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Homotopy Analysis Method for Solving Fuzzy Fractional Volterra-Fredholmintegro-differential Equations

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Abstract: The fuzzy fractional Volterra-Fredholm integro-differential equation is introduced by using the fuzzy Caputo derivative under the generalized Hukuhara difference, and the existence and uniqueness of the solution of this equation are proved by using the fixed point theorem. The homotopy analysis method is used to study the numerical solutions of linear and nonlinear fuzzy fractional integro-differential equations. Several numerical examples are given to illustrate the effectiveness and applica-bility of the method.

Keywords: Volterra-Fredholm integro-differential Equations; Fixed Point Theorem; Homotopy Analysis Method; Caputo Fractional Derivative.

1. Introduction

The motivation of fractional calculus research is its application in biology, electrochemistry, fractal theory, control theory, fluid flow and viscoelasticity [1–3]. In 1823, the application of fractional derivatives was introduced by Abel, who applied fractional calculus to solve the integral equation in the Tautochrone problem [4]. The fixed point theorem is often used to study the existence and uniqueness of solutions of fractional calculus equations [5–9], and its numerical methods are also widely studied. Because of its wide application of fractional calculus equations, in many models, such as the novel coronavirus infection system model in recent years, researchers often face parameter uncertainties. In order to solve this problem, fuzzy concepts are introduced. Zadeh [10,11] introduced the concept of fuzzy numbers and its arithmetic operations, Mizumoto and Tanaka [12] further enriched the related concepts of fuzzy numbers. Dubois and Prade [13] introduced the concept of fuzzy function sets. The Hukuhara derivative of fuzzy valued functions and fuzzy initial value problems was proposed in [14] and studied in [15]. In 1992, Liao proposed a general analytical method for solving linear and nonlinear problems by using the basic idea of homotopy in topology and differential geometry [16], which is called homotopy analysis method (HAM). Hamoud and Ghadle applied HAM to solve fuzzy Volterra-Fredholm integro-differential equations [17]. Hussain [18] used HAM to study fuzzy integral-differential equations. Hanan [19] uses HAM to solve MultiFractional order integro-differential equations. Adomian decomposition method (ADM) [20,21], Laplace transform method [22,23], implicit finite difference method [24], variational iteration method [25] and other numerical methods are also applied to solve fuzzy integral, differential and integro-differential equations. In this paper, we mainly study HAM processing fuzzy fractional Volterra-Fredholm integro-differential equation.

In addition, the existence of solutions of fuzzy equations is also the focus of re- searchers. In 2015, Arshad et al [26] studied the existence and uniqueness of solutions for Riemann-Liouville fuzzy fractional differential equations. In [27], the existence and uniqueness of solutions for a class of fractional differential equations with fuzzy initial values defined in the sense of fuzzy Caputo were discussed. Ahmad and Ullah [28] also studied the existence and uniqueness of solutions of fuzzy fractional Volterra-Fredholm integro-differential equation in the sense of Caputo. Allahviranloo et al. [29] studied the fuzzy fractional Volterra-Fredholm integro-differential equation in the sense of fuzzy Caputo derivative under the generalized Hukuhara difference(gH-difference), and proved the existence and uniqueness of the solution of the equation. Armand and Gouyandeh [30] introduced the existence and uniqueness of solutions of nonlinear fuzzy fractional Fredholm integral-differential equations under generalized fuzzy Caputo derivative. In this paper, We will discuss the existence and uniqueness of fuzzy fractional Volterra-Fredholm integro-differential equation solution in the sense of fuzzy Caputo under gH-difference.

$$\begin{aligned} {}^{c}D^{\gamma} w(x,r) &= g(x,r) + a(x)w(x,r) \\ &+ \int_0^x k_1(x,t)F_1(w(t,r))dt + \int_0^1 k_2(x,t)F_2(w(t,r))dt, \end{aligned} \tag{1}$$

$$w(0,r) = [\underline{w}(0,r), \overline{w}(0,r)], \tag{2}$$

${}^{c}D^{\gamma}$ denotes the Caputo fractional generalized derivative of order γ, $0<\gamma\le 1$, $w(x,r)=[\underline{w}(x,r),\overline{w}(x,r)]$ is a fuzzy function. $k_1 \in (\Delta, E), \Delta = \{(x,t): 0 \le t \le x \le 1\}$, $k_2 \in C([0,1]\times[0,1],E), F_i \in C([0,1]\times E, E)(i=1,2)$ and E denotes the fuzzy number space.

