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The Mean Capture Time on Horizontal Divided Nested Networks

Zuodong Xiang[1, *]

[1] School of Mathematical Sciences, Jiangsu University, Zhenjiang, 212013, China
[*] Corresponding author: Zuodong Xiang (Email: zuodongxiang24@163.com)

Abstract: In this paper, we consider the division of nested networks with a horizontal division line l_k, where k is the division coefficient. The problem of capture on the surplus network obtained after the division is studied. In addition, by studying the structure of the surplus network, we obtain the relationship between the transmission efficiency on the surplus network and the division coefficient k. When k is larger, the capture time is shorter and the network transmission efficiency is higher. At the same time, we also solved the mean capture time of each node on the bottom edge of the surplus network.

Keywords: Nested network; horizontal division line; division coefficient; mean capture time.

1. Introduction

In recent years, complex networks have been studied by scholars in different fields. They have been applied more and more widely, such as sociology, computer science, biology, physics, and so on[1-4]. For example, in power systems, applications of complex network theory include design optimisation of power grids, fault detection and recovery, and power demand forecasting[5-8]. In logistics and supply chain management, complex network theory can be used to optimise the design and management of logistics networks and improve the efficiency of supply chains[9,10]. The study of dynamic processes has gradually become the focus of research, and random walk is the basis and important tool of our research. Many scholars in various fields have conducted in-depth research on random walk[11-14]. As we know, the mean capture time (MCT) as an important index obtained through the random walk model on the network. MCT can be defined as average of the mean first passage time (MFPT) of all nodes on the network of the wanderer to reach the trap node. MFPT is defined as the average time for a walker to reach the trap node for the first time from the node.

Complex networks are networks with some or all of the properties in self-similarity, self-organisation, attractors, small worlds, and scale-free. Therefore, fractal networks with the important property of self-similarity have attracted a great deal of attention in various fields, such as flower network[15,16], Koch curve[17,18], Sierpinski Gasket triangle network[19], Vicsek fractal[20] and so on. Among the many fractal networks, Sierpinski Gasket network is one of the most popular network models. Many existing research papers have explored the MCT on Sierpinski Gasket network and its derivative network. Wu et al.[21,22] studied the MCT on three-level Sierpinski gasket network and half-Sierpinski gasket. Zhang et al.[23,24] studied the MCT on three-dimensional horizontally segmented Sierpinski gasket network and three-dimensional Sierpinski gasket gasket network. Hu et al.[25] studied the MCT of a three-level horizontally segmented Sierpinski gasket network. Sun et al.[26] studied the MCT on a network extended by a Sierpinski gasket. Han et al.[27] studied the MCT when performing non-nearest-neighbour jumps on a nested network derivative by a Sierpinski gasket network. On the basis of nested networks, we consider a horizontal division of them to obtain a surplus network model (SSN). Solve the analytical expression of MCT on the SSN, which contains the number of iterations and the division coefficients. At the same time, we analysed the effect of changes in division coefficients on the efficiency of the random walk. In addition, we solve the MCT when each node on the bottom edge of the SSN is used as a trap node.

The rest of the paper is as follows: in Section 2, we present the method for constructing SSN and auxiliary networks (ASN) by nested networks. In Section 3, we solve the exact expression for the MCT on the SSN. In Section 4, we solve for the MCT of each node on the bottom edge of the SSN. Finally, in Section 5, we conclude the full paper.

2. Preliminaries

In this section, we provide a brief description of the nested network model and introduce the definition and construction of the horizontal division line l_k, the surplus network SSN and the auxiliary network (ASN).

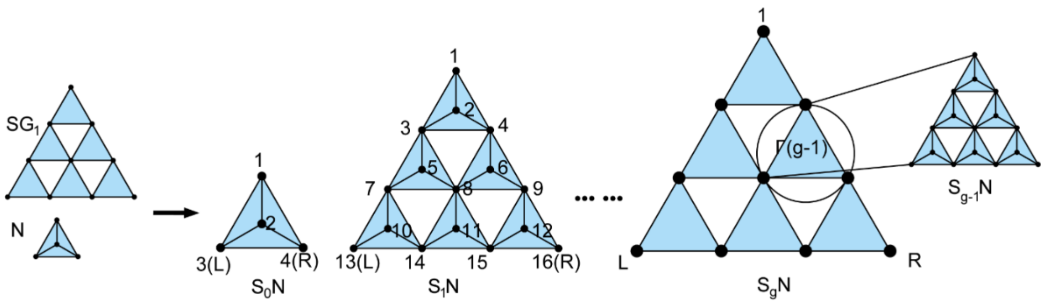

Figure 1. The structure of S_gN.

2.1. Structure of S_gN

S_gN is an extension of the Sierpinski Gasket, also with self-similarity. First, we denote the equilateral triangle and the three-regular graph as S and N, respectively. where S is a structural graph of S by taking a trisection on each side of the triangle, and connecting all pairs of trisections parallel to the initial side. Then, six copies of the three-regular graph N are embedded into S to form the 1st generation network, denoted as S_1N. When $n = 2$, the six $S_{g-1}N$ are embedded into S to form S_gN, as shown in figure 1.

According to the construction method of S_gN, we can divide S_gN into six parts, denoted as $\Gamma_i(g-1)$, $i = (1,2,\dots,6)$. We label the nodes on the network in the top-to-bottom order, and the three corner nodes are denoted as denoted as 1, L, and R. Where each region consists of many S_0N of the 0th generation, and we denote the number of S_0N of the gth generation as Δ^g. It is easy to get the number of S_0N as 6^g. By the structure of the nested network, we denote the set of all nodes as N(g) and the set of all edges as E(g), can be calculated.

$$\begin{cases} |N(g)| = \dfrac{12}{5}\cdot 6^g + \dfrac{8}{5}, \\ |E(g)| = 6^{g+1}. \end{cases} \tag{1}$$

Here we use the symbol $|\cdot|$ for the number of elements in the set.

