It is given by all elements where $\vec{v}=\vec{0}$ and $b=0$. Its orbits are the planes of constant t which we denote by Σ_t. Note that boosts act like translations in each Σ_t separately, but scaled with a factor t. Only time-translations permute the planes Σ_t. The general law is $g\cdot\Sigma_t=\Sigma_{t+b}$.

3.1 Galilean Simultaneity

Let S be a $G=\mathtt{IGal}$–invariant, non-trivial equivalence relation. We choose a hyperplane Σ_t and a point $p\in\Sigma_t$.

First we assume that a point $q\neq p$ exists on Σ_t such that $S(p,q)$. For the moment we forget about $\mathcal{M}$ and restrict attention to Σ_t which we regard as $\mathbb{R}^3$ with standard inner product and norm $\|\cdot\|$. $\mathtt{Euc}\subset\mathtt{IGal}$ acts transitively on Σ_t

[12] $\mathtt{IGal}$ can be embedded in $\mathtt{Gl}(5,\mathbb{R})$ as follows:

$$g(R,\vec{v},\vec{a},b)\longrightarrow\begin{pmatrix} R & \vec{v} & \vec{a} \\ 0 & 1 & b \\ 0 & 0 & 1\end{pmatrix} \quad \in \mathtt{Gl}(5,\mathbb{R}) \tag{13}$$

which also gives (14,15). In this picture $\mathcal{M}$ is identified with the 4-dimensional hyperplane $x^5=1$, which then leads to (16,17).

by standard Euclidean motions. The stabiliser subgroup $\texttt{Euc}_p$ of p consists of all rotations centred at p. The orbit $\texttt{Euc}_p \cdot q$ is a 2-sphere around p of radius $\|p-q\|$. Now, let $g \in \texttt{Euc}$ be a translation by a vector of norm less then $2\|p-q\|$. Clearly $S(p', q')$ for $p' = g \cdot p$ and $q' = g \cdot q$. Moreover, since the distance between p and p' is less then $2\|p-q\|$, the $\texttt{Euc}_p$-orbit of q and the $\texttt{Euc}_{p'}$-orbit of q' intersect, which implies $[p] = [p']$; compare discussion surrounding (10). Since any point on Σ_t can be reached by a finite number of translations of norm less than $2\|p-q\|$, we only need to iterate this argument to show that all points of Σ_t lie in the same equivalence class.

Next we assume that a point $p' \notin \Sigma_t$ exists such that $S(p, p')$. Let p' be a member of, say, $\Sigma_{t'}$, where $t \neq t'$. Consider the stabiliser subgroup $\texttt{IGal}_{p'}$ of p'. Besides certain rotations (which need not concern us at the moment), it contains the 3-dimensional subgroup which is given by the combinations of translations and boosts for which $\vec{a} = -t'\vec{v}$. By construction this subgroup fixes $\Sigma_{t'}$ pointwise, but acts transitively on Σ_t via translations of the form $\vec{x} \mapsto \vec{x} + (t - t')\vec{v}$. This already proves that $[p](= [p'])$ contains both hyperplanes, Σ_t and $\Sigma_{t'}$. The latter is seen by just reversing the rôles of p and p' in the argument.

So far our arguments show that, for any $p \in \mathcal{M}$, $[p]$ is a union of planes Σ_t one of which contains p. It is consistent with $\texttt{IGal}$-invariance to choose for $[p]$ just the single plane containing p since $g \cdot \Sigma_t = \Sigma_{t+b}$ implies (9). We may thus call this the *minimal* or *finest* non-trivial $\texttt{IGal}$-invariant equivalence relation on $\mathcal{M}$. If $[p]$ contains more than one plane, say Σ_t and $\Sigma_{t'}$, then for $\texttt{IGal}$ invariance (here only time-translations matter) it is necessary that all planes $\Sigma_{t+n(t'-t)}$ for $n \in \mathbb{Z}$ are also contained in $[p]$. Therefore, if λ denotes the infimum of all time-differences of planes contained in $[p]$, then $[p]$ is the union of all planes $\Sigma_{t+n\lambda}$ for $n \in \mathbb{Z}$, where $p \in \Sigma_t$. If this infimum were zero we would obtain the trivial equivalence relation where all of $\mathcal{M}$ is a single class. Hence we have

Theorem 2 *Let S be a non-trivial $\texttt{Aut} = \texttt{IGal}$ – invariant equivalence relation on $\mathcal{M}$. Then the possible equivalence classes $[p]$ are given by:*
(i) the plane Σ_t containing p;
(ii) the union over $n \in \mathbb{Z}$ of planes $\Sigma_{t+n\lambda}$, where $0 < \lambda \in \mathbb{R}$ and $p \in \Sigma_t$.