Inspired by the above literature, the main contributions of this paper are as follows. We know that HAM is often used to solve linear and nonlinear differential- integral problems. In this paper, HAM is used to solve the approximate solutions of linear and nonlinear fuzzy fractional Volterra-Fredholm integro-differential equation, and the effectiveness and applicability of the method are illustrated by numerical examples. Another contribution is to prove the existence and uniqueness of solution by using fixed point theorem. Based on the parametric representation of fuzzy numbers, the set of one-dimensional fuzzy numbers can be regarded as a closed convex set in Banach space.

The structure of this paper is as follows: Section 2 mainly introduces the fuzzy definition and fixed point theorem.

Section 3 introduces the iterative scheme of HAM and its application in fuzzy fractional Volterra-Fredholm integro-differential equation. In Section 4, the existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equation are studied. We provide several numerical examples and analysis of numerical results in Section 5. Finally, Section 6 gives a brief conclusion.

2. Preliminaries

In this section, we will give the related concepts of fuzzy numbers, and some important theorems and symbols used in the article.

Definition2.1. [31,32] $\mathcal{F}(R)$ denotes the set of all fuzzy sets on R. Let $h \in \mathcal{F}(R)$, if h satisfies

(i) h is a normal fuzzy set, i.e., there exists $s_0 \in R$ such that $h(s_0) = 1$,

(ii) h is a convex fuzzy set, i.e., $h(\delta s_1 + (1-\delta)s_2) \geq \min\{h(s_1), h(s_2)\}$ for all $s_1, s_2 \in R$ and $\delta \in [0,1]$,

(iii) h is an upper semi-continuous function,

(iv) The closure of the support of h is compact, i.e., $[h]^0$ is compact;

then h is called as a fuzzy number. The set of all fuzzy numbers is known as the fuzzy number space, denoted by E.

Definition2.2. [33] Given $0 \leq r \leq 1$, a fuzzy number h in parametric form is represented by an ordered function pairs $(\underline{h}(r), \overline{h}(r))$ satisfying

(i) $\underline{h}(r)$ is a bounded left continuous non decreasing function,

(ii) $\overline{h}(r)$ is a bounded left continuous non increasing function,

(iii) $\underline{h}(r) \leq \overline{h}(r)$.

For $h = (\underline{h}, \overline{h}), v = (\underline{v}, \overline{v}) \in E$ and $\delta \in R$, the sum of $v + h$ and the scalar multiplication δh can be defined by

$$(\underline{v+h})(r) = \underline{v}(r) + \underline{h}(r), \quad (\overline{v+h})(r) = \overline{v}(r) + \overline{h}(r), \quad \forall r \in [0,1],$$

and

$$\delta h = \begin{cases} (\delta\underline{h}, \delta\overline{h}), \delta \geq 0, \\ (\delta\overline{h}, \delta\underline{h}), \delta \leq 0. \end{cases}$$

Definition2.3. [29] For any two fuzzy numbers w and h, defin $D_r: E \times E \to R^+ \cup \{0\}$by

$$D_r(v,h) = \sup_{r\in[0,1]} \max\{| \underline{v}(r) - \underline{h}(r) |, | \overline{v}(r) - \overline{h}(r) |\}.$$

where $v = [\underline{v}(r), \overline{v}(r)], h = [\underline{h}(r), \overline{h}(r)]$. It has the following useful propertie.

For $\forall w, h, v, \alpha \in E$, there are

(i) (E, D_r) is a complete metric space,

(ii) $D_r(w+v, h+v) = D_r(w,h)$,

(iii) $D_r(w,h) \leq D_r(w,v) + D(v,h)$,

(iv) $D_r(\alpha w, \alpha h) =\| \alpha \| D_r(w,h)$, (see [23]).

(v) $\| w \| = D_r(w, \tilde{0})$, (see [24]).

(vi) $D_r\left(\int_J w(s)ds, \int_J h(s)ds\right) \leq \int_J D_r(w(s), h(s))ds$,

(vii) $D_r(w \tilde{*} h, \tilde{0}) = D_r(w, \tilde{0}) D_r(h, \tilde{0})$ with the fuzzy multiplication $\tilde{*}$ is based on the extension principle that can be proved by α -cuts of fuzzy numbers $w, h \in E$. Here $\tilde{0} \in E$ is defined by (see [25])

$$\tilde{0}(t) = \begin{cases} 1, s = 0, \\ 0, elsewhere. \end{cases}$$

3. Homotopy Analysis Method

To illustrate the basic idea of HAM, consider the following differential equation

$$N[w(x)] = 0, \tag{3}$$

where N is a nonlinear operator and $w(x)$ is an unknown function.