2.2. Structures of $SSN(g,k)$ and $ASN(g,n)$

First, we define the horizontal division line l_1 as the horizontal line where nodes 7, 8 and 9 are located (at $\frac{1}{3}$ of the line segment between corner nodes 1 and L). The network S_gN is divided into two parts by the horizontal line l_1, preserving the lower network of the division line. Notice that the lower network preserves the points on the division line. We call the lower network the surplus network notated as $SSN(g,1)$. Similarly, the second horizontal division line l_2 is defined as the line segment identified through corner nodes 1 and L at $\frac{1}{3^2}$ (near node L) and parallel to l_1. The retained surplus network is therefore denoted as $SSN(g,2)$. Based on the above surplus network construction method, we denote the division coefficients by the integration number k. When the network is divided using the division line l_k, $SSN(g,k)$ is obtained as shown in Figure 2. Note that the division coefficients here need to satisfy: $1 \le k \le g$. It is easy to see that $SSN(g,k)$ consists of 3^k $S_gN(g-k)$.

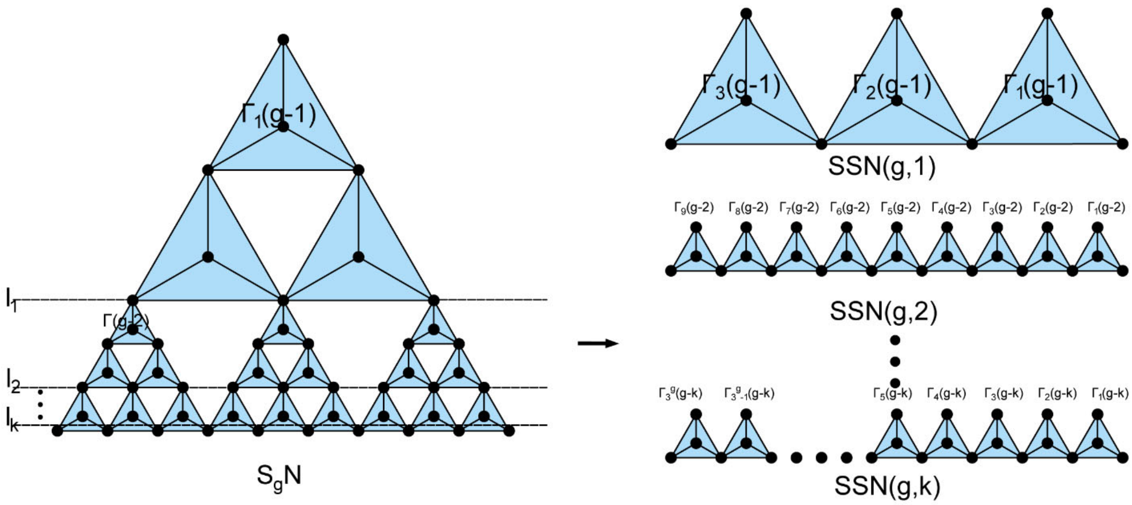

Figure 2. The structure of $SSN(g,k)$.

Therefore, we give here the definition of the auxiliary network $ASN(g,n)$, that is, n S_gN can be obtained by connecting them in series in order to obtain $ASN(g,n)$.

For a clearer representation, we denote these n regions as $\Gamma_1(g)$, $\Gamma_2(g)$,…, and $\Gamma_n(g)$ from right to left. For any region $\Gamma_j(g)(1 \le j \le n)$, we denote the corner nodes as (A,j), (B,j), (C,j), as shown in Figure 3. Note that since some of the nodes connect two regions, there exists (B,i) = (C,i+1), where $1 \le i \le n$.

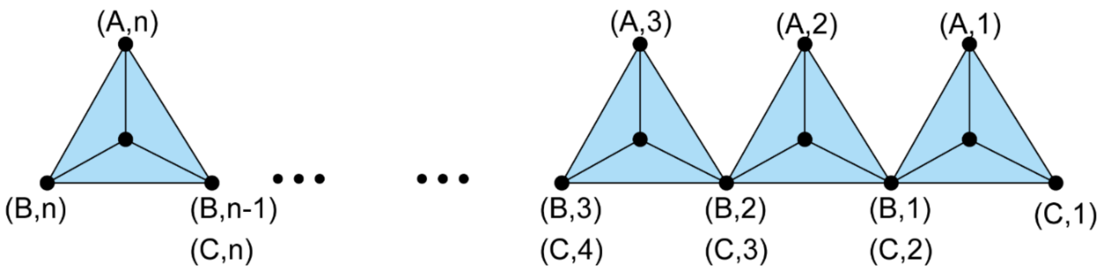

Figure 3. The structure of $ASN(g,n)$.

Then, similar to the definitions of the set of nodes and the set of edges given earlier. We denote the set of all nodes and the set of all edges of $\mathrm{SSN}(g,k)$ and $\mathrm{ASN}(g,n)$ as $N^S(g,k)$, $N^A(g,n)$ and $E^S(g,k)$, $E^A(g,n)$ respectively. Based on the structure of each of the two networks, we can obtain:

$$\begin{cases} |N^A(g,n)| = & n\cdot N(g)-(n-1)=\dfrac{12}{5}n\cdot 6^g+\dfrac{3}{5}n+1, \\ |E^A(g,n)| = & n\cdot E(g)=n\cdot 6^{g+1}; \end{cases} \tag{2}$$

$$\begin{cases} |N^S(g,k)| = & 3^k\cdot N(g-k)-(3^k-1)=\dfrac{3^k}{5}(12\cdot 6^{g-k}+3)+1, \\ |E^S(g,k)| = & E^A(g-k,3^k)=3^k\cdot 6^{g-k+1}. \end{cases} \tag{3}$$

3. Analytic Expression for The Mean Capture Time on Network $\mathrm{SSN}(g,k)$

In this section, we will first compute the MCT on the network S_gN with corner node L as the trap node. Then, compute the MCT on $\mathrm{ASN}(g,n)$ when node (C,1) is used as a trap node. Finally, the equivalence between the network $\mathrm{ASN}(g,n)$ and $\mathrm{SSN}(g,k)$ is obtained through the network $\mathrm{ASN}(g,n)$. This results in an analytic expression for the MCT of $\mathrm{SSN}(g,k)$.

