ON WEAK EXTENSIVE MEASUREMENT*

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Extensive measurement is called weak if the axioms allow two objects to have the same scale value without being indifferent with respect to the order. Necessary and/or sufficient conditions for such representations are given. The Archimedean and the non-Archimedean case are dealt with separately.

1. Introduction. In the measurement of extensive quantities one generally considers a triple $\langle A, \succsim, \bigcirc \rangle$ of primitives. The symbol A denotes some nonempty set of objects; $a \succsim b$ for $a, b \in A$ means that b does not exceed a, for the attribute under study (i.e. $\succsim$ is a binary relation on A); $(a,b) \rightarrow a \bigcirc b$ denotes the concatenation of objects a and b in A forming a new object $a \bigcirc b \in A$ (i.e. $\bigcirc$ is a binary operation on A). For this triple, *strong extensive measurement* (s.e.m.) is defined as the construction of a nonconstant real-valued function f on A such that $f(a) \geq f(b)$ if and only if $a \succsim b$, and $f(a \bigcirc b) = f(a) + f(b)$. *Weak extensive measurement* (w.e.m.) is defined[1] as the construction of a nonconstant real-valued function f on A such that $f(a) \geq f(b)$ if (but not necessarily only if) $a \succsim b$ and $f(a \bigcirc b) = f(a) + f(b)$. Thus, weak extensive measurement differs from strong only in permitting two elements to have the same numerical value without being indifferent with respect to the relation.

The axioms for strong extensive measurement often turn out to be too restrictive in practice. When pairs of objects similar in size (in a sense to be defined below) may exist among the objects under study, weak extensive measurement is more appropriate. For a more thorough discussion of the problems of application the reader is referred to [4]. In this paper, axiom systems for w.e.m. are stated. The Archimedean and the non-Archimedean case are dealt with separately. It can be shown that for both weak and strong extensive measurement, representations are unique up to similarity transformations (see [3]).

2. Definitions. Certain notational conventions must now be introduced.

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[1]This notion has been introduced in [3].

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The above function f is sometimes called a *strong* (resp. *weak*) *extensive measurement representation.* The concatenation symbol $\bigcirc$ may be omitted for the sake of brevity: ab will stand for $a \bigcirc b$. For any positive integer n, the notation a^n is defined recursively, such that $a^1 = a$ and $a^{n+1} = a \bigcirc a^n$. For $a \succsim b$ and $a \succ b$ we sometimes write $b \precsim a$ and $b \prec a$, respectively; $\succ$ and $\simeq$ are the asymmetric and symmetric parts of $\succsim$. $\succsim$ is called a *weak order* iff it is transitive and connected ("iff" is short for "if and only if"). Finally, A may stand for the triple $\langle A, \succsim, \bigcirc \rangle$ if no ambiguity arises. An element $a \in A$ is called *positive,* or *negative,* or *null* iff for any $b \in A$, we have $ab \succ b$ and $ba \succ b$, or $ab \prec b$ and $ba \prec b$, or $ab \simeq b$ and $ba \simeq b$, respectively. The sets of positive, negative, and null elements are called P, N, and O, respectively. These sets are pairwise disjoint. If they constitute a partition on A, A is called *sign consistent.* A is *nontrivial* iff it contains at least a null and a non-null element. For $a, b \in A$, a is said to be *Archimedean equivalent to b* ($a \sim b$) iff there is a positive integer n such that at least one of the following inequalities holds:

$$a \precsim b \precsim a^n,\ b \precsim a \precsim b^n,\ a^n \precsim b \precsim a, \text{ or } b^n \precsim a \precsim b.$$

$\sim$ can be shown to be an equivalence relation on A if sign consistency holds. $[a]$ will denote the Archimedean equivalence class (AEC) generated by a. Archimedean axioms are conditions that somehow restrict the number of $AECs$, e.g. postulate that P consist of a single class. A pair of elements a, b in A is called *anomalous* if $a \not\simeq b$ and either, for all $n \in IN$, $a^n \succ b^{n+1}$ and $b^n \succ a^{n+1}$ or, for all $n \in IN$, $b^{n+1} \succ a^n$ and $a^{n+1} \succ b^n$. Intuitively, two elements are Archimedean equivalent if none is "infinitely greater or smaller" than the other. Analoguously, two elements form an anomalous pair if their "difference" is "infinitely small" (with respect to "differences" of other pairs of elements[2]). On the real line with usual order, anomalous pairs do not exist and thus, their prohibition is necessary for s.e.m.

3. Axiom Systems. Numerous axiom systems for s.e.m. have been proposed. The reader may consult [4], for detailed reference. Few systems for w.e.m. have appeared in the literature, however. On the other hand, identification of anomalous pairs and thus proof of their nonexistence will often be extremely difficult or even impossible in practice. Their prohibition may be unduly restrictive. Therefore,

[2]For results on the relation between Archimedean equivalence and anomalous pairs in ordered commutative semigroups see [1].

w.e.m. representations that do not differentiate between elements of an anomalous pair (i.e., map anomalous pairs onto the same number) seem to be more appropriate in many practical cases.