The following homotopy can be constructed when $N[w(x)] = 0$.

$$\begin{aligned}&(1-q)L[\varphi(x;q) - w_0(x)] - qhH_1(x)N[\varphi(x;q)] \\ &= H[\varphi(x;q), w_0(x), H_1(x), h, q],\end{aligned} \tag{4}$$

where $w_0(x)$ is the initial conjecture of the exact solution $w(x)$, h, H_1 are auxiliary parameters and auxiliary functions, respectively. L is an auxiliary linear operator. When $w(x) = 0, L[w(x)] = 0$, $q \in [0,1]$ is the embedding parameter.

Let homotopy Eq. (4) be zero, that is

$$H[\varphi(x;q), w_0(x), H_1(x), h, q] = 0. \tag{5}$$

The Zero-order deformation equation from Eq.(4)and Eq.(5)

$$(1-q)L[\varphi(x;q) - w_0(x)] = qhH_1(x)N[\varphi(x;q)], \tag{6}$$

when $q = 0$, from Eq.(6)

$$\varphi(x;0) = w_0(x), \tag{7}$$

and when $q = 1$, due to $h \neq 0, H_1 \neq 0$, the Eq.(6) is equivalent to

$$\varphi(x;1) = w(x). \tag{8}$$

It can be obtained from Eq.(7) and Eq.(8) that as the embedding parameter q increases from 0 to 1, $\varphi(x;q)$ changes continuously from the initial conjecture $w_0(x)$ to the exact solution, which is called homotopy deformation.

Using Taylor 's theorem, $\varphi(x;q)$ is expanded to the following power series with respect to q

$$\varphi(x;q) = w_0(x) + \sum_{n=1}^{\infty} w_n(x) q^n, \tag{9}$$

where

$$w_n(x) = \frac{1}{n!} \frac{\partial^n \varphi(x;q)}{\partial q^n}\Big|_{q=0}. \tag{10}$$

By choosing appropriate w_0, h, H_1 , etc.to make $\varphi(x;q)$ converge at $q = 1$, then under these assumptions, we have the following series solution

$$w(x) = \varphi(x;1) = w_0(x) + \sum_{n=1}^{\infty} w_n(x). \tag{11}$$

Substituting Eq.(9) into Eq.(6) yields

$$\begin{aligned}&(1-q)L[\varphi(x;q) - w_0(x)] \\ &= (1-q)L[\sum_{n=1}^{\infty} w_n(x) q^n] \\ &= qhH_1(x)N[\varphi(x;q)],\end{aligned} \tag{12}$$

Simplifying Eq.(12) yields

$$L[\sum_{n=1}^{\infty} w_n(x)q^n] - qL[\sum_{n=1}^{\infty} w_n(x)q^n] = qhH_1(x)N[\varphi(x;q)], \quad (13)$$

The n-th differential of Eq(13) with respect to q is calculated and assigned at $q=0$.

$$\left\{L[\sum_{n=1}^{\infty} w_n(x)q^n] - qL[\sum_{n=1}^{\infty} w_n(x)q^n]\right\}^n = qhH_1(x)N[\varphi(x;q)]^n = n!L[w_n(x) - w_{n-1}(x) = hH_1(x)n\frac{\partial^{n-1}\varphi(x;q)}{\partial q^{n-1}}\Big|_{q=0}. \quad (14)$$

Therefore

$$L[w_n(x) - \chi_n w_{n-1}(x)] = hH_1(x)\xi_n(\vec{w}_{n-1}(x)), \quad (15)$$

where

$$\xi_n(\vec{w}_{n-1}(x)) = \frac{1}{(n-1)!}\frac{\partial^{n-1}N[\varphi(x;q)]}{\partial q^{n-1}}\Big|_{q=0}, \quad \chi_n = \begin{cases}0, n \le 1,\\ 1, n > 1.\end{cases} \quad (16)$$

Application of HAM to fuzzy fractional Volterra-Fredholm integro-differential equations.