3.1. MCT on the S_gN

In this section, we will fix the trap node as the corner node L, thus considering the MCT on S_gN. We consider unbiased Markov random roaming on the network. During the roaming, the directions are random and the step sizes are equal. First, the random walker starts from any non-trap node. It wanders along the network only one unit step at a time until it is captured by a trap node and the wandering ends. The transfer probability at each roam is the inverse of the degree of that node. Here, we denote the degree of node u as d_u. Denoting the transfer probability from node u to v as $p_{u,v}$, we obtain:

$$p_{u,v}=\begin{cases} \dfrac{1}{d_u}, & if\ u\sim v\ and\ u\neq 2 \\ 0, & others. \end{cases}$$

Then, we define the mean first passage time (MFPT), that is, the mean time for a random wanderer to reach a trap node v for the first time from any node u is denoted as $T_{u,v}$. When node u is not a trap node and there exists a trap node α, the MPFT from node u to the trap node, is denoted as $T_u=T_{u,\alpha}$. In particular, if the initial node is the same as the trap node, the we have $T_{u,u}=0$. Denoting T_{sum} as the sum of the MFPT of the wanderers starting from all nodes, we obtain the following equation:

$$T_{sum}=\sum_{u\in N(g)}T_u=\sum_{u\in N(g)}T_{u,\alpha}. \tag{4}$$

From this, we obtain the MCT on S_gN, denoted $\bar{T}_{sum}$, which can be obtained:

$$\bar{T}_{sum}=\frac{T_{sum}}{|N(g)|-1}. \tag{5}$$

According to previous scholars we can get a MCT on S_gN containing $\lambda(q)=\frac{7(8-7q)}{56+79q}\in[\frac{7}{135},1]$ [27]:

$$\begin{aligned} T_{sum}(g)=&\frac{611\lambda(q)}{640}(\frac{135}{7})^g\cdot 6^{g+1} \\ &+\frac{3993\lambda(q)}{155}(\frac{135}{7})^g-\frac{4011}{1984}\frac{135}{7}\lambda(q)\cdot 6^g \\ &+\frac{11}{4}\cdot 6^{g+1}. \end{aligned}$$

$$\begin{aligned} \bar{T}_{sum}(g)=&\frac{1}{3968(12\cdot 6^g+3)} \\ &\cdot[18941\lambda(q)(\frac{135}{7})^g 6^{g+1}+511104\lambda(q)(\frac{135}{7})^g \\ &+5(10912-25785\lambda(q))6^{g+1}]. \end{aligned}$$

When we make $q=0$ that is, $\lambda(q)=1$, what is going on at this point is what we define as a random wandering, and we can get:

$$\begin{aligned} T_{sum}(g)=&\frac{611}{640}(\frac{135}{7})^g\cdot 6^{g+1} \\ &+\frac{3993}{155}(\frac{135}{7})^g-\frac{77355}{1984}6^g+\frac{11}{4}\cdot 6^{g+1}. \end{aligned} \tag{6}$$

$$\begin{aligned} \bar{T}_{sum}(g)=&\frac{1}{3968(12\cdot 6^g+3)}[18941(\frac{135}{7})^g 6^{g+1} \\ &+511104(\frac{135}{7})^g-74365\cdot 6^{g+1}]. \end{aligned} \tag{7}$$

3.2. MCT on the $\mathrm{ASN}(g,n)$

The above discussion of MCT on S_gN are based on when corner nodes L are used as trap nodes, but are not sufficient to support the computation of MCT on $\mathrm{ASN}(g,n)$. We also need to obtain parsing expressions for MCT when two and three corner nodes are used as traps, which can be used to compute parsing expressions for MCT on the network $\mathrm{ASN}(g,n)$.

Here, we use the corner nodes L and R as trap nodes on the network S_gN. At this point any node u reaches the MFPT of the absorbing node, and the total capture time and MCT are denoted as $T_u^2(g)$, $T_{sum}^2(g)$, and $\bar{T}_{sum}^2(g)$, respectively. Similarly, we use corner nodes 1, L and R as trap nodes on the network S_gN. At this point any node u reaches the MFPT of the absorbing node, and the total capture time and MCT are denoted as $T_u^3(g)$, $T_{sum}^3(g)$, and $\bar{T}_{sum}^3(g)$, respectively.

Based on the symmetry of the nested network S_gN., we can obtain the following relation:

$$T_1^2(g)=\frac{135}{7}T'(g). \tag{8}$$

$$T_{sum}^2(g)=\frac{71}{128}(\frac{135}{7})^g 6^{g+1}+\frac{749}{31}(\frac{135}{7})^g -\frac{44619}{1984}6^g. \quad (9)$$

From this, we can obtain the MCT when there are two trap nodes on S_gN, that is:

$$\bar{T}_{sum}^2(g)=\frac{T_{sum}^2(g)}{|N(g)|-2} =\frac{\frac{355}{128}(\frac{135}{7})^g 6^{g+1}+\frac{3745}{31}(\frac{135}{7})^g-\frac{223095}{1984}6^g}{12\cdot 6^g-2}.$$

The above conclusions are solved in detail in Appendix A.

Next, we compute the parsing expression of the MCT on the auxiliary network $ASN(g,n)$ with node (C,1) as the capture node. We let the set of all nodes on $\Gamma_j(g)(1\le j\le n)$ be N_A^j. Denote the set of nodes containing only (B,j) and (C,j) as $\hat{N}_A^j$. Furthermore, let $\hat{N}_A^j=\hat{N}_A^1\cup\hat{N}_A^2\dots\cup\hat{N}_A^n$, $\bar{N}_A^j=N_A^j/\hat{N}_A^j$ and $\bar{N}_A=N_A/\hat{N}_A$. We let the MFPT from node (a,b) to node (c,d) on the auxiliary network $ASN(g,n)$ be denoted as $T_{(a,b),(c,d)}^A(g,n)$. Let the sum capture time on the auxiliary network $ASN(g,n)$ be $T_{sum}^A(g,n)$ and the mean capture time be $\bar{T}_{sum}^A(g,n)$.

