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Free vibration Analysis of Beams with Functional Gradient Materials with Cracks

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Abstract: The article presents the development of a dynamic model for a functionally graded material (FGM) beam incorporating cracks. Initially, it assumes an exponential distribution of material properties along the thickness direction of the beam and simulates the opening crack using a zero-mass rotational spring model. This approach enables the calculation of bending stiffness and local flexibility at the cracked section. Subsequently, drawing upon Timoshenko beam theory and von Kármán geometric nonlinear theory, the study formulates the energy equation of the beam and establishes the partial differential control equations for the cracked FGM using Hamilton's principle. The method of separation of variables is employed to discretize the partial differential motion equations into ordinary differential motion equations. Beam functions serve as mode functions, whose unknown coefficients are determined according to the boundary and continuity conditions, thereby yielding the natural frequencies and mode shapes of the cracked FGM beam. Numerical analysis is conducted to evaluate the impact of boundary conditions, the relative position of cracks, and the length-to-thickness ratio on the natural frequencies of the cracked FGM beam.

Keywords: Functional gradient materials, crack, Free vibration.

1. Introduction

Since the 1930s, the vibration problem of beams with cracks has drawn the attention of researchers, and extensive research has been conducted in this field over time. The exploration process typically includes selecting the method for modeling cracks, establishing dynamic equations, applying various algorithms, and investigating vibration characteristics, excitation responses, stability, and crack parameter identification.

Irwin GR[1] analyzed the dynamic characteristics of metal plates with cracks through the theory of local flexibility, laying the foundation for subsequent research. Sourki R et al.[2] utilized the classical Euler-Bernoulli beam theory and different methods such as the modified stress couple theory, Fourier series, and Galerkin method to analyze the free lateral vibration of cracked beams, obtaining the frequency equations for cracked beams. By employing Hamilton's principle, they derived control equations and related boundary conditions, taking into account the additional strain energy caused by cracks and the discontinuity in bending slope due to cracks. The study examined the impact of different crack positions, crack depths, material length scale parameters, and Poisson's ratio on the system's natural frequencies. Based on the first-order shear deformation theory, Torabi K et al.[3] modeled the free lateral vibration of cracked beams using a generalized delta function and variation coefficients. They derived the control equations for cracked beams under both symmetric and asymmetric boundary conditions and analytically solved these equations using basic standard trigonometric and hyperbolic functions, analyzing the free vibration characteristics of the system and investigating the effects of different system conditions on its dynamic behavior. Taima MS and El-Sayed TA et al.[4] studied the lateral vibration of isotropic and thick beams with cracks, comparing the analysis of Timoshenko beam theory and Reddy beam theory for beams with cracks. By dividing the beam into multiple elements and deriving stiffness and mass matrices for each beam element based on Reddy beam theory equations, they simulated the impact of cracks and analyzed the effects of boundary conditions, slenderness ratio, crack position, and depth on lateral vibration. The accuracy of the model results was validated through experiments and finite element analysis. Zhao et al.[5] used the Green's function method and the superposition principle, combined with the separation of variables method, Laplace transforms, and other mathematical methods, to obtain explicit expressions for the steady-state forced vibration of fixed cross-section and variable cross-section Euler-Bernoulli beams with multiple cracks under harmonic loads. They analyzed the results, demonstrating that the geometric shape of the crack (depth and position) and section parameters significantly impact the system's dynamic behavior. Based on this, X. Zhao et al.[6] studied the steady-state forced vibration of Timoshenko beams with multiple cracks, verifying that the influence of shear deformation and rotational inertia on beams with a certain thickness under forced vibration cannot be ignored. Loya JA et al.[7] and Chen et al.[8] established crack beam models based on Winkler foundation, deriving the differential equations for free and forced vibration of beams with multiple cracks through corresponding boundary conditions and compatibility conditions at the crack sections. They examined the impact of different support types, support stiffness, crack position, and crack depth on the system's natural frequencies.

All the above studies focus on the dynamic analysis of beams with cracks made of homogeneous materials. This article will analyze the free vibration characteristics of functionally graded materials with cracks.

2. Material Properties of Functionally Graded Beans and Eauivalent Spring Models

This functionally graded material is composed of two different constituent materials, with the Young's modulus E(z)and ρ(z)density varying exponentially in the thickness direction (z-direction) based on the volume fraction of each material component. This gradient variation can be expressed specifically as:

$$E(z)=E_0 e^{\beta z},\quad \rho(z)=\rho_0 e^{\beta z} \tag{1}$$

In the equation, E_0 and ρ_0 represent the Young's modulus and density at the mid-plane (z=0) of the functionally graded beam, while E_1 and E_2 correspond to the Young's modulus of the upper and lower materials, respectively. Similarly, ρ_1 and ρ_2 correspond to the density of the upper and lower materials. The constant denotes the gradient change of material properties along the thickness direction, and the beam is assumed to be isotropic and homogeneous. Since the Poisson's ratio ν has a minimal effect on the stress intensity factor at the crack tip, ν is taken as a constant.