$$N[\varphi(x,r;q)] = {}^cD^\gamma\varphi(x,r;q) - g(x,r) - a(x)\varphi(x,r;q) - \int_0^x k_1(x,t)F_1(\varphi(t,r;q))dt - \int_0^1 k_2(x,t)F_2(\varphi(t,r;q))dt, \quad (17)$$

where $\varphi(x,r;q) = [\underline{\varphi}(x,r;q), \overline{\varphi}(x,r;q)], w_0(x,r) = [\underline{w_0}(x,r), \overline{w_0}(x,r)]$ Construct the following zero-order deformation equation

$$(1-q){}^cD^\gamma[\varphi(x,r;q) - w_0(x,r)] = qhN[\varphi(x,r;q], \quad (18)$$

And meet the following initial conditions

$$w_0(x,r) = \varphi(0,r;q) = w(0,r) \quad (19)$$

The next steps are similar to (3)-(19), and now by selecting the appropriate parameters, we can make converge $\varphi(x,r;q)$ to $q=1$, and obtain the following series solution

$${}^cD^\gamma[w_n(x,r) - \chi_n w_{n-1}(x,r)] = h\xi_n(\vec{w}_{n-1}(x,r)), \quad (20)$$

where

$$\xi_n(\vec{w}_{n-1}(x,r)) = \frac{1}{(n-1)!}\frac{\partial^{n-1}N[\varphi(x;q)]}{\partial q^{n-1}}\Big|_{q=0} = {}^cD^\gamma w_{n-1}(x,r) - a(x)w_{n-1}(x,r) - (1-\chi_n)g(x,r) - \int_0^x k_1(x,t)F_1(w_{n-1}(t,r))dt - \int_0^1 k_2(x,t)F_2(w_{n-1}(t,r))dt. \quad (21)$$

Acting on operator J^γ on both sides of Eq. (20), there is

$$w_n(x,r) = (\chi_n + h)w_{n-1}(x,r) - hJ^\gamma[a(x)w_{n-1}(x,r) + (1-\chi_n)g(x,r) + \int_0^x k_1(x,t)F_1(w_{n-1}(t,r))dt + \int_0^1 k_2(x,t)F_2(w_{n-1}(t,r))dt].$$

4. Existence and Uniqueness of the Solution

In this section, we will give the existence and uniqueness results of the solutions of Eq. (1), and prove it. The following assumptions are given before proof.

H(1) $g(x,r), a(x):[0,1] \to E$, assume that g(x), f(x) are continuous functions, and write $\zeta = \sup_{x\in[0,1]}|a(x)|$.

H(2) There exist $L_1 > 0, L_2 > 0,$ such that for any $w_1(x,r), w_2(x,r) \in C_F[0,1]$ satisfying.

$$D(F_1(w_1(x,r)), F_1(w_2(x,r))) \le L_1 D(w_1(x,r), w_2(x,r)),$$
$$D(F_2(w_1(x,r)), F_2(w_2(x,r))) \le L_2 D(w_1(x,r), w_2(x,r)).$$

H(3) There exist two continuous functions $M_1 > 0, M_2 > 0$ satisfying

$$M_1 = \sup_{x\in[0,1]}\int_0^x |k_1(x,t)|dt < \infty, M_2 = \sup_{x\in[0,1]}\int_0^1 |k_1(x,t)|dt < \infty.$$

Next, the uniqueness of the solution of the equation will be proved using Balach's fixed point theorem.

Theorem 4.1. Suppose that the hypotheses H(1)-H(3) hold, if

$$\frac{\zeta + L_1M_1 + L_2M_2}{\Gamma(\gamma+1)} < 1,$$

then Eqs. (1)-(2) has a unique solution $w(x,r) \in$. $C_F[0,1]$.

Proof. Let the definition of operator $T: C_F[0,1] \to C_F[0,1], \forall w_1, w_2 \in E, \forall x \in [0,1]$

$$(Tw_i)(x,r) = w(0,r) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}g(s,r)ds + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}a(s)w_i(s,r)ds + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_i(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w_i(\tau,r))d\tau)ds, (i=1,2).$$