Based on the above classification of nodes, we divide the path from any node $(i,j)\in\bar{N}_A$ to the capture node (C,1) into two steps:

(1) (i,j) reaches node (B,j) or (C,j) after time $T_i^2(g)$.

(2) Departing from node (B,j) or (C,j) is finally captured at node (C,1).

So, we have the following equation:

$$\begin{aligned}T_{sum}^A(g,n)&=\sum_{(i,j)\in N(g)}T_{(i,j),(C,1)}^A(g,n)\\&=\sum_{j=1}^n\sum_{(i,j)\in\bar{N}_A^j}T_{(i,j),(C,1)}^A(g,n)\\&\quad+\sum_{(i,j)\in\hat{N}_A}T_{(i,j),(C,1)}^A(g,n).\end{aligned}\quad (10)$$

The nodes (B,j) and (C,j) in $\bar{N}_A^j$ are symmetric in the network $\Gamma_j(g)$, so the following equation can be obtained in any $j\in[1,n]$:

$$\begin{aligned}&\sum_{(i,j)\in\bar{N}_A^j}T_{(i,j),(C,1)}^A(g,n)\\&=\sum_{(i,j)\in\bar{N}_A^j}T_{(i,j)}^2(g)\\&+\sum_{(i,j)\in\bar{N}_A^j}[\frac{1}{2}T_{(B,j),(C,1)}^A(g,n)+\frac{1}{2}T_{(C,j),(C,1)}^A(g,n)]\\&=T_{sum}^2(g)+\frac{1}{2}(|N(g)|-2)\\&[T_{(B,j),(C,1)}^A(g,n)+T_{(C,j),(C,1)}^A(g,n)].\end{aligned}\quad (11)$$

So substituting Eq.(11) into Eq.(10) yields the following equation:

$$\begin{aligned}&T_{sum}^A(g,n)\\&=n\cdot T_{sum}^2(g)+\frac{1}{2}(|N(g)|-2)\\&\sum_{j=1}^n[T_{(B,j),(C,1)}^A(g,n)+T_{(C,j),(C,1)}^A(g,n)]+\\&\sum_{(i,j)\in\hat{N}_A}T_{(i,j),(C,1)}^A(g,n).\end{aligned}\quad (12)$$

Since we labelled the nodes (B,j) and (C,j) in the corner nodes (B,j) and (C,j) on the network $ASN(g,n)$ in such a way as to know that (B,j) = (C,j+1), where $1\le j\le n$, we can convert Eq.(12) into the following equation:

$$\begin{aligned}T_{sum}^A(g,n)&=n\cdot T_{sum}^2(g)+(|N(g)|-2)\\&\quad[\sum_{(i,j)\in\hat{N}_A}T_{(i,j),(C,1)}^A(g,n)-\frac{1}{2}T_{(B,n),(C,1)}^A(g,n)]\\&\quad+\sum_{(i,j)\in\hat{N}_A}T_{(i,j),(C,1)}^A(g,n)\\&=n\cdot T_{sum}^2(g)\\&\quad+(|N(g)|-1)\sum_{(i,j)\in\hat{N}_A}T_{(i,j),(C,1)}^A(g,n)\\&\quad-\frac{1}{2}(|N(g)|-2)T_{(B,n),(C,1)}^A(g,n).\end{aligned}\quad (13)$$

Next, we consider the computation $T_{(i,j),(C,1)}^A(g,n)$, $(i,j)\in\hat{N}_A$, where $T_{(C,1),(C,1)}^A(g,n)=0$ when (i,j)=(C,1). So we only need to consider nodes (B,j), $j\in[1,n]$. The random walks we consider are equal probability walks, so a walk starting at node (B,j) $1\le j\le n$ will reach node (B,j-1) or node (B,j+1) for the first time after time $T_{1,L}(g)$ with equal probability. Thus, computing $T_{(i,j),(C,1)}^A(g,n)(i,j)\in\hat{N}_A$ reduces the model to a random walk model on a one-dimensional finite lattice with the node (C,1) as the capture node, denoted $L(n)$, with a length of n and a length of unit length for each edge. We label it from right to left as 0 to n, a total of $n+1$ nodes. Denote the commuting time from any of these nodes a to node b as $T_{a\leftrightarrow b}(n)$. So, by the finite resistance principle, we can get the following equation[28]:

$$T_{a\leftrightarrow b}(n)=T_{a,b}^L(n)+T_{b,a}^L(n)=2nR_{a,b}.$$

where $T_{a,b}^L(n)$ denotes the MFPT from node a to node b on $L(n)$ and $R_{a,b}$ denotes the effective resistance from node a to node b on $L(n)$. So we can easily get $R_{a,b}=b-a$ when $a=0$:

$$T_{0\leftrightarrow b}(n)=T_{0,b}^L(n)+T_{b,0}^L(n)=2nR_{0,b}=2nb.$$

Since $b\le n$, we have: $T_{0,b}^L(n)=T_{0,b}^L(b)=T_{b,0}^L(b)$. From this, we we can obtain:

$$T_{0,b}^L(n)=T_{0,b}^L(b)=\frac{1}{2}T_{0\leftarrow b}(n)=b^2.$$

On this basis, we can get:

$$T_{b,0}^{L}(n)=2nb-T_{0,b}^{L}(n)=2nb-b^2.$$

So from the above results and analyses on the one-dimensional finite lattice, we can get the following relation:

$$T_{(B,n),(C,1)}^{A}(g,n)=T_{n,0}^{L}\cdot T_{1,L}(g)=2n^2(\frac{135}{7})^g. \quad (14)$$

$$\begin{aligned}\sum_{(i,j)\in\hat{N}_A}T_{(i,j),(C,1)}^{A}(g,n)&=\sum_{b=1}^{n}T_{b,0}^{L}(n)\cdot T_{1,L}(g)\\&=2\cdot(\frac{135}{7})^g\sum_{b=1}^{n}(2nb-b^2) \quad (15)\\&=\frac{1}{3}n(n+1)(4n-1)(\frac{135}{7})^g.\end{aligned}$$