The cracked beam can be divided into two sub-beams at the crack location, with the two sub-beams connected by a zero-mass, zero-length rotational spring at the crack cross-section. Therefore, the crack cross-section is modeled as a zero-mass elastic rotational spring, as shown in Figure 1. The beam contains a crack of depth a located at a distance $L1$ from the left end, with a bending stiffness of:

$$K_T=\frac{1}{G} \tag{2}$$

Where G represents the local flexibility caused by the crack.

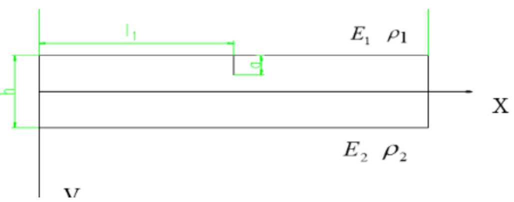

Figure 1. Functional graded beam model with crack

According to fracture mechanics theory, the flexibility G caused by the presence of a crack is related to the stress intensity factor at the crack tip. The additional strain energy U caused by the crack can be expressed as:

$$U=\int_A \frac{\left(1-\nu^2\right)}{E(a)} K_1^2\,\mathrm{d}A \tag{3}$$

Where A is the cross-sectional area, KI represents the stress intensity factor (SIF) at the crack tip under bending load in a mode I crack state, and M is the bending moment at the crack cross-section. The local flexibility G at the material's crack cross-section can be expressed as:

$$G=\frac{\partial^2 U}{\partial M^2} \tag{4}$$

For a functionally graded beam system with a mode I crack, the magnitude of the stress intensity factor is obtained from the data provided by Erdogan and Wu[9] using Lagrange interpolation techniques.

$$K_1=\frac{6M\sqrt{\pi h\varsigma}}{h^2}F(\varsigma),\quad \varsigma=\frac{a}{h}\quad (\varsigma\leqslant 0.7) \tag{5}$$

3. The Equations of Motion for Functionally Graded Beams with Crack

According to Timoshenko theory, the displacements along the x-axis and z-axis at any point in a functionally graded beam with cracks can be expressed:

$$\begin{aligned}\bar{U}(x,z,t)&=U(x,t)+z\Psi(x,t)\\ \bar{W}(x,z,t)&=W(x,t)\end{aligned} \tag{6}$$

Where $u0$ and $w0$ are the longitudinal and lateral displacement components at the mid-plane (z=0), and ϕ is the rotation angle of the beam cross-section, with t representing time.

According to von Kármán's nonlinear strain-displacement relationship, the normal strain and shear strain can be derived as follows:

$$\varepsilon_x=\frac{\partial U}{\partial x}+z\frac{\partial \Psi}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2,\quad \gamma_{xz}=\frac{\partial W}{\partial x}+\Psi \tag{7}$$

According to the linear elastic constitutive relations, the normal stress σ and shear stress τ can be expressed as:

$$\begin{aligned}\sigma_{xx}&=Q_{11}(z)\left(\frac{\partial U}{\partial x}+z\frac{\partial \Psi}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right),\\ \tau_{xz}&=Q_{55}(z)\left(\frac{\partial W}{\partial x}+\Psi\right)\end{aligned} \tag{8}$$

Where:

$$Q_{11}(z)=\frac{E(z)}{1-\nu^2},\quad Q_{55}(z)=\frac{E(z)}{2(1+\nu)} \tag{9}$$

The axial force N, bending moment M, and transverse shear force V acting on the functionally graded material beam with cracks can be expressed in the following forms:

$$\begin{aligned}N_x&=\int_{-h/2}^{h/2}\sigma_{xx}\,\mathrm{d}z\\ M_x&=\int_{-h/2}^{h/2}\sigma_{xx}z\,\mathrm{d}z\\ Q_x&=\kappa\int_{-h/2}^{h/2}\tau_{xz}\,\mathrm{d}z\end{aligned} \tag{10}$$