cording to the definition 2.3 and assumptions H(1)-H (3), there is

$$\begin{aligned}
&D(Tw_1(x,r), Tw_2(x,r)) \\
&= D(w(0,r) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}g(s,r)ds + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}a(s)w_1(s,r)ds \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_1(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w_1(\tau,r))d\tau)ds, w(0,r) \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}g(s,r)ds + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}a(s)w_2(s,r)ds \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_2(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w_2(\tau,r))d\tau)ds) \\
&\le D(\frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}a(s)w_1(s,r)ds, \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}a(s)w_2(s,r)ds) \\
&+ D(\frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_1(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w_1(\tau,r))d\tau)ds, \\
&\frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_2(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w_2(\tau,r))d\tau)ds) \\
&\le \frac{1}{\Gamma(\gamma)}\int_0^x |(x-s)^{\gamma-1}| \sup_{s\in[0,1]}|a(s)| D(w_1(s,r), w_2(s,r))ds \\
&+ \frac{1}{\Gamma(\gamma)}\int_0^x |(x-s)^{\gamma-1}| D(\int_0^s k_1(s,\tau)F_1(w_1(\tau,r))d\tau, \int_0^s k_1(s,\tau)F_1(w_2(\tau,r))d\tau)ds \\
&\frac{1}{\Gamma(\gamma)}\int_0^x |(x-s)^{\gamma-1}| D(\int_0^1 k_2(s,\tau)F_2(w_1(\tau,r))d\tau, \int_0^1 k_2(s,\tau)F_2(w_2(\tau,r))d\tau)ds
\end{aligned}$$

$$\leq \frac{\zeta}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(w_1(s,r), w_2(\tau,r))ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(\int_0^s k_1(s,\tau)F_1(w_1(\tau,r))d\tau)ds,$$

$$\int_0^s k_1(s,\tau)F_1(w_2(\tau,r))d\tau)ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(\int_0^1 k_2(s,\tau)F_2(w_1(\tau,r))d\tau,$$

$$\int_0^1 k_2(s,\tau)F_2(w_2(\tau,r))d\tau)ds$$

$$:= \phi_1 + \phi_2 + \phi_3.$$

where

$$\phi_1 = \frac{\zeta}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(w_1(s,r), w_2(\tau,r))ds \leq \frac{\zeta}{\Gamma(\gamma+1)} D(w_1(s,r), w_2(\tau,r).$$

$$\phi_2 \leq \frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| \left|\int_0^s |k_1(s,\tau)| d\tau D(F_1(w_1(\tau,r)), F_1(w_2(\tau,r)))ds\right.$$

$$\leq \frac{P_1 Q_1}{\Gamma(\gamma+1)} D(w_1(s,r), w_2(\tau,r).$$

$$\phi_3 \leq \frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| \left|\int_0^1 |k_2(s,\tau)| d\tau D(F_2(w_1(\tau,r)), F_2(w_2(\tau,r)))ds\right.$$

$$\leq \frac{P_2 Q_2}{\Gamma(\gamma+1)} D(w_1(s,r), w_2(\tau,r).$$

Substitute ϕ_1, ϕ_2 and ϕ_3 for it

$$D(Tw_1(x,r), Tw_2(x,r))$$

$$\leq \frac{\zeta + L_1 M_1 + L_2 M_2}{\Gamma(\gamma+1)} D(w_1(s,r), w_2(\tau,r),$$

The above equation shows that T is a compressive map, and the equation has a unique solution by Banach's fixed point theorem.

5. Numerical Results

Example1 Consider the application of HAM to solve the following linear fuzzy fractional Volterra-Fredholm integro-differential equations

$$\begin{cases} \begin{cases} {}^cD^{1/2}[\underline{w}(x,r)] = \dfrac{rx^{1/2}}{6\Gamma(1/2)} - \dfrac{rx^3}{72} - \dfrac{x^4}{3}\underline{w}(x,r) \\ \qquad + \displaystyle\int_0^x x^2 t \underline{w}(t,r)dt + \int_0^1 x^3(1-t)\underline{w}(t,r)dt, \\ \underline{w}(0,r) = 0, \end{cases} \\ \begin{cases} {}^cD^{1/2}[\overline{w}(x,r)] = \dfrac{(2-r)x^{1/2}}{6\Gamma(1/2)} - \dfrac{(2-r)x^3}{72} - \dfrac{x^4}{3}\overline{w}(x,r) \\ \qquad + \displaystyle\int_0^x x^2 t \overline{w}(t,r)dt + \int_0^1 x^3(1-t)\overline{w}(t,r)dt, \\ \overline{w}(0,r) = 0. \end{cases} \end{cases}$$

The exact solution is

$$w(x,r) = [\underline{w}(x,r), \overline{w}(x,r)] = [\frac{r}{12}x, \frac{2-r}{12}x], r \in [0,1]$$

