Academic Journal of Science and Technology
ISSN: 2771-3032 | Vol. 16, No. 3, 2025

Global Sensitivity Analysis of Structural Random Response Based on Polynomial Chaotic Expansion

Qi Cheng[1, *], Zhen Cheng[1]

[1]School of Urban Construction, Yangtze University, Jingzhou 434023, China
[*] Corresponding author

Abstract: With the growing significance of stochastic uncertainties in structural design, the quantification of uncertainty and sensitivity analysis of static displacement responses have gained considerable attention. This study employs polynomial chaos expansion (PC) to perform a global sensitivity analysis of the tip deflection of a cantilever beam, aiming to identify the random variable that contributes most significantly to the output variance. Compared with the Monte Carlo simulation approach, the PC method demonstrates superior computational accuracy and efficiency. The results indicate that PC not only accurately captures the sensitivity ranking of the variables but also substantially reduces computational cost, thus offering an effective tool for stochastic analysis and optimization in structural engineering.

Keywords: Polynomial chaos expansion, Global sensitivity analysis, Sobol.

1. Introduction

With the continuous advancement of technology and the rapid economic development, the safety of static displacement in building structures has garnered increasing attention. In practical construction and engineering practice, inherent random variations in material properties, geometric dimensions, and boundary conditions often lead to significant discrepancies between the actual structural response and the design predictions. Therefore, systematically analyzing the static displacement response while accounting for the randomness of structural parameters holds substantial theoretical and practical significance for enhancing structural reliability and facilitating optimized design.

In reality, uncertainty is widespread in building structures. The quantification of uncertainty and parameter sensitivity analysis of random structures have always been hot research issues. Parameter sensitivity analysis can identify key variables that have a significant impact on the uncertainty of structural responses, thereby effectively reducing the number of random variables and providing a basis for structural optimization. Sensitivity analysis methods mainly include two categories: local sensitivity and global sensitivity. Local sensitivity can only reflect the influence of the variation of parameters within the mean neighborhood on the output, and its effectiveness highly depends on the linearity of the input-output relationship. Global sensitivity can evaluate the comprehensive impact of parameter variations throughout the probability space on output uncertainty, and thus is widely applied in strongly nonlinear and non-monotonic systems.

Among numerous global sensitivity indicators, the Sobol sensitivity indicator based on variance decomposition[1] has been widely applied in various engineering problems due to its clear concept, intuitive calculation, and the ability to quantify the contribution of each input variable to the output variance. Traditionally, this indicator is often estimated using Monte Carlo simulation (MCS), but due to its double-layer cyclic sampling feature, the computational cost is high. Later, proxy model methods such as polynomial chaotic expansion (PC)[2] were proposed to solve the Sobol indicator.

This article mainly introduces the application of the PC method in global sensitivity analysis. Based on the cantilever beam example, this paper systematically analyzes the ranking of the importance of each random variable on the deflection at the beam end. Through comparison with the MCS method, it highlights the significant advantages of the PC method in terms of computational efficiency.

2. Polynomial Chaos Expansion

In traditional methods, the estimation of sensitivity indicators usually relies on MCS. Although this method is easy to implement, in some cases, it requires a large number of model evaluations and has a relatively high computational cost. Proxy models can serve as an effective alternative for estimating sensitivity indicators due to their low construction costs, high evaluation efficiency, and often having more accuracy advantages than traditional Monte Carlo methods under small to medium sample sizes[3]. Common surrogate models include linear and nonlinear regression models, cubic splines, artificial neural networks, Gaussian processes, and orthogonal polynomials, etc. Among them, PC is a type of random variable series expansion method based on orthogonal polynomial bases, and its polynomial type depends on the pre-set PDF[4]. For instance, Hermite polynomials are often used to represent normal random variables. In the global sensitivity analysis of scalar output, an important advantage of PC lies in the fact that the Sobol exponent can be directly derived from its coefficients, a property proposed and refined by Sudret[2].