By substituting equation 6 into equation 11, we obtain:

$$
\begin{aligned}
N_{xi} &= A_{11}\frac{\partial U_i}{\partial x}+\frac{1}{2}A_{11}\left(\frac{\partial W_i}{\partial x}\right)^2+B_{11}\frac{\partial \Psi_i}{\partial x}\\
M_{xi} &= B_{11}\frac{\partial U_i}{\partial x}+\frac{1}{2}B_{11}\left(\frac{\partial W_i}{\partial x}\right)^2+D_{11}\frac{\partial \Psi_i}{\partial x}\\
Q_{xi} &= \kappa A_{55}\left(\frac{\partial W_i}{\partial x}+\Psi_i\right)
\end{aligned}
\tag{11}
$$

In the equation, κκ represents the shear correction factor for the Timoshenko beam, the precise value of which is calculated as a function of material properties and beam cross-sectional parameters and requires complex computations. For functionally graded material beams with rectangular cross-section, the value is typically taken as κ=5/6.

The stiffness coefficients and inertia terms for functionally graded Timoshenko beams with cracks can be defined as:

$$
\begin{aligned}
\left(A_{11},B_{11},D_{11}\right) &= \int_{-h/2}^{h/2} Q_{11}(z)\left(1,z,z^2\right)\mathrm{d}z\\
A_{55} &= \int_{-h/2}^{h/2} Q_{55}(z)\mathrm{d}z
\end{aligned}
\tag{12}
$$

$$
\left\{I_1,I_2,I_3\right\} = \int_{-h/2}^{h/2}\rho(z)\left\{1,\mathrm{z},\mathrm{z}^2\right\}\mathrm{d}z
$$

Based on Hamilton's principle, continuity conditions of force and displacement, and the relationship between crack-induced additional rotation, and by non-dimensionalizing the equations, the system's free vibration equation is derived as:

$$
\begin{aligned}
a_{11}\frac{\partial^2 U_i}{\partial \zeta^2}+b_{11}\frac{\partial^2 \psi_i}{\partial \zeta^2} &= I_1\frac{\partial^2 U_i}{\partial \tau^2}+I_2\frac{\partial^2 \psi_i}{\partial \tau^2}\\
\kappa a_{55}\left(\frac{\partial^2 W_i}{\partial \zeta^2}+\eta\frac{\partial \psi_i}{\partial \zeta}\right) &= I_1\frac{\partial^2 W_i}{\partial t^2}
\end{aligned}
\tag{13}
$$

$$
b_{11}\frac{\partial^2 U_i}{\partial \zeta^2}+d_{11}\frac{\partial^2 \psi_i}{\partial \zeta^2}-\kappa a_{55}\eta\left(\frac{\partial W_i}{\partial \zeta}+\eta\psi_i\right)=I_2\frac{\partial^2 U_i}{\partial t^2}+I_3\frac{\partial^2 \psi_i}{\partial t^2}
$$

The boundary conditions corresponding to three different support types can be stated as follows:

(a) One end complained, one end free (C-F):

$$
\begin{aligned}
&\text{在}\zeta=0\text{处：}u_1=0,\quad w_1=0,\quad \psi_1=0\\
&\text{在}\zeta=1\text{处：}N_{x2}=0,\quad Q_{x2}=0,\quad M_{x2}=0
\end{aligned}
\tag{14}
$$

(b) Hinged at both ends (H-H):

$$
\begin{aligned}
&\text{在}\zeta=0\text{处：}u_1=0,\quad w_1=0,\quad M_{x1}=0\\
&\text{在}\zeta=1\text{处：}u_2=0,\quad w_2=0,\quad M_{x2}=0
\end{aligned}
\tag{15}
$$

(c) Both ends complained (C-C):

$$
\begin{aligned}
&\text{在}\zeta=0\text{处：}u_1=0,\quad w_1=0,\quad \psi_1=0\\
&\text{在}\zeta=1\text{处：}u_2=0,\quad w_2=0,\quad \psi_2=0
\end{aligned}
\tag{16}
$$

By considering different boundary conditions and combining the expressions for N, M, and Q, 12 homogeneous nonlinear algebraic equations can be obtained, which can be written in matrix form as:

$$
\mathbf{H}(\omega)\boldsymbol{\chi}=\mathbf{0} \tag{17}
$$

In the equation, H(ω) is a matrix that is associated with the system's natural frequencies ω and exhibits nonlinear characteristics, while χ represents a vector composed of the 12 unknown coefficients, defined, In order for the equations to have a nontrivial solution, the determinant of H(ω) must be zero, as:

$$
\det[\mathbf{H}(\omega)]=0 \tag{18}
$$

4. Numerical Results and Discussion

In the numerical simulations presented in this paper, the following geometric and material parameters were used: the thickness of the beam is h=0.1 m, and the ratio of elastic moduli $E2$ /$E1$ is set to 0.2, 1, and 5. The case $E2$ /$E1$=1 corresponds to an isotropic homogeneous beam. The upper surface of the beam consists of 100% aluminum, with the material properties being.