It can be written from the recursive

$$\underline{w}_n(x,r) = (\chi_n + h)\underline{w}_{n-1}(x,r) - hJ^\gamma[-\frac{x^4}{3}\underline{w}_{n-1}(x,r)$$

$$+(1-\chi_n)(\frac{rx^{1/2}}{6\Gamma(1/2)} - \frac{rx^3}{72})$$

$$+\int_0^x x^2 t \underline{w}_{n-1}(t,r)dt + \int_0^1 x^3(1-t)\underline{w}_{n-1}(t,r)dt],$$

$$\overline{w}_n(x,r) = (\chi_n + h)\overline{w}_{n-1}(x,r) - hJ^\gamma[-\frac{x^4}{3}\overline{w}_{n-1}(x,r)$$

$$+(1-\chi_n)(\frac{(2-r)x^{1/2}}{6\Gamma(1/2)} - \frac{(2-r)x^3}{72})$$

$$+\int_0^x x^2 t \overline{w}_{n-1}(t,r)dt + \int_0^1 x^3(1-t)\overline{w}_{n-1}(t,r)dt].$$

Select $h = -1, r = 0.4, n = 3$. The left and right boundary errors between the approximate solution and the exact solution are obtained in Table 1 and Table 2.

Table 1. left bound of error

x	exact	HAM	error
0	0	0	0
0.2	6.6667e-03	6.6665e-03	2.1330e-07
0.4	1.3333e-02	1.3331e-02	2.3185e-06
0.6	2.0000e-02	1.9993e-02	7.4545e-06
0.8	2.6667e-02	2.6665e-02	1.9953e-06
1.0	3.3333e-02	3.3425e-02	9.1408e-05

Table 2. right left bound of error

x	exact	HAM	error
0	0	0	0
0.2	2.6667e-02	2.6666e-02	8.5319e-07
0.4	5.3333e-02	5.3324e-02	9.2740e-06
0.6	8.0000e-02	7.9970e-02	2.9818e-05
0.8	1.0667e-01	1.0666e-01	7.9813e-06
1.0	1.3333e-01	1.3370e-01	3.6563e-04

Example 2 Consider the following nonlinear fuzzy fractions: Waltra-Fredholm integral differential equations

$$\begin{cases} \begin{cases} {}^cD^{3/4}[\underline{w}(x,r)] = \dfrac{4rx^{1/4}}{5\Gamma(1/4)} - \dfrac{r^2x^2}{100} - \dfrac{rx^3}{15}\underline{w}(x,r) \\ \qquad + \displaystyle\int_0^x x\underline{w}^2(t,r)dt + \int_0^1 x^2 t\underline{w}^2(t,r)dt, \\ \underline{w}(0,r) = 0, \end{cases} \\ \begin{cases} {}^cD^{3/4}[\overline{w}(x,r)] = \dfrac{4(2-r)x^{1/4}}{5\Gamma(1/4)} - \dfrac{(2-r)^2x^2}{100} \\ \qquad - \dfrac{(2-r)x^3}{15}\overline{w}(x,r) + \displaystyle\int_0^x x\overline{w}^2(t,r)dt + \int_0^1 x^2 t\overline{w}^2(t,r)dt, \\ \overline{w}(0,r) = 0. \end{cases} \end{cases}$$

The exact solution is

$$w(x,r) = [\underline{w}(x,r), \overline{w}(x,r)] = [\frac{r}{5}x, \frac{2-r}{5}x], r \in [0,1]$$

\

It can be written from the recursive

$$\underline{w}_n(x,r)=(\chi_n+h)\underline{w}_{n-1}(x,r)-hJ^{\gamma}[-\frac{rx^3}{15}\underline{w}_{n-1}(x,r)$$

$$+(1-\chi_n)(\frac{4rx^{1/4}}{5\Gamma(1/4)}-\frac{r^2x^2}{100})$$

$$+\int_0^x x\underline{w}^2{}_{n-1}(t,r)dt+\int_0^1 x^2t\underline{w}^2{}_{n-1}(t,r)dt],$$

$$\overline{w}_n(x,r)=(\chi_n+h)\overline{w}_{n-1}(x,r)-hJ^{\gamma}[-\frac{(2-r)x^3}{15}\overline{w}_{n-1}(x,r)$$

$$+(1-\chi_n)(\frac{4(2-r)x^{1/4}}{5\Gamma(1/4)}-\frac{(2-r)^2x^2}{100})$$

$$+\int_0^x x\overline{w}^2{}_{n-1}(t,r)dt+\int_0^1 x^2t\overline{w}^2{}_{n-1}(t,r)dt].$$

Select $h=-1, r=0.5, n=3$. The left and right boundary errors between the approximate solution and the exact solution are obtained in Table 31 and Table 4.