The PC decomposition form of the random variable Y is as follows:

$$Y(x) = \sum_{j=0}^{\infty} \beta_j \psi_j(x) \tag{1}$$

where β_j represents the expanded covariance, ψ_j is the Hermite polynomial, and $x = (x_1, \cdots, x_n)$ is an independent standard normal distribution random variable.

The number of terms for polynomial chaos expansion varies with the specific problem. If the order of the polynomial is set to M and the number of input variables is n, then the total number P+1 of all terms in the expansion that are not higher than the order M is determined by the following formula:

$$P+1=\frac{(M+n)!}{M!n!} \tag{2}$$

Even if the input variables do not follow a normal distribution, the polynomial chaotic expansion of the output variable Y can still be constructed using Hermite polynomials. In such cases, the method of series expansion of random variables based on orthogonal polynomials corresponding to the input variable PDF is called generalized Polynomial Chaos[5]. The most significant advantage of the PC method lies in its rigorous mathematical foundation, theoretical convergence guarantee, as well as its simple form and clear concept. Therefore, it has been widely applied in many engineering and scientific fields.

3. Global Sensitivity Analysis

Let the functional functions of the structure be $G=Y(X)$, and $X=(X_1, X_2, \dots, X_n)$ represents an n-dimensional input random variable. According to the high-dimensional model expansion theory[6], when the random variables input to the model are independent of each other, the functional function $Y(X)$ can be expanded as:

$$\begin{aligned} Y(X) &= Y_0+\sum_{i=1}^{n}Y_i(X_i)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}Y_{ij}(X_i,X_j)+ \\ &\cdots+Y_{1,2,\cdots,n}(X_1,\cdots,X_n) \end{aligned} \tag{3}$$

where Y_0 is the mean of $Y(X)$, $Y_i(X_i)$ is a univariate function with respect to X_i, $Y_{ij}(X_i,X_j)$ is a bivariate function with respect to X_i and X_j, and so on.

Find the variance on both sides of Eq. (3) simultaneously:

$$Var(G)=\sum_{i=1}^{n}G_i+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}G_{ij}+\cdots+G_{1,2\cdots,n} \tag{4}$$

where G_i is the variance of $Y_i(X_i)$, G_{ij} is the variance of $Y_{ij}(X_i,X_j)$, and so on. The specific expression is:

$$\begin{aligned} G_i &= Var(E(G\mid X_i)) \\ G_{ij} &= Var(E(G\mid X_i,X_j))-G_i-G_j \\ &\dots \end{aligned} \tag{5}$$

where Var represents the variance operation and E represents the expectation operation.

Therefore, the expression of the first-order variance global sensitivity index S_i of variable X_i is:

$$S_i=\frac{G_i}{Var(G)}=\frac{Var\left(E\left(G\mid X_i\right)\right)}{Var(G)} \tag{6}$$

Sort all the calculated first-order Sobol exponents by their numerical magnitudes. The higher the first-order sensitivity index is, the greater the individual contribution of this input variable to the output variance.

4. Cantilever Beam Calculation Example

As shown in Figure 1, there is a reinforced concrete cantilever beam with one end fixed and the other end free. The cantilever beam is 6 meters long, with an elastic modulus of 40×10^6 kN/m^2. The concentrated load at the end is 5kN. The finite element model consists of 60 beam elements.

Let the elastic modulus of concrete follow a lognormal distribution, with the mean and mean square deviation taken as 0 and 0.1 respectively. The randomness of the material within every 2-meter length range is described by one variable. The entire model consists of three independent random variables, corresponding to X_1, X_2, and X_3 respectively.