Anomalous pairs may exist even if certain Archimedean properties hold (see, e.g., [2], p. 163). In what follows, we shall give representations admitting anomalous pairs. First, we deal with the Archimedean case, then the Archimedean condition will be deleted entirely.

Theorem 1 (Archimedean Case). Let $\gtrsim$ be a binary relation and let $\circ$ be a binary operation on a nontrivial set A such that for all $a, b, c \in A$

(1) $a \gtrsim b$ or $b \gtrsim a$; $a \gtrsim b$ and $b \gtrsim c$ imply $a \gtrsim c$.
(2) $(ab)c \simeq a(bc)$.
(3) $a \gtrsim b$ implies $ac \gtrsim bc$ and $ca \gtrsim cb$;

then the following condition is necessary and sufficient for the existence of a w.e.m. representation of A such that two distinct elements of A (i.e., different with respect to $\gtrsim$) are mapped onto the same number iff they form an anomalous pair:

(4) Any non null element $b \in A$ satisfies one of the following statements: for any a, there are $m, n \in IN$ such that $b^m > a$ and $b^n a \gtrsim b$; or, for any a, there are $m, n \in IN$ such that $b^m < a$ and $b^n a \lesssim b$.

The proof bearing on a theorem in [3] is deferred until the end of the paper.

Condition (4) is an Archimedean axiom stating that any two elements are "comparable" with respect to the order. It is stronger than a corresponding condition in [3] but, as Theorem 1 indicates, it is a necessary condition for the representation to hold. It includes an unboundedness assumption by use of strict inequalities. Since (4) holds in any *AEC*, Theorem 1 implies that a fully ordered semigroup consisting of one *AEC P* of positive elements is order-homomorphic to a subsemigroup of the additive positive real numbers such that two distinct elements have the same image iff they form an anomalous pair. This eliminates cancellativity in the theorem of Hion (1957) reported by Fuchs ([2], p. 170). Moreover, positivity being omitted, Condition (4) is necessary and sufficient for an analoguous homomorphy into a subsemigroup of the additive real numbers.

Non-Archimedean extensive structures have been investigated in [5] and [6]. The former, extending the notion of s.e.m. has shown that these structures can be embedded into non standard models of the reals. If only representations into IR are considered the following statement can be derived:

Theorem 2 (Non-Archimedean Case). Suppose that all conditions of Theorem 1 except (4) hold on $\langle A, \succsim, \bigcirc \rangle$ and that A is sign consistent (i.e., $A = P + N + O$); then there is a real-valued function s on A such that for all $x, y \in A$,

(i) $x \sim y$ implies $s(x \bigcirc y) = s(x) + s(y)$;
(ii) $x \sim y$ and $x \precsim y$ imply $s(x) \leq s(y)$;
(iii) $x \prec y$ and not $x \sim y$ imply $s(x \bigcirc y) = s(y)$ for $x \bigcirc y \in P$ and $s(x \bigcirc y) = s(x)$ for $x \bigcirc y \in N$;

moreover, for all $z \in A$,

(iv) for all $x, y \in P$, $x \sim y$ and $s(x) = s(y)$ and $x \bigcirc z \precsim y$ implies not $z \sim x$, and
for all $x, y \in N$, $x \sim y$ and $s(x) = s(y)$ and $x \bigcirc z \succsim y$ implies not $z \sim x$.

The function s constitutes w.e.m. on every *AEC*. If two elements x, y do not belong to the same *AEC*, $x \bigcirc y$ is assigned the s-value of the greater or the smaller element depending on whether $x \bigcirc y$ is positive or negative. Moreover, if two Archimedean equivalent elements get the same s-value, their "difference" cannot belong to the same *AEC*.

Theorem 2 extends a result in [5] (Theorem 5.8, p. 389) by assuming sign consistency instead of positivity and by removing commutativity of $\bigcirc$. Clearly, commutativity does not follow from the representation. Violations of commutativity, however, will hardly be detected in practice: it follows from Theorem 1 that $a \bigcirc b \succ b \bigcirc a$ in an *AEC* iff $a \bigcirc b$, $b \bigcirc a$ form an anomalous pair. It is interesting to note that unlike [5], in the proof of Theorem 2 no model-theoretic methods need to be used. Rather the theorem follows almost trivially from Theorem 1 by use of the Axiom of Choice.