Therefore, substituting Eq.(9), Eq.(14) and Eq.(15) into Eq.(13), we get:

$$\begin{aligned}T_{sum}^{A}(g,n)&=(\frac{16}{5}n^3+\frac{809}{320}n)\cdot 6^g\cdot(\frac{135}{7})^g\\&+(\frac{4}{5}n^3+n^2+\frac{3714}{155}n)\cdot(\frac{135}{7})^g \quad (16)\\&-\frac{44619}{1984}\cdot n\cdot 6^g.\end{aligned}$$

Hence we can solve for MCT on the auxiliary network ASN(g, n):

$$\begin{aligned}\bar{T}_{sum}^{A}(g,n)&=\frac{T_{sum}^{A}(g,n)}{|N^A(g,n)|-1}\\&=\frac{1}{12\cdot 6^g+3}\cdot[(16n^2+\frac{809}{64})6^g\cdot(\frac{135}{7})^g \quad (17)\\&+(4n^2+5n+\frac{3714}{31})\cdot(\frac{135}{7})^g-\frac{223095}{1984}6^g].\end{aligned}$$

3.3. MCT on the SSN(g, k)

In the previous section we have solved the analytic expression for MCT on the auxiliary network ASN(g, n). Through the equivalence between the surplus network SSN(g, k) and the auxiliary network ASN(g, n), we can easily solve the following equation:

$$\begin{aligned}T_{sum}^{S}(g,k)&=T_{sum}^{A}(g-k,3^k)\\&=(\frac{16}{5}\cdot 3^{3k}+\frac{809}{320}\cdot 3^k)\cdot 6^{g-k}\cdot(\frac{135}{7})^{g-k}\\&+(\frac{4}{5}\cdot 3^{3k}+3^{2k}+\frac{3714}{155}\cdot 3^k)(\frac{135}{7})^{g-k} \quad (18)\\&-\frac{44619}{1984}\cdot 3^k\cdot 6^{g-k}.\end{aligned}$$

From this, we can solve the analytic expression for MCT on the surplus network SSN(g, k) with corner nodes L as trap nodes:

$$\begin{aligned}\bar{T}_{sum}^{S}(g,k)&=\frac{T_{sum}^{S}(g,k)}{|N^S(g,k)|-1}\\&=\frac{1}{12\cdot 6^{g-k}+3}\{(16\cdot 3^{2k}+\frac{809}{64})6^{g-k}\cdot(\frac{135}{7})^{g-k} \quad (19)\\&+(4\cdot 3^{2k}+5\cdot 3^k+\frac{3714}{31})(\frac{135}{7})^{g-k}-\frac{223095}{1984}\cdot 6^{g-k}\}.\end{aligned}$$

Make $k=0$ to verify the above equation and we get $\bar{T}_{sum}^{S}(g,0)=\bar{T}_{sum}(g)$. We take the principal term of $\bar{T}_{sum}^{S}(g,k)$ to satisfy the following equation:

$$\begin{aligned}\bar{T}_{sum}^{S}(g,k)&\sim\frac{(16\cdot 3^{2k}+\frac{809}{64})\cdot 6^{g-k}\cdot(\frac{135}{7})^{g-k}}{12\cdot 6^{g-k}+3} \quad (20)\\&\sim\frac{3}{4}\cdot(\frac{63}{135})^k\cdot(\frac{135}{7})^g.\end{aligned}$$

Thus, we can know that $\bar{T}_{sum}^{S}(g,k)$ is proportional to the number of iterations, which is inversely proportional to the division coefficient.

In order to be more intuitive we carry out numerical simulation of $\bar{T}_{sum}^{S}(g,k)$ with respect to g and k, see Figure. 4.

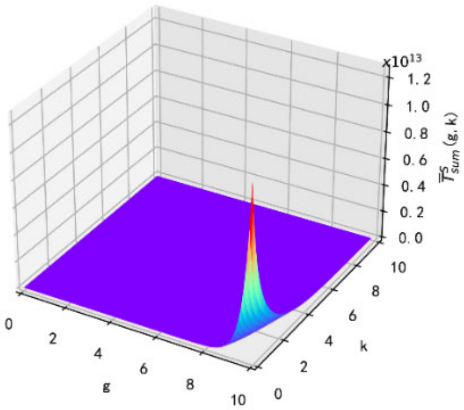

Figure 4. Numerical simulation plot of $\bar{T}_{sum}^{S}(g,k)$ with respect to variables g and k.

In order to better see the trend of $\bar{T}_{sum}^{S}(g,k)$ with respect to the number of iterations g and the division coefficient k, we take the time when k=10, 30 and 50, respectively. The trend of $\bar{T}_{sum}^{S}(g,k)$ with respect to the number of iterations g is shown in Figure. 5.

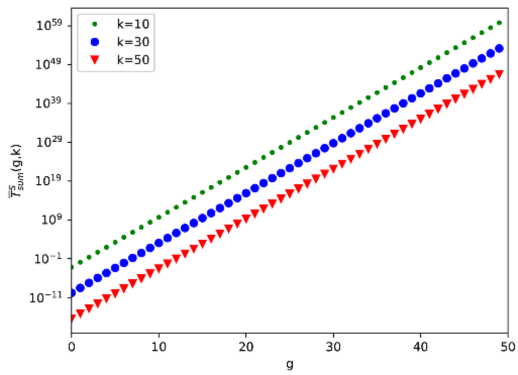

Figure 5. It shows that $\bar{T}_{sum}^{S}(g,k)$ increases as g increases, that is, it is positively correlated, by fixing k = 10, 30 or 50, respectively.