According to Figure 2, when the functionally graded beam has material parameters $E2$ /$E1$=0.2, E2 /E1 =0.2, a length-to-width ratio of 8, and a relative crack depth of 0.2, the impact of different crack positions and different boundary conditions on the first three natural frequency ratios of the system can be observed. From Figure 2(a), it can be seen that under one fixed end and one free end boundary conditions, the first natural frequency ratio of the system increases as the relative position of the crack shifts from the fixed end towards the free end, reaching its maximum when the crack is entirely at the free end. The second and third natural frequency ratios show an irregular fluctuation trend as the crack position increases, reaching their maximum values when the relative crack position $L1$ /L is at 0.2, 1 and 0.3, 0.5, 1 respectively. The natural frequency ratios for the first three modes are all minimized when the crack is located at the fixed end, with the first mode being the smallest and the second mode following closely.

From Figures 2(b) and 2(c), it can be observed that when the functionally graded beam with cracks is under either both ends fixed or both ends hinged boundary conditions, the curves of the natural frequency ratios with respect to the crack position exhibit symmetry. For functionally graded beams under both ends fixed conditions, the natural frequency ratios for the first three modes are minimized when the crack is located at the fixed ends, following a similar trend to the one fixed end condition. For the both ends hinged boundary condition, the first natural frequency ratio of the system is minimized when $L1$ /L=0.5. The second natural frequency ratio is minimized at $L1$ /L=0.25 and 0.75, while the third natural frequency ratio is minimized when L1 /L=0.15 and 0.85. In all three boundary conditions, the pattern remains consistent: the first mode frequency ratio is the smallest, followed by the second mode frequency ratio.

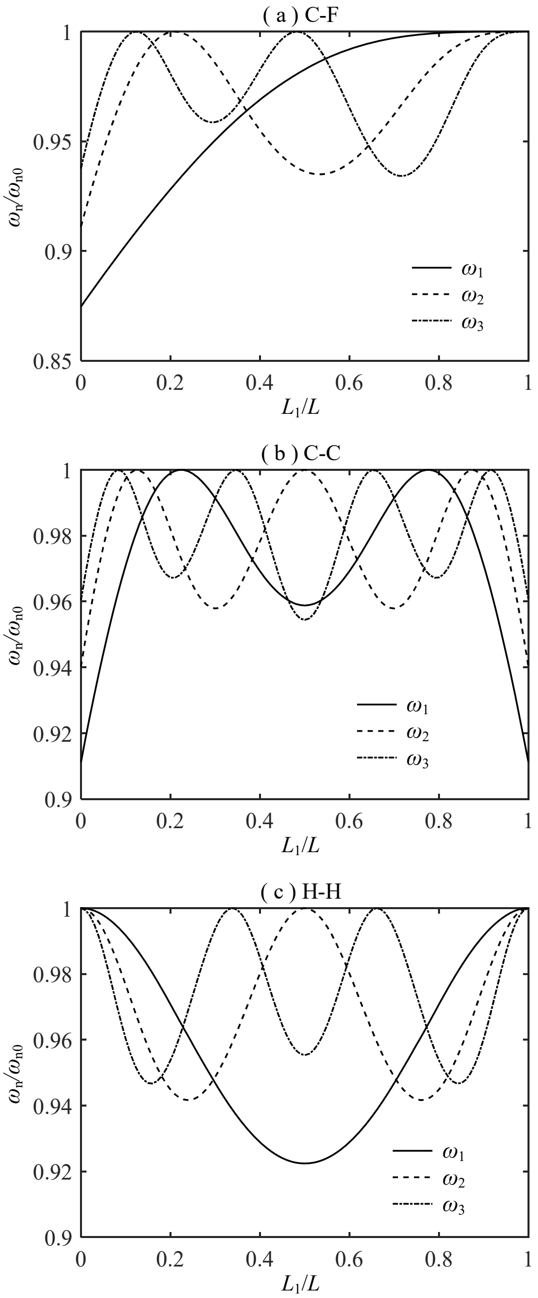

Figure 2. The first three natural frequency ratios of functionally graded material (FGM) beams with edge cracks at different positions

As seen from Figure 3, when the elastic modulus ratio of the functionally graded beam with cracks is 5 and the relative crack depth is 0.2, the impact of different length-to-thickness ratios on the third mode natural frequency ratio under different boundary conditions and crack positions can be observed.