Next to the mathematical space $\mathcal{M}$ that represents physical spacetime, one may also associate a mathematical space $\mathcal{T}$ that simply represents time, namely the set of equivalence classes given by the quotient

$$\mathcal{T} := \mathcal{M}/S\,. \tag{18}$$

In case the equivalence classes in $\mathcal{M}$ just consist of single hypersurfaces Σ_t, $\mathcal{T}$ is isomorphic to $\mathbb{R}$. The action of $\texttt{IGal}$ on $\mathcal{T}$ is just $(g, t) \mapsto t + b$. In case there are more than one Σ_t in each equivalence class, $\mathcal{T}$ is isomorphic to the circle $S^1 = \mathbb{R}/\{\text{identification mod } \lambda\}$. In this sense time is periodic with period $\lambda < \infty$. Note however that this does not mean that we may make periodic identifications in $\mathcal{M}$ and represent spacetime by $\mathcal{M}_\lambda := \mathcal{M}/\mathbb{Z}$, where $\mathbb{Z} \subset \texttt{IGal}$ is represented by the discrete time translations $t \mapsto t + n\lambda\,, n \in \mathbb{Z}$. The point being that this space (homeomorphic to $S^1 \times \mathbb{R}^3$) would not support an action of $\mathtt{IGal}$, since the boost transformations $(t, \vec{x}) \mapsto (t, \vec{x} + t\vec{v})$ are incompatible with such an identification. This is due to the group-theoretic fact that time-translations do not form a *normal* subgroup in $\mathtt{IGal}$, in contrast to spatial translations. For example, periodic *spatial* identifications by some integer lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$, which certainly does form a normal subgroup, results in a closed spatial space whose topology is that of the 3-torus, T^3, and a spacetime $\mathcal{M}' = \mathbb{R} \times T^3$ which still carries an action of $\mathtt{IGal}$, though not an effective one, since $\mathbb{Z}^3$-valued spatial translations now act trivially. We summarise the results of this section in

Theorem 3 *Standard Galilean simultaneity is the unique absolute simultaneity satisfying Requirement 1 for* $\mathtt{Aut} = \mathtt{IGal}$. *It is also the unique non-trivial* $\mathtt{IGal}$-*invariant equivalence relation on* $\mathcal{M}$ *for which time is non-cyclic, or for which the equivalence classes in* $\mathcal{M}$ *are connected.*

4 Lorentzian Relativity

We now wish to explore the consequences of replacing $\mathtt{IGal}$ by the inhomogeneous Lorentz group $\mathtt{ILor}$ (also known as Poincaré group). This group has not only quite a different group-structure than $\mathtt{IGal}$, but also very different orbits in $\mathcal{M}$. This results essentially from the way boost-transformations are implemented, which are now not allowed to boost beyond a finite limit-velocity c, usually taken to be the velocity of light, but the value of c is unimportant for us as long as $0 < c < \infty$. In the following we choose units such that $c = 1$.

$\mathtt{ILor}$ is obtained by choosing for $\mathtt{Aut}^* \subset \mathtt{Gl}(4, \mathbb{R})$ the homogeneous (proper, orthochronous) Lorentz group, $\mathtt{Lor}$. To define it, consider the real 4×4-matrices L which leave the diagonal-matrix – called the Minkowski metric – $\eta := \mathrm{diag}(1, -1, -1, -1)$ invariant in the following sense:

$$L \eta L^\top = \eta \,. \tag{19}$$

Those L with determinant $+1$ form the proper Lorentz group which is also denoted by $SO(1,3)$, a notation with an obvious meaning in view of (19). This group has two components: one where the time-time component $L^0_0 \geq 1$, the other where it is ≤ -1. The former is the identity component and leads to our group $\mathtt{Lor}$ of proper orthochronous Lorentz transformations. As for the Galilei group we excluded reflections of space or time.