When we take g=10, 30, and 50, respectively, the trend of $\bar{T}^S_{sum}(g,k)$ relative to the division coefficient k is shown in Figure 6.

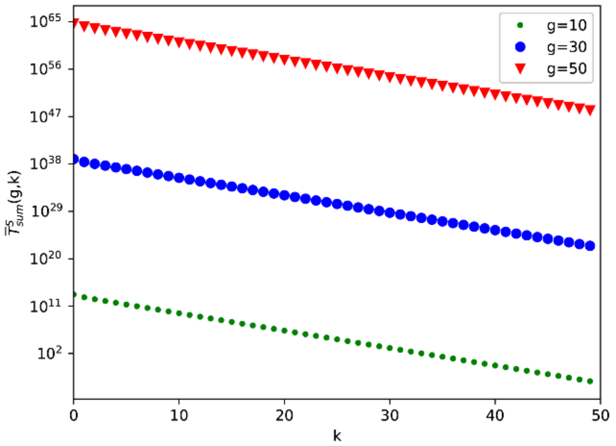

Figure 6. It is fixed $g = 10$, 30 or 50 respectively, which gives $\bar{T}^S_{sum}(g,k)$ decreases with increasing k, that is, a negative correlation.

In order to see that with the change of division coefficients k, the surplus network $\mathrm{SSN(g,k)}$ on MCT versus MCT on the uncut front network $\mathrm{S_gN}$. The trend of $\bar{T}^S_{sum}(g,k)$ versus $\bar{T}_{sum}(g)$ with the number of iterations g is shown in Figure. 7 when we take k=5, 25 and 50. It is easy to find that as the division coefficients k gets smaller, the network retains a higher degree of completeness $\bar{T}^S_{sum}(g,k)$ which converges to $\bar{T}_{sum}(g)$.

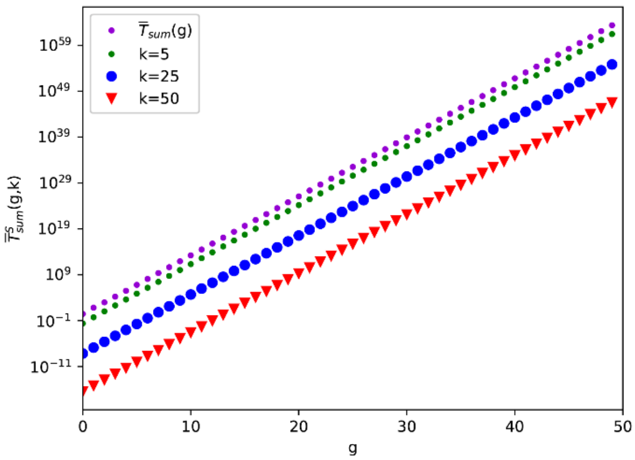

Figure 7. Numerical simulation plots of MCT with iteration coefficients g for the network $\mathrm{SSN(g,k)}$ versus the network $\mathrm{S_gN}$, which are taken to be$k$ = 5, 25 or 50.

Moreover, it is easy to know that $\bar{T}^S_{sum}(g,k)$ takes the minimum value when $k = g$, which we can obtain:

$$\bar{T}^S_{sum}(g,g) = \frac{4}{3}\cdot 3^{2g} = \frac{1}{15}\cdot 3^g + \frac{4}{3}$$

$$\sim |N^S(g,g)|^2 \sim [\bar{T}_{sum}(g)]^{\frac{2\ln 3}{\ln\frac{135}{7}}}.$$

Similarly, $\bar{T}^S_{sum}(g,k)$ takes the maximum value when $k = 0$, which we can obtain:

$$\bar{T}^S_{sum}(g,g) = \frac{\frac{1833}{64}(\frac{135}{7})^g 6^g + \frac{3869}{31}(\frac{135}{7})^g - \frac{223095}{1984}6^g}{12\cdot 6^g + 3}$$

$$\sim |N^S(g,0)|^{\frac{\ln\frac{135}{7}}{\ln 6}}.$$

4. MCT of the Other Trap Nodes on SSN(g, k)

In the previous section, we solved the MCT on the auxiliary network $\mathrm{ASN(g,n)}$ with node (C,1) as the trap node. In this section, instead of considering only node (C,1) as a trap node, we consider the parsing expression for MCT when all nodes on the bottom edge of $\mathrm{ASN(g,n)}$ are individually used as trap nodes. Based on the previous way of labelling the nodes on $\mathrm{ASN(g,n)}$, we know that (B,j) and (C,j+1) denote a single node, so we unify the labelling on the bottom edge of $\mathrm{ASN(g,n)}$ and set it to (C,j), where $j \in [1, n+1]$. At this time, we set the trap node as (C,h) where $1 \le h \le n+1$. Hence, based on the network structure of the auxiliary network $\mathrm{ASN(g,n)}$, we denote the network to the left of the node (C,h) as $\mathrm{ASN(g,n-h+1)}$, and the network to the right as $\mathrm{ASN(g,h-1)}$, see Figure 8.