4. Proofs. *Proof of Theorem 1.* (Sufficiency) Theorem 1 in [3] states that the existence of a single non-null element $x \in A$ satisfying the conditions on non-null elements in Condition (4) is sufficient for the construction of a w.e.m. representation f of A. All we need to know about this construction is that $f(x) = 1$ or -1; A being nontrivial we may take an $a \in A \setminus O$ in order to get a w.e.m. representation f with $f(a) = 1$ or -1. Suppose $f(b) = 0$ for $b \in A \setminus O$. If g is the w.e.m. representation attained by taking b instead of a, then $g(b) = 1$ or -1. But by uniqueness of the representation there must be a real $\alpha > 0$, such that for all $c \in A$ $\alpha f(c) = g(c)$, thus $g(b) = 0$. This contradiction implies $f(b) \neq 0$ for all $b \in A \setminus O$.

Now suppose $c, d \in A$ do form an anomalous pair; $c^n \prec d^{n+1}$

and $d^n \prec c^{n+1}$ for all $n \in IN$ imply $nf(c) \leq (n+1)f(d)$ and $nf(d) \leq (n+1)f(c)$ and thus $n/(n+1) \leq f(c)/f(d) \leq (n+1)/n$ for all $n \in IN$, i.e., $f(c) = f(d)$. The case $c^n \succ d^{n+1}$ and $d^n \succ c^{n+1}$ follows in the same way. Now suppose c, $d \in A$ do not form an anomalous pair. It remains to show $f(c) \neq f(d)$. There are four possible cases:

(a) there are m and p with $d^m \precsim c^{m+1}$ and $d^{p+1} \precsim c^p$,
(b) there are m and q with $d^m \precsim c^{m+1}$ and $c^{q+1} \precsim d^q$,
(c) there are 1 and p with $c^1 \precsim d^{1+1}$ and $d^{p+1} \precsim c^p$,
(d) there are 1 and q with $c^1 \precsim d^{1+1}$ and $c^{q+1} \precsim d^q$,

with 1, m, p, and $q \in IN$. We distinguish two possibilities:

(I) $d \notin O$; then

(a) implies $f(d^m) - f(d) \leq f(c^m)$ and $f(d^p) + f(d) \leq f(c^p)$;
(b) implies $f(d^m) - f(d) \leq f(c^m)$ and $f(c^q) \leq f(d^q) - f(d)$;
(c) implies $f(c^1) \leq f(d^1) + f(d)$ and $f(d^p) + f(d) \leq f(c^p)$;
(d) implies $f(c^1) \leq f(d^1) + f(d)$ and $f(c^q) \leq f(d^q) - f(d)$.

For $f(d) > 0$, we have $f(d^p) < f(c^p)$ and thus $f(d) < f(c)$ in (a) and (b); similarly, we have $f(c^q) < f(d^q)$ and thus $f(c) < f(d)$ in (b) and (d); for $f(d) < 0$, we have $f(d^m) < f(c^m)$ and thus $f(d) < f(c)$ in (a) and (b); similarly, we have $f(c^1) < f(d^1)$ and thus $f(c) < f(d)$ in (c) and (d).

(II) $c \notin 0$; because of the obvious symmetry of c and d in (a)—(d), this case can be dealt with analoguously and we omit it here.

(Necessity) Suppose $f(b) = 0$ for $b \in A \setminus O$, then b must form an anomalous pair with a null element; this is prohibited by the definition of anomalous pairs, however; for $f(b) > 0$, we have: for all $a \in A$ there exist m, $n \in IN$ such that $mf(b) > f(a)$ and $nf(b) + f(a) > f(b)$; thus $f(b^m) > f(a)$ and $f(b^n a) > f(b)$ implying $b^m \succ a$ and $b^n a \succsim b$; for $f(b) < 0$ we get $b^m \prec a$ and $b^n a \precsim b$, in the same way.

Proof of Theorem 2. The relation $\sim$ induces a unique partition of A into AECs; $a \in P$ (or N or O) implies $[a] \subset P$ (or N or O), obviously. Thus it is readily seen that Condition (4) of Theorem 1 holds in every *AEC*. This yields for any $k \in K$, K the class of all *AECs* of A, a w.e.m. representation f_k with the properties stated in Theorem 1. For $k = 0$, there is the trivial representation $f_0 = 0$. Now the function s on A may be defined as follows: for $x \in k$, $s(x) = f_k(x)$, and for $x \in k$, $y \in k'$, $x \prec y$ $(k \neq k')$ $s(x \bigcirc y) = s(y)$ for $x \bigcirc y \in P$, $s(x \bigcirc y) = s(x)$ for $x \bigcirc y \in N$, and $s(x \bigcirc y) = 0$ for $x \bigcirc y \in O$; (i)—(iii) of Theorem 2 are easily verified; for (iv), suppose $z \in A$, $x, y \in P$, $x \sim y$, $s(x) = s(y)$, $x \bigcirc z \precsim y$, and $z \sim x$; thus $x \bigcirc z \in [x]$ and $s(x \bigcirc z) = s(x) + s(z) \leq s(y)$; this implies $s(z) \leq 0$; however, since $[z] \subset P$, $s(z) > 0$; the case $x, y \in N$ is similarly shown. This completes the proof.

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