To see the group-theoretic difference between $\mathtt{Gal}$ and $\mathtt{Lor}$, recall that $\mathtt{Gal}$ is a semi direct product $\{\text{boosts}\} \rtimes \{\text{rotations}\}$, which implies that boosts and rotations separately form subgroups, and that the boost-subgroup is normal (i.e., invariant under conjugations in $\mathtt{Gal}$; compare (3)). $\mathtt{Lor}$, on the other hand, is a *simple* group, that is, it does not contain any normal subgroups other than the identity and the whole group. Rotations still form a subgroup, but boosts do not. In general, a boost multiplied by a boost is a boost times a non-trivial rotation (the latter being the origin of 'Thomas-Precession'). Requirement (3) still holds, of course, and merely means that boosts form an invariant *set* under conjugation with rotations but not under all conjugations in `Lor`.

The translation subgroup in `ILor` acts on $\mathcal{M}$ just in the same way the translations in `IGal` do. This is also true for spatial rotations, which together with boosts make up `Lor`. The former correspond to matrices of the form

$$\begin{pmatrix} 1 & \vec{0}^\top \\ \vec{0} & R \end{pmatrix} , \tag{20}$$

where $R \in SO(3)$. On the other hand, boost with velocity $\vec{v} = v\vec{n}$, where $v := \|\vec{v}\| < 1$, correspond to matrices of the form ($\gamma := 1/\sqrt{1-v^2}$):

$$\begin{pmatrix} \gamma & \gamma\vec{v}^\top \\ \gamma\vec{v} & \mathbf{1} + (\gamma-1)\vec{n}\otimes\vec{n}^\top \end{pmatrix} , \tag{21}$$

which act like

$$\vec{x} \mapsto \vec{x}' \quad = \quad \vec{x} + \gamma\vec{v}t + (\gamma-1)(\vec{n}\cdot\vec{x})\vec{n}\,, \tag{22}$$

$$t \mapsto t' \quad = \quad \gamma(t + \vec{v}\cdot\vec{x})\,. \tag{23}$$

Whereas (22) is merely a deformation of the boost action in (16), (23) differs significantly from (17). Thinking of $\mathcal{M}$ as $\mathbb{R}^4$, it means that the family of parallel planes $t =$ const., which can be characterised by their normal-direction (here w.r.t. standard Euclidean metric of $\mathbb{R}^4$) given by $(1, \vec{0})$, will be transformed into the family of tilted planes with normal direction given by $(1, \vec{v})$, i.e. tilted by the angle $\tan\alpha = v$. Hence *any* two planes from the first and second family respectively intersect.

With respect to Theorem 1 we also remark that in $\mathbb{R}^4 \rtimes \texttt{Aut}^*$ the action of $\texttt{Aut}^* = \texttt{Lor}$ on $\mathbb{R}^4$ is now irreducible, hence no subspace of translations is left invariant, as it was the case for spatial translations when $\texttt{Aut}^* = \texttt{Gal}$. This implies that $\texttt{Aut}^* = \texttt{Lor}$ and all its conjugations by translations – which make up the stabiliser subgroups $\texttt{Aut}_p$ – are maximal subgroups of `Aut`. From Theorem 1 we can therefore anticipate that there will be no $\texttt{Aut} = \texttt{ILor}$-invariant non-trivial equivalence relation on $\mathcal{M}$. Below we prefer to give a simple direct proof of this fact.

4.1 Lorentzian Simultaneity

Since spacetime translations and spatial rotations in `IGal` and `ILor` act identically on $\mathcal{M}$, all the arguments which were given in the framework of Galilean relativity and which did not use boost do also apply in the present case. In particular, each equivalence classes of any non-trivial `ILor`-invariant equivalence relation S contains one or more of the planes Σ_t. But now comes the point: since boosts transform the family of planes Σ_t to a tilted family $\Sigma'_{t'}$, any member of which intersects any member of the former, all the planes must be in the same equivalence class. This implies

Theorem 4 *The only* $\mathtt{ILor}$*-invariant equivalence relation on* $\mathcal{M}$ *is the trivial one where* $\mathcal{M}$ *is the only equivalence class. Hence absolute simultaneity satisfying Requirement 1 does not exist in Lorentzian relativity.*

The proof is almost trivial: since Σ_t lies within a single class, and since boosts in $\mathtt{ILor}$ map classes to classes, the image Σ'_t of Σ_t under a boost also lies within one class. But it intersects *all* Σ_t and hence all other classes, which implies that there is only one class.