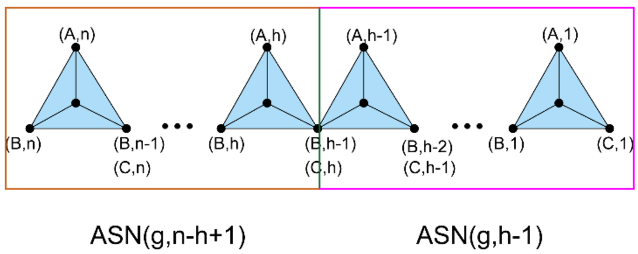

Figure 8. The structure of $\mathrm{ASN(g,n-h+1)}$ and $\mathrm{ASN(g,h-1)}$

Immediately, we can get the total capture time of the auxiliary network $\mathrm{ASN(g,n)}$ on the auxiliary network $\mathrm{ASN(g,n)}$ containing the trap node (C,h) by Eq.(16), denoted as $T^A_{sum}(g,n,h)$, so that we can get:

$$\begin{aligned} T^A_{sum}(g,n,h) &= T^A_{sum}(g,n-h+1) + T^A_{sum}(g,h-1) \\ &= [\frac{16}{5}(n^3 - 3n^2h + 3nh^2 + 3n^2 - 6nh + 3n) + \frac{809}{320}\cdot n] \\ &\cdot 6^g \cdot (\frac{135}{7})^g + (\frac{4}{5}\cdot n^3 - \frac{12}{5}\cdot n^2 h + \frac{12}{5}\cdot nh^2 + \frac{17}{5}\cdot n^2 \\ &- \frac{34}{5}\cdot nh + \frac{4396}{155}\cdot n + 2h^2 - 4h + 2)\cdot(\frac{135}{7})^g \\ &- \frac{44619}{1984}\cdot n \cdot 6^g. \end{aligned} \quad (21)$$

Based on the equivalence between the network $\mathrm{ASN(g,n)}$ and the network $\mathrm{SSN(g,k)}$, we can solve for the total capture time of the other trap nodes on the network $\mathrm{SSN(g,k)}$, which is denoted as $T^S_{sum}(g,k,h)$, and so we get:

$$
\begin{aligned}
&T_{sum}^{S}(g,k,h)=T_{sum}^{A}(g-k,3^{k},h)\\
&=[\frac{16}{5}(3^{3k}-3^{2k+1}h+3^{k+1}h^{2}+3^{2k+1}-6\cdot 3^{k}h+3^{k+1})+\frac{809}{320}\cdot 3^{k}] \\
&\cdot 6^{g-k}(\frac{135}{7})^{g-k}+(\frac{4}{5}\cdot 3^{3k}-\frac{12}{5}\cdot 3^{2k}h+\frac{12}{5}\cdot 3^{k}h^{2}+\frac{17}{5}\cdot 3^{2k}\\
&-\frac{34}{5}\cdot 3^{k}h+\frac{4396}{155}\cdot 3^{k}+2h^{2}-4h+2)\\
&\cdot(\frac{135}{7})^{g-k}-\frac{44619}{1984}\cdot 3^{k}\cdot 6^{g-k}.
\end{aligned} \tag{22}
$$

From this, we can obtain analytic expressions for the MCT of the other trap nodes on the network $\mathrm{SSN(g,k)}$:

$$
\begin{aligned}
&\bar{T}_{sum}^{S}(g,k,h)=\frac{T_{sum}^{S}(g,k,h)}{|N^{S}(g,k)|-1}\\
&=\frac{1}{12\cdot 6^{(g-k)}+3}\{[16(3^{2k}-3^{k+1}h+3h^{2}+3^{k+1}-6h+3)+\frac{809}{64}] \\
&\cdot 6^{g-k}\cdot(\frac{135}{7})^{g-k}+(4\cdot 3^{2k}-12\cdot 3^{k}h+12\cdot h^{2}+17\cdot 3^{k}-34h\\
&+\frac{4396}{31}+(10h^{2}-20h+10)3^{-k})\cdot(\frac{135}{7})^{g-k}-\frac{223095}{1984}\cdot 6^{g-k}\}.
\end{aligned} \tag{23}
$$

We can find that the parsed expression of $\bar{T}_{sum}^{S}(g,k,h)$ is still positively correlated with the number of iterations g and negatively correlated with the division coefficient k.

5. Conclusion

In this paper, we obtain the surplus network $\mathrm{SSN(g,k)}$ based on nested networks, which are divided by a horizontal division coefficient k. Firstly, we obtain the mean capture time of the fixed nodes on the nested network. Secondly, we solve for the mean capture time of the fixed nodes on the auxiliary network $\mathrm{ASN(g,n)}$, and then we obtain the mean capture time on the surplus network $\mathrm{SSN(g,k)}$ based on the equivalence between the surplus network $\mathrm{SSN(g,k)}$ and $\mathrm{ASN(g,n)}$. We perform analytical numerical simulations of the MCT on $\mathrm{SSN(g,k)}$. The results show that the network $\mathrm{SSN(g,k)}$ is only locally self-similar, and the larger the division coefficient k is, the smaller the MCT is, that is, the network becomes more and more efficient in transmission during successive destructions. Finally, we also solve the analytical expression for MCT when any node on the bottom edge of the surplus network $\mathrm{SSN(g,k)}$ is used as a trap node.

Appendix A. MCT of Two Trap Nodes on $\mathrm{S_gN}$

First, we consider the iterative relationship between the nodes corresponding to the first generation of the network in the network $\mathrm{S_gN}$ as they wander at random. According to the labels in Figure 1, the average first pass time of node 1 in $\mathrm{S_0N}$ when it arrives at a node with node L or R as the trap node is denoted as T'. Then, L and R in the network $\mathrm{S_1N}$ are set as trap nodes.