Table 3. left bound of error

x	exact	HAM	error
0	0	0	0
0.2	2.0000e-02	2.0000e-02	2.1184e-07
0.4	4.0000e-02	3.9999e-02	1.4309e-06
0.6	6.0000e-02	5.9996e-02	4.4309e-06
0.8	8.0000e-02	7.9990e-02	1.0140e-05
1.0	1.0000e-01	9.9980e-02	2.0064e-05

Table 4. right left bound of error

x	exact	HAM	error
0	0	0	0
0.2	6.0000e-02	5.9994e-02	5.6700e-06
0.4	1.2000e-01	1.1996e-01	3.8297e-05
0.6	1.8000e-01	1.7988e-01	1.1858e-04
0.8	2.4000e-01	2.3973e-01	2.7120e-04
1.0	3.0000e-01	2.9946e-01	5.3540e-04

From the error results of Table 1-Table 4, the error between the exact solution and the approximate solution is small, so HAM is effective and practical for dealing with linear and nonlinear fuzzy fractional integral and differential problems.

6. Conclusion

In this paper, a class of fuzzy fractional order mixed integro-differential equations is studied. The fixed point theorem is used to prove the existence and uniqueness of the solution. The appropriate auxiliary parameters and initial conditions are selected to solve the fuzzy fractional Volterra-Fredholm integro-differential equation by using the method of homotopy analysis. The validity and applicability of the method can be obtained by several numerical examples and their results. In the future, the fuzzy nonlinear fractional Volterra-Fredholm integral-differential will be studied. Based on the HAM, the nonlinear term will be approximated and its stability will be studied.

References

[1] A. Carpinteri, F. Mainardi. Fractals and fractional calculus in continuum me- chanics, Springer, Vienna, 378(1997), 277-290.

[2] D. Baleanu, K. Diethelm, E. Scalas, J. Trujillo. Fractional calculus: models and numerical methods, Series on Complexity, Nonlinearity and Chaos, 3(2012), 39- 133.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory and applications of frac- tional differential equations. Elsevier, Amsterdam, 204(2006), 1-523.

[4] N. H. Abel. Solution de quelques problemes a l´aide d´ıntegrales definites, Christiania Grondahl, Norway, 1(1881), 16-18.

[5] A. A. Hamoud, I. Y. A. Alasadi. Existence and stability results for fractional Volterra-Fredholm integro-dierential equation with mixed conditions, Advances in Dynamical Systems and Applications, 16(2021), 217-236.

[6] S. Andr´as. Weakly singular Volterra and Fredholm-Volterra integral equations, Studia Universitatis Babes-Bolyai. Mathematica, 48(2003), 147-155.

[7] S. Abbas, M. Benchohra. Fractional order Riemann-Liouville integral equations with multiple time delays, Applied Mathematics E-Notes [electronic only], 12(2012), 79-87.

[8] A. Atangana, N. Bildik. Existence and numerical solution of the Volterra fractional integral equations of the second kind, Mathematical Problems in Engineering, 2013(2013), 1-11.

[9] V. Laksmikantam. Theory of fractional functional differential equations, Non- linear Analysis: Theory, Methods and Applications, 69(2008), 3337-3343.

[10] S. S. L. Chang, L. A. Zadeh. On fuzzy mapping and control, IEEE Transactions on Systems, Man, and Cybernetics, 2(1972), 30-34.

[11] L. A. Zadeh. The concept of linguistic variable and its application to approximate reasoning. Information Sciences, 8 (1975), 199-249.

[12] [M. Mizumoto, K. Tanaka, The four operations of arithmetic on fuzzy numbers, Systems Comput Controls, 7(1976), 73-81.

[13] D. Dubois, H. Prade. Towards fuzzy differential calculus, Part I: integration of fuzzy mappings, class of second-order. Fuzzy Sets and Systems, 8(1982), 1-17.

[14] M. Puri, D. Ralescu. Differentials of fuzzy functions. Journal of Mathematical Analysis and Applications, 91 (1983), 552-558.