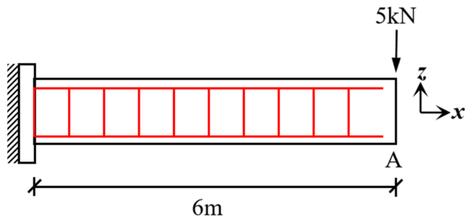

Figure 1. Plan view of the cantilever beam

As shown in Figure 2, this figure presents the global sensitivity index analysis results of the deflection at point A at the end of the cantilever beam, which were calculated using Monte Carlo Simulation (MCS) and PC methods respectively. The results of the global sensitivity analysis show that the sensitivity ranking of each variable is $S_1 > S_2 > S_3$, indicating that variable X_1 contributes the most to the deflection at point A, and the influence of X_2 is significantly greater than that of X_3, that is, the elastic modulus of the concrete farther from point A has a more significant impact on the deflection. By comparing the calculation results of PC and MCS, the two are highly consistent, verifying that the PC method has a relatively high calculation accuracy in such problems. Furthermore, in terms of computational efficiency, MCS takes 13.44 seconds, while PC only requires 0.48 seconds. The former is approximately 28 times that of the latter, fully demonstrating that the PC method has significant efficiency while ensuring accuracy.

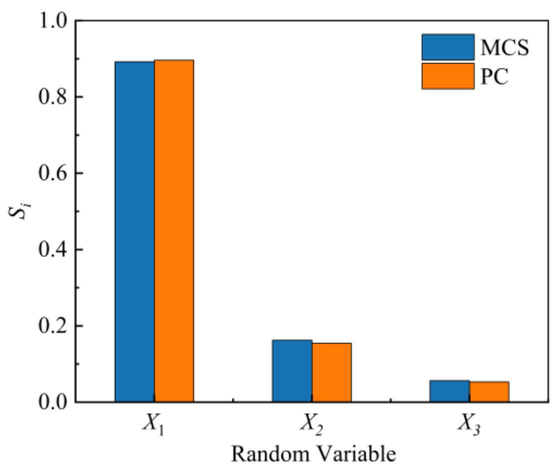

Figure 2. Global sensitivity index of deflection at point A

5. Conclusion

This paper, through the example of a cantilever beam, systematically analyzes the influence of the randomness of the elastic modulus of concrete on the deflection at the beam end, and conducts a global sensitivity analysis using the PC method. The research results show that: (1) The variable X_1 has the most significant influence on deflection, followed by X_2 and X_3, indicating that the material parameters farther from the free end contribute more to deflection. (2) The PC method was highly consistent with the MCS results when calculating the Sobol sensitivity index, verifying its accuracy. (3) In terms of computational efficiency, the time consumption of the PC method is only 1/28 of that of MCS, demonstrating remarkable efficiency. In conclusion, the PC method, as an efficient surrogate model technology, is suitable for the global sensitivity calculation in the random response analysis of structures and has a promising engineering application prospect. In the future, it can be further extended to more complex structural systems and nonlinear problems.

References

[1] Sobol' IM. On sensitivity estimation for nonlinear mathematical models [J]. Matematicheskoe Modelirovanie. 1990, 2: 112-118.

[2] Sudret Bruno. Global sensitivity analysis using polynomial chaos expansions[J]. Reliability Engineering and System Safety. 2008, 93(7): 964-979.

[3] Storlie B C , Swiler P L , Helton C J , et al. Implementation and evaluation of nonparametric regression procedures for sensitivity analysis of computationally demanding models[J]. Reliability Engineering and System Safety, 2009, 94(11): 1735-1763.

[4] Kim D, Debusschere B J, Najm H N. Spectral Methods for Parametric Sensitivity in Stochastic Dynamical Systems[J]. Biophys. 2007, 92(2): 379–393.

[5] Xiu D, Karniadakis G E. The Wiener --Askey Polynomial Chaos for Stochastic Differential Equations[J]. Slam Journal on Scientific Computing. 2002, 24(2): 619-644.

[6] Sobol IM. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates [J]. Mathematics and Computers in Simulation, 2001, 55(1-3): 271-280.