Based on the definition of unbiased Markov random walk and the symmetry of the nested network, the MFPT of each node is denoted as T_i^2 $i=(1,2,\ldots,16)$, where $T_3^2=T_4^2$, $T_5^2=T_6^2$, $T_7^2=T_9^2$, $T_{10}^2=T_{12}^2$, $T_{14}^2=T_{15}^2$, and $T_{13}^2=T_{16}^2=0$, and we can set up the following equation:

$$
\begin{cases}
T_1^2=\frac{1}{3}(T'+T_2^2)+\frac{2}{3}(T'+T_1^2)\\
T_2^2=\frac{1}{3}(T'+T_1^2)+\frac{2}{3}(T'+T_1^2)\\
T_3^2=\frac{1}{6}(T'+T_1^2)+\frac{1}{6}(T'+T_2^2)+\frac{1}{6}(T'+T_4^2)+\frac{1}{6}(T'+T_5^2)+\frac{1}{6}(T'+T_7^2)+\frac{1}{6}(T'+T_8^2)\\
T_5^2=\frac{1}{3}(T'+T_3^2)+\frac{1}{3}(T'+T_7^2)+\frac{1}{3}(T'+T_8^2)\\
T_7^2=\frac{1}{6}(T'+T_3^2)+\frac{1}{6}(T'+T_5^2)+\frac{1}{6}(T'+T_8^2)+\frac{1}{6}(T'+T_{10}^2)+\frac{1}{6}(T'+T_{14}^2)+\frac{1}{6}T'\\
T_8^2=\frac{1}{9}(T'+T_3^2)+\frac{1}{9}(T'+T_4^2)+\frac{2}{9}(T'+T_5^2)+\frac{2}{9}(T'+T_7^2)+\frac{1}{9}(T'+T_{11}^2)+\frac{2}{9}(T'+T_{14}^2)\\
T_{10}^2=\frac{1}{3}(T'+T_7^2)+\frac{1}{3}(T'+T_{14}^2)+\frac{1}{3}T'\\
T_{11}^2=\frac{1}{3}(T'+T_8^2)+\frac{2}{3}(T'+T_{14}^2)\\
T_{14}^2=\frac{1}{6}(T'+T_7^2)+\frac{1}{6}(T'+T_8^2)+\frac{1}{6}(T'+T_{10}^2)+\frac{1}{6}(T'+T_{11}^2)+\frac{1}{6}(T'+T_{15}^2)+\frac{1}{6}T'
\end{cases}
$$

Solving the system of equations gives:

$$\lambda=T_1^2=\frac{135}{7}T'.$$

Here λ is the time difference between the nodes of two consecutive generations to reach the trap node.

Since the network $\mathrm{S_gN}$ has self-similarity, $T'(g)=T_1^2(g-1)$. The MFPT to reach the trap node L or R from node 1 in the network $\mathrm{S_0N}$ is 1, denoted as $T_1^2(0)=1$. So we can get the following equation:

$$T_1^2(g)=(\frac{135}{7})^{g}. \tag{A.1}$$

Then, by the symmetry of the network $\mathrm{S_gN}$, we know that nodes L and R have the same capture probability. Therefore, the average first pass time for a wanderer to reach node L from node 1, that is, the time it takes for a walker to reach either L or R after $T_1^2(g)$ from node 1. It ends if it reaches node L, and vice versa, it reaches R after another $T_1^2(g)$ time. The following relation is satisfied:

$$T_{1,L}(g)=\frac{1}{2}T_1^2(g)+\frac{1}{2}(T_1^2(g)+T_{R,L}(g)).$$

From the symmetry of the network $\mathrm{S_gN}$, we know that: $T_{1,L}(g)=T_{R,L}(g)$. From this, we get the following equation:

$$T_{1,L}(g)=T_{R,L}(g)=2T_1^2(g)=2\cdot(\frac{135}{7})^{g}.$$

In the following we will consider the relationship between $T_{sum}^3(g)$ and $T_{sum}(g)$ when node 1, nodes L and R are all used as capture nodes. When only the corner node L is used as the capture node, it takes two steps to reach the capture node L from any node $i\in N(g)/\{1,L,R\}$:

(1) The walker reaches the corner node for the first time, and the walk ends if the node is node L, otherwise, the second step is performed.

(2) The first step is repeated with a random wander starting from the corner node when the first reached node is 1 or R.

When $\{1,L,R\}$ is used as a capture node, $i\in N(g)/\{1,L,R\}$ is captured by $\{1,L,R\}$ with equal probability. We can obtain the following equation:

$$\begin{aligned}
T_{sum}(g) &= \sum_{i\in N(g)} T_{i,L}(g) \\
&= \sum_{i\in N(g)} [T_i^3(g) + (1-\delta_{q,L})T_{1,L}(g)] \\
&= \sum_{i\in N(g)} T_i^3(g) + 2[\frac{1}{3}(|N(g)|-3)+1]T_{1,L}(g) \\
&= T_{sum}^3(g) + \frac{4}{3}|N(g)|T_1^2(g).
\end{aligned}$$

where $q \in \{1, L, R\}$ serves as the capture node and the random walk starts from any node i, we can get the following relation:

$$\begin{cases} \delta_{q,L} = 1, & q = L \\ \delta_{q,L} = 0, & q = 1/R. \end{cases}$$

Since $T_{sum}(g)$ is known, we can obtain the following equation:

$$\begin{aligned}
T_{sum}^3(g) &= T_{sum}(g) - \frac{4}{3}|N(g)|T_1^2(g) \\
&= \frac{809}{1920}(\frac{135}{7})^g 6^{g+1} + \frac{10987}{465}(\frac{135}{7})^g - \frac{44619}{1984}6^g.
\end{aligned}$$

Similarly, with the symmetry of the network S_gN, we get:

$$\begin{aligned}
T_{sum}^2(g) &= \sum_{i\in N(g)} T_i^2(g) \\
&= \sum_{i\in N(g)} T_i^3(g) + +[\frac{1}{3}(|N(g)|-3)+1]T_1^2(g) \qquad (A.2)\\
&= T_{sum}^3(g) + \frac{1}{3}|N(g)|T_1^2(g) \\
&== \frac{71}{128}(\frac{135}{7})^g 6^{g+1} + \frac{749}{31}(\frac{135}{7})^g - \frac{44619}{1984}6^g.
\end{aligned}$$

$$\begin{aligned}
\overline{T}_{sum}^2(g) &= \frac{T_{sum}^2(g)}{|N(g)|-2} \\
&= \frac{\frac{355}{128}(\frac{135}{7})^g 6^{g+1} + \frac{3745}{31}(\frac{135}{7})^g - \frac{223095}{1984}6^g}{12\cdot 6^g - 2}.
\end{aligned}$$

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