4.2 Lorentzian Simultaneity Relative to an Inertial Frame

What we have just learned is that $\mathtt{Aut}^* = \mathtt{SO}(1,3)$ does not leave spacetime with enough structure to be able to define absolute simultaneity. But what about relative simultaneity? For this we have to add some further structure X. Clearly, if we choose X so that $\mathtt{Aut}$ gets broken down completely, $\mathtt{Aut}_X$-invariance will be an empty requirement and *any* equivalence relation will do. As outlined in the introduction, the task is to choose X rich enough to ensure existence but otherwise as symmetry-preserving as possible. If this leads to a sufficiently big $\mathtt{Aut}_X$, the residual invariance requirement may ensure uniqueness.

The structure X we wish to consider here is an *inertial reference frame*, which in our case (flat geometry) corresponds to a foliation of $\mathcal{M}$ by (necessarily parallel) timelike straight lines. Let now X stand for such a foliation by lines which are all parallel to the four-vector $v = (1, \vec{v})$. The foliation X is obviously invariant (meaning lines are transformed to lines) under all spacetime translations, and obviously not invariant under any non-trivial boost. In fact, we can w.l.o.g. assume $\vec{v} = \vec{0}$, for otherwise let $B(\vec{v})$ denote the boost which maps the t-axis to a line in X, which we call the t'-axis, and refer the whole situation to t' and the planes $\Sigma'_{t'}$ perpendicular (w.r.t. Minkowski metric) to the t'-axis. The full stabiliser group $\mathtt{Aut}_X = \mathtt{ILor}_X$ is now seen to consist of time translations and the group $\mathtt{Euc}$ of spatial Euclidean motions: $\mathtt{Aut}_X = \mathbb{R} \times \mathtt{Euc}$.

First, we can now use the spatial rotations in $\mathtt{Euc}$ to argue exactly as in Section 3.1: if in some Σ_t there exist two different points for which $S(p,q)$, then the argument there shows that $[p]$ contains Σ_t. Since time translations can map Σ_t to any other hyperplane in this family, each hypersurface Σ_t is contained in some equivalence class.

Next suppose that for a $p \in \Sigma_t$ some $p' \in \Sigma_{t'}$ for $t' \neq t$ exists so that $S(p,p')$. Now we cannot proceed as in Section 3.1 since $\mathtt{Aut}_X$ contains no boosts. Instead we argue as follows: let $\ell_{p'}$ denote the straight line in $\mathcal{M}$ through p' which is parallel to the t-axis, and let r be its point of intersection with Σ_t. The stabiliser $\mathtt{Euc}_{p'}$ of p' in $\mathtt{Euc}$ consists of rotations which in $\Sigma_{t'}$ rotate about p' and in Σ_t rotate about r.

If $p \neq r$ we can move p by an element of $\mathtt{Euc}_{p'}$ to get another point q in Σ_t for which $S(p,q)$. Hence we are back to the case above which now shows that $[p] (= [p'])$ contains Σ_t and $\Sigma_{t'}$, the latter again by reversing the rôles of p and p' in the argument.

This conclusion is avoided iff $r = p$, i.e., iff p lies on $\ell_{p'}$. The conclusion that each equivalence class contains some hyperplanes is avoided iff *any* two different p, p' for which $S(p, p')$ lie on the same straight line parallel to the t-axis. There clearly are non-trivial $\mathtt{Aut}_X$-invariant equivalence relations whose classes are contained in the straight lines parallel to the t-axis. These are readily classified: $S(p, q)$ iff either p, q are on the same such line, or $\tau_\lambda^n \cdot p = q$ for some $n \in \mathbb{Z}$, where $\tau_\lambda \in \mathtt{Aut}_X$ is the time translation $\tau \mapsto t + \lambda$. This leads to the following classification of all possible equivalence relations:

Theorem 5 *Let X be a foliation of $\mathcal{M}$ by timelike straight lines and S a non-trivial $\mathtt{Aut}_X = \mathtt{ILor}_X$ – invariant equivalence relation on $\mathcal{M}$. Then the possible equivalence classes $[p]$ are given by:*
(i) the plane $\Sigma_t \ni p$ perpendicular (Minkowski metric) to the timelike lines X;
(ii) the union over $n \in \mathbb{Z}$ of planes $\Sigma_{t+n\lambda}$, where $0 < \lambda \in \mathbb{R}$ and $p \in \Sigma_t$;
(iii) the line in X through p;
(iv) the union over $n \in \mathbb{Z}$ of points $\tau_\lambda^n \cdot p$, where τ_λ is the translation by an amount $\lambda > 0$ along the line in X through p.

In case (ii) each straight timelike line intersects each equivalence class a countably infinite number of times. In case (iii) each timelike line in X is its own equivalence class and hence intersects an equivalence class in uncountably many points. In case (iv) the same is true for countably many points. Therefore, these cases do not define a notion of relative simultaneity satisfying Requirement 2, since 'physically realizable trajectories' will certainly include all timelike straight lines (inertial motion). On the other hand, case (i) does satisfy the condition that each equivalence class is cut at most (in fact: exactly) in one point by each physically realizable trajectory, which here, for definiteness, we may e.g. specify to be all timelike piecewise differentiable curves. Hence we have

Theorem 6 *Einstein simultaneity is the unique relative simultaneity satisfying Requirement 2 for $\mathtt{Aut}_X = \mathtt{ILor}_X$, where X denotes an inertial frame (=foliation of $\mathcal{M}$ by timelike straight lines).*

5 Relation to Work of Others

Here we shall basically focus on the work of Malament (1977) and Sarkar & Stachel (1999). Both are directly concerned with uniqueness issues, but neither gives a systematic classification of invariant equivalence relations. Malament proves uniqueness of Lorentzian simultaneity relative to a single observer. In this case X is a single timelike straight line. But instead of spacetime automorphisms $\mathtt{Aut}$ he takes all *causal* automorphisms, which we denote by $\mathtt{Aut}^{\mathrm{c}}$, by which he understands all bijections f of spacetime such that $p - q$ is non spacelike iff $f(p) - f(q)$ is non spacelike. It has been proven by Alexandrov (1975) that any such transformation is a combination of transformations in $\mathtt{ILor}$, time reflections $(t, \vec{x}) \mapsto (-t, \vec{x})$, space reflections $(t, \vec{x}) \mapsto (t, -\vec{c})$, and dilatations $p \mapsto \lambda p$

with $\lambda \in \mathbb{R}_+$ (positive real numbers).[13] More precisely, Malament proved the following

Theorem 7 (Malament 1977) *Let X be an initial observer, i.e., a timelike straight line. Let S be an $\mathtt{Aut}^c_X$-invariant non-trivial equivalence relation, which also satisfies the following condition: there exists a point $p \in X$ and a point $q \notin X$ such that $S(p,q)$. Then S is given by standard Einstein simultaneity.*

In a recent review, Anderson et al. (1998, 124-125) claim Malament's proof to be technically incorrect. We disagree, as do Sarkar & Stachel (1999) and apparently also Janis (1999). However, it is true that the proof presented by Malament (1977) leaves out some details. To settle this technical issue, we present an alternative and somewhat more detailed proof in the Appendix which uses the language developed in previous sections.

Let us look at the same situation from our point of view, where instead of $\mathtt{Aut}^c$ we take $\mathtt{Aut} = \mathtt{ILor}$. We may w.l.o.g. take X to be the time axis; otherwise we boost and translate the selected observer to rest at the origin and take the conjugate of all subgroups to be mentioned by that combination of a boost and a translation. Then $\mathtt{Aut}_X = \mathbb{R} \times \mathtt{SO}(3)$, where $\mathbb{R}$ consist of pure time translations and $\mathtt{SO}(3)$ are the rotations in $\mathtt{Euc}$ fixing the t-axis. Picking a single inertial observer out of an inertial reference frame eliminates the space translations in $\mathtt{Euc}$. This distinguishes the present case from that discussed above and makes a big difference concerning the question of uniqueness. Consider the two-parameter family of subsets of $\mathcal{M}$:

$$\sigma(\tau, r) := \{(t, \vec{x}) \in \mathcal{M} \mid t = \tau,\ \|\vec{x}\| = r\}, \tag{24}$$

given by the points (for $r = 0$) of the t-axis and all concentric 2-spheres about the spatial origin in each $t =$ const. plane. This is already an $\mathtt{Aut}_X$-invariant partition of $\mathcal{M}$ so that taking the $\sigma(\tau, r)$ as equivalence classes would define a notion (though not a very reasonable one) of relative simultaneity satisfying Requirement 2. However, it obviously violates the condition in Malament's theorem, since no point on X is related to a point off X. But this can be easily cured: just take $\mathtt{Aut}_X$-invariant unions of sets $\sigma(\tau, r)$ which connect points on X with points off X, and define these unions as new equivalence classes. For example, in each $t =$ const. hypersurface, we can take a central ball and spherical shells of radius 1:

$$\sigma'(\tau, n) := \bigcup_{r \in [n-1, n)} \sigma(\tau, r)\,, \tag{25}$$

where n is a positive integer. Another possibility would be to unite sets $\sigma(\tau, r)$ in different hyperplanes $t =$ const. We just have to take care that no two sets

[13] The same is true for any bijection which in both directions preserves just 'lightlike' or just 'timelike' separations. Moreover, in the time oriented case, the same statements hold if one restricts to just future (or past) oriented separations and if time reflections are eliminated from the list of possible transformations. Note that, mathematically speaking, the particular non-trivial aspect of these results lies in the lack of any initial continuity requirement for the bijective maps; the listed requirements suffice to imply continuity.

which are causally related are in the same equivalence class. For example, as slight modification of (25), we may take:

$$\sigma''(\tau, n) := \bigcup_{r \in [n-1, n)} \sigma(\tau + mr, r)\,, \tag{26}$$

which also consist of an inner ball and concentric spherical shells, but now taken from the half-cones $C^{\pm}(\tau, \alpha)$ with vertex on X at time τ and opening angle $\alpha = |\tan^{-1}(1/m)|$. They open to the future (+ sign) for $m > 0$ and to the past ($-$ sign) for $m < 0$. For the half-cones to be acausal, i.e. spacelike hypersurfaces, we need opening angles bigger than $\pi/2$, i.e., $|m| < 1$. $m = 1$ gives future light-cones, $m = -1$ past light-cones.

This abundance of possibilities does not violate Malament's theorem, since the onion-like partitions of the spatial hypersurfaces Σ_τ or $C(\tau, \alpha)$ is not invariant under scale transformations; only the partition of $\mathcal{M}$ into the Σ_τ or $C(\tau, \alpha)$ (α fixed) is. But, in turn, the latter is not invariant under time reflections. This is why Malament's theorem works. (See the appendix for more details.)

Sarkar & Stachel pointed out that time reflections had to be included in order to prove Malament's theorem, and that without them one could still have lightlike half-cones as equivalence classes (half-cones of other opening angles are also possible, or course). They assert – and we agree with this – that there is no physical reason to require invariance under time reflections. But likewise is there no physical reason to include dilatations, which they do include in their definitions, although in footnote 11 of their paper Sarkar & Stachel (1999) explicitly state that their 'considerations are independent of a requirement of invariance under scale transformations' (i.e. dilatations). They do not mention that dropping dilatations adds a plethora of new invariant equivalence relations, like those in (25)(26) (where e.g. the shell thickness is totally arbitrary and may, in addition, depend on r).

Finally Sarkar & Stachel (1999) consider the case where X is an inertial frame and try to show uniqueness of standard Einstein simultaneity. Expressed in our terminology they argue as follows: if X is an inertial frame, $\mathtt{Aut}_X$ contains spatial translations perpendicular (w.r.t. Minkowski metric) to the lines in X. The orbit of any point p under these translations is the plane Σ_t containing p. Since 'they [the spatial translations] are not to affect the simultaneity relation, these translations must take each simultaneity hypersurface to itself' (Sarkar & Stachel 1999, 217), which are therefore given by the Σ_t's. But this does not provide a proof, since the underlined part does not follow. It is not true that the equivalence classes of a G-invariant equivalence relation must separately be G-invariant sets; only the *partition* must be G-invariant, but G may well permute the equivalence classes. The precise statement is given in equation (9). From our Theorem 5 it is also clear that classifying invariant equivalence classes is not quite sufficient, one also needs to impose some condition which eliminates those relations whose classes contain timelike related points. No such condition is mentioned by Sarkar & Stachel (1999).