In Figure 3(a), under the boundary condition of one fixed end and one free end, the natural frequency ratio is minimized when the crack is located at the fixed end. As the crack position shifts towards the free end, the frequency ratio increases steadily. In Figure 3(b), under both ends fixed boundary conditions, the natural frequency ratio is also minimized when the crack is at either fixed end, with a noticeable increase in the minimum frequency ratio compared to the one fixed end condition. The frequency ratio fluctuates as the relative crack position increases, reaching a minimum when the crack is at the center of the beam. From Figure 3(c), under both ends hinged boundary conditions, the natural frequency ratio is minimized when the crack is at the center of the beam. The natural frequency ratio is maximized when the crack is at either end of the beam. As the relative crack position increases, the natural frequency ratio for different length-to-thickness ratios first decreases and then increases. When the functionally graded beam with cracks is under either both ends hinged or both ends fixed boundary conditions, the curves of different relative crack depths and crack positions show geometric symmetry.

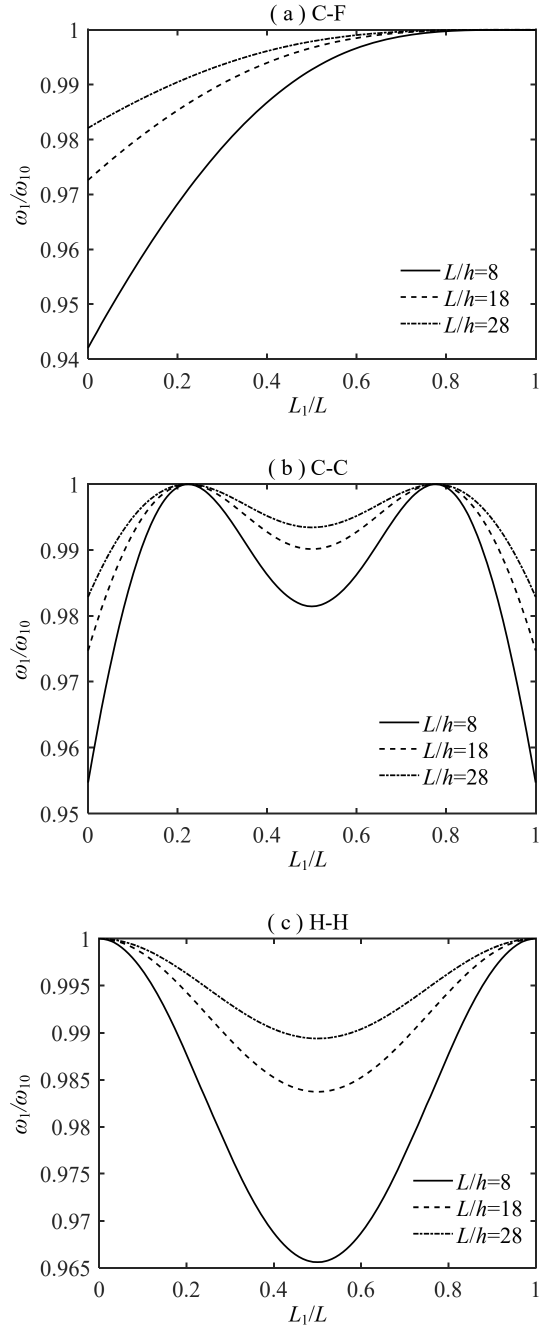

Figure 3. Variation in the first mode natural frequency of cracked functionally graded material (FGM) beams with different length-to-thickness ratios

From Figure 3, under three different boundary conditions, when the crack is at any position, the system with a higher length-to-thickness ratio tends to have a lower natural frequency ratio. When the length-to-thickness ratio of the system decreases from 18 to 8, there is a significant decrease in the natural frequency ratio, indicating that the first mode natural frequency of functionally graded beams with lower length-to-thickness ratios is more sensitive to cracks.

5. Conclusion

This paper studies the nonlinear static response of functionally graded material Timoshenko beams with cracks. A nonlinear mechanical model of functionally graded material Timoshenko beams with cracks was established using the first-order shear deformation theory and von Kármán nonlinear theory. The influence of crack position and beam length-to-thickness ratio on the system's free vibration characteristics under different boundary conditions was analyzed. Numerical results indicate:

Under three different boundary conditions, the first mode natural frequency is always less than the second mode natural frequency, which in turn is less than the third mode natural frequency for different relative crack positions in the cracked beam. Additionally, when the relative crack position takes specific values, the natural frequency ratio equals 1, indicating that the crack at the current position does not impact a certain mode of natural frequency in the beam. The increase in the length-to-thickness ratio also does not change the relative crack positions at which the minimum and maximum natural frequency ratios occur for each mode.

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