[15] B. Bede, S. G. Gal. Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equations. Fuzzy Sets and Systems, 151(2005), 581-599.

[16] S. J. Liao. On the homotopy analysis method for nonlinear problem, Applied Mathematics and Computation, 147(2004), 499-513.

[17] A. A. Hamoud, K. P. Ghadle. Homotopy analysis method for the first order fuzzy Volterra-Fredholm integro-differential equations. Indonesian Journal of Electrical Engineering and Computer Science, 11(2018). 857-867.

[18] E. Hussain, A. Ali. Homotopy analysis method for solving fuzzy integro- differential equations, Modern Applied Science, 7 (2013), 15-25.

[19] I. K. Hanan. The Homotopy analysis method for solving multi-fractional orderintegro-differential equations, Journal of Al-Nahrain University Science, 14(2011), 139-143.

[20] A. A. Hamoud, K. P. Ghadle, S. M. Atshan. The approximate solutions offractional integro-differential equations by using modified Adomian decomposition method. Khayyam Journal of Mathematics, 5(2019), 21-39.

[21] M. SH. Bani Issa, A. A. Hamoud, K. P. Ghadle. Numerical solutions of fuzzy integro-differential equations of the second kind. Journal of Mathematics and Computer Science, 23(2021), 67-74.

[22] S. Salahshour, A. Ahmadian, C. S. Chan. A novel technique for solving fuzzy differential equations of fractional order using laplace and integral transforms, 2016 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), 2016, 1473-1477.

[23] S. Salahshour, T. Allahviranloo, S. Abbasbandy. Solving fuzzy fractional differ- ential equations by fuzzy Laplace transforms, Commun Nonlinear Sci Numer Simulat, 17(2012), 1372-1381.

[24] H. Zureigat, A. I. Ismail, S. Sathasivam. Numerical solutions of fuzzy fractional diffusion equations by an implicit finite difference scheme, Neural Computing and Applications, 31 (2018), 4085-4094.

[25] A. A. Hamoud, A. D. Azeez, K. P. Ghadle. A study of some iterative methods for solving fuzzy Volterra-Fredholm integral equations. Indonesian Journal of Electrical Engineering and Computer Science, 11(2018), 1228-1235.

[26] S. Arshad, V. Lupulescu. On the fractional differential equations with uncer- tainty, Nonlinear Analysis: Theory, Methods and Applications, 74(2011), 85-93.

[27] S. Arshad, V. Luplescu. Fractional differential equation with fuzzy initial conditon, Electronic Journal of Differential Equations, 34(2011), 1-8.

[28] N. Ahmad, A. Ullah, A. Ullah, S. Ahmad, K. Shah , I. Ahmad. On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation. Alexandria Engineering Journal, 60(2020), 1827-1838.

[29] T. Allahviranloo, A. Armand, Z. Gouyandeh, H. Ghadiri. Existence and unique- ness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equa- tions. Journal of Fuzzy Set Valued Analysis, 2013(2013), 1-9.

[30] A. Armand, Z. Gouyandeh. Fuzzy fractional integro-differential equations under generalized Capotu differentiability, Annals of Fuzzy Mathematics and Infor- matics, 10(2015), 789-798.

[31] S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu, P. Agarwal. On analytical solutions of the fractional differential equation with uncertinity: Application to the basset problem, Entropy, 17(2015), 855-902.

[32] D. Dubois, H. Prade. Operations on fuzzy numbers. International Journal of Systems Science, 9(1978), 613-626.

[33] S. Hajighasemi. Fuzzy Fredholm-Volterra integral equations and existance and uniqueness of solution of them, Australian Journal of Basice and Applied Sci- ence, 5(2011), 1-8.

[34] B. Bede, L. Stefanini. Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems, In press, 230(2013), 119-141.

[35] T. Allahviranloo, S. Salahshour, S. Abbasbandy. Explicit solutions of fractional differential equations with uncertainty. Soft Comput, 16(2012), 297-302.

[36] R. L. Pouso. Schauder's fixed-point theorem: new applications and a new ver- sion for discontinuous operators. Boundary Value Problems. 2012(2012), 1-14.

[37] M. R. M. Shabestari, R. Ezzati, T.Allahviranloo . Numerical solution of fuzzy fractionalintegro-differential equation via two-dimensional Legendre Wavelet method, Journal of Intelligent and Fuzzy Systems, 34(2018), 2453-2465.