Appendix: Proof of Theorem 7

By hypothesis there exist $p \in X$ and $q \notin X$ such that $S(p, q)$. W.l.o.g. we may take X to be the t-axis and p to be the origin; otherwise we can boost the observer to rest and translate p to the origin, and then consider the conjugates of all subgroups to be mentioned by that combination of a boost and a translation. We set $q = (t', \vec{x}')$.

From Alexandrov's (1975) results we know that $\mathtt{Aut}^{\mathrm{c}}$ is as specified above and can hence infer that the stabiliser subgroup $G := \mathtt{Aut}^{\mathrm{c}}_X$ consist of the following transformations and combinations thereof: 1) time-translations, 2) time reflections about any moment in time, 3) the group $\mathtt{O}(3)$ of all orthogonal spatial transformations (i.e. including reflections), and 4) all dilatations about points on X. Consider the subgroup $G_p \subset G$ that fixes p (the origin). It consists of (and combinations thereof): 2') the single time reflections about time zero: $t \mapsto -t$, 3') all of $\mathtt{O}(3)$, and 4') dilatations about p: $r \mapsto \lambda r\,, \forall r \in \mathcal{M}$ and $\lambda > 0$.

We know that $G_p \cdot q \subseteq [p]$, hence we are interested in the G_p-orbit of q. The orbit of q under transformations 4'), including the point p, consists of the half-line $L : \lambda \mapsto \lambda q$, $\lambda \geq 0$. The shape of the orbit of this half-line under all remaining transformations in G_p crucially depends on whether $t' \neq 0$ or $t' = 0$ (i.e. whether or not p and q lie in the same hyperplane perpendicular to X).

Case 1: $t' \neq 0$. The angle between X and the half line L ending on X is $\alpha = \tan^{-1}(\|\vec{x}'\|/|t'|)$ with $0 < \alpha < \pi/2$. Hence the orbit of L under all transformations in $\mathtt{O}(3)$ – which is the same as the orbit under $\mathtt{SO}(3)$, so spatial reflections do not add anything new – is a half-cone with vertex p and opening angle α, which opens to the future if $t' > 0$ and to the past if $t' < 0$. Acting with the remaining time reflection 2') results in a (full) cone with same vertex and opening angle, which we call $C(p, \alpha)$.

Case 2: $t = 0$. Now L is perpendicular to X ($\alpha = \pi/2$). The orbit of L under $\mathtt{O}(3)$ is the plane $\Sigma_{t=0}$. The main difference to Case 1 is that now the time reflection 2') adds nothing new since it leaves Σ_0 (pointwise) fixed. Hence the full G_p-orbit is still Σ_0.

Consider Case 1; then $[p] \supseteq C(p, \alpha)$ and hence by (9) $[g \cdot p] \supseteq g \cdot C(p, \alpha) = C(g \cdot p, \alpha)$ for all $g \in \mathtt{Aut}^{\mathrm{c}}$. (The last equality is obvious if one writes g as a linear transformation in $\mathtt{Aut}^{\mathrm{c}}$, which leaves the light cone at the origin invariant, followed by a translation. But any g can be so written since the translations form a normal subgroup.) Taking all spacetime translations for g shows $[p] \supseteq C(p, \alpha)$ for all $p \in \mathcal{M}$. But for $0 < \alpha < \pi/2$ the cones $C(p, \alpha)$ and $C(p', \alpha)$ for any two p, p' necessarily intersect. This is indeed easy to see and needs not be proven here. Hence all equivalence classes intersect and S is trivial.

Finally consider Case 2; then $[p]$ contains the hypersurface Σ_0. If it contains any other point we are back to case 1 and S is trivial. Hence a non-trivial S would have $[p] = \Sigma_0$. Using time translations in (9) then shows that $[p]$ would likewise be given by the hyperplane perpendicular to X containing p. But this proves the theorem since the partition of $\mathcal{M}$ into the hyperplanes Σ_t is indeed $\mathtt{Aut}^{\mathrm{c}}$-invariant and hence defines a non-trivial equivalence relation.

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