

THE SCIENTIFIC LEGACY OF MILOSLAV HAVLÍČEK

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ABSTRACT. We review the main research achievements of Miloslav Havlíček in algebraic methods in Quantum Theory, his extensive work on realisations of Lie algebras and superalgebras and their representation theory, quantum groups and differential equations.

KEYWORDS: Realisations of Lie algebras, Lie gradings, quantum groups, differential equations, quantum mechanics.

1. INTRODUCTION. ACADEMIC AND PROFESSIONAL TRAJECTORY

It is certainly not an easy task to condense the life and work of Miloslav Havlíček (1938–2024) to few pages, nor is to emphasise his influence at the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University in Prague and the Faculty of Mathematics and Physics at Charles University, where he has, for many decades, trained dozens of leading specialists, firmly contributing to the solid reputation of these institutions as centres of excellence in mathematical physics. It is our purpose to recall some of the main achievements of Miloslav in algebraic methods in quantum theory, emphasising various problems that have determined and motivated important research topics that are still of current interest and application.

Havlíček's academic and professional trajectory has been marked for many years by the difficulties and mistrust of the government and university officials, plenty of bitter experiences and privations, as well as some endearing anecdotes, like the confiscation of his doctoral dissertation by Soviet custom officers due to the extremely suspicious name of “Li” in the title (with Ли being the Russian form of Sophus Lie's surname, written and spelled as the Chinese surname Li), in a time where the confrontation of the People's Republic of China and the Soviet Union was at its highest point. This arbitrary intervention did, however, not prevent Miloslav from demonstrating his exceptional scientific ability during the examination.

Despite all obstacles, Miloslav never lost faith that education should be devoid of any political interventionism and favoritism, and based solely on personal qualifications. These were the precepts he maintained while being the Dean of the Faculty, as well as a member of the various scientific associations to which he belonged, such as the International Association of Mathematical Physics (IAMP) or the Union of Czech Mathematicians and Physicists (JČMF). Miloslav also served as Scientific Council Member of the ČVUT for

the terms 1990–1994 and 2000–2006, of the Faculty of Mathematics and Physics at Charles University for the terms 1999–2002 and 2002–2005, and of the Institute of Informatics of the Czech Academy of Sciences (AV ČR) between 2001 and 2005, also acting as Vice-Chair of the Academic Assessment Board at this institution between 1993 and 1999. His efforts, devotion, and altruistic dedication were finally recognised officially in 1998, when Miloslav was awarded with the Prize of the Minister of Education (1st degree). Rigour and scientific excellence were also the guiding principles in the foundation of the Doppler Institute in 1993, where Miloslav's efforts, among other colleagues, managed to finally found a research institute that crystallised the high scientific level of the seminars they had led for many years.

To offer a glimpse into the captivating and humble personality of Miloslav, who always avoided the spotlight and the pursuit of fame, let us recall his response when asked about his experiences as Dean of the faculty in the terms 1990–1994 and 2000–2006, Havlíček answered, “The faculty actually runs itself”.

As a teacher, he was well-known to be strict and demanding, but impartial and fair. He was perfectly aware that, in order to create a really selective, representative, and competitive scientific school, rigor was the first commandment. In his own words, on the occasion of the 50th anniversary of FJFI, he stated: “We don't pretend to be an easy school, but for the effort, we offer individual attention, quick involvement in faculty research teams and an emphasis on our students' own creative work. These students are then able to establish themselves at the top workplaces in the Czech Republic and abroad”. At the same occasion, and concerning the social relevance of the rôle that higher education had to play in the development of the country, deprived from any political fanaticism, he declared that “You may be surprised, but applied natural sciences penetrate into virtually all components of human activities, from theoretical physics and research into the construction of matter through medicine, energy, environmental protection,

monument research, food science and many others. We have a good historical basis, excellent scientific and technological level and capable people, our students are among the elite of the Czech intelligentsia – what other field can say this?”

Despite the fact that Havlíček was by no means considered a political activist “by the book”, a fact which would cause difficulties for him until 1989, but without bending either his spirit or convictions, he was allowed to defend the DrSc. Dissertation in the Laboratory of Theoretical Physics at JINR in Dubna, showing that his work had truly impressed the authorities there, as well as in Prague. In this context, it is not surprising, that, despite his political convictions, Miloslav was awarded the First Prize for Theoretical Physics at the JINR in Dubna in 1985, jointly with Jiří Blank, Jaroslav Dittrich, Pavel Exner and Genrikh Ivanovich Kolerov, in recognition of their important work on operator methods in quantum physics, with Miloslav’s contributions to realisations of Lie algebras being part of the awarded collection of papers. The Dissertation, dealing with canonical realisations of classical Lie algebras [1], was brilliantly defended in Dubna in 1979. The opponents of the thesis were three highly reputed and influential scientists, Vladas Vladovich Vanagas from the Lithuanian Academy of Sciences, Alexander Aleksandrovich Kirillov from the Moscow State University, and Yakov Abramovich Smorodinsky from JINR, all of them world leading experts in group theoretical methods in physics.

The wide range of Miloslav’s scientific interests is well reflected by enumerating his (main) collaborators, also illustrating that, despite the disparity of specific subjects, the nucleus was always related to his passion for group theory. Among the collaborators whose main affiliation belongs to an establishment in Prague¹, we mention (given in alphabetical order): Martin Baccovský, Miroslav Bednář, Jiří Blank, Čestmír Burdík, Goce Chadzitaskos, Pavel Exner, Jiří Hořejší, Jan Kotrbatý, Ondřej Navrátil, Edita Pelantová, Severin Pošta, Jiří Tolar, Ivan Úlehla, and Jan Votruba.

Besides them, Miloslav actively worked with Jiří Patera and Pavel Winternitz from the Centre de Recherches Mathématiques in Montréal (Canada), Anatoly Ul’yanovich Klimyk from the Institute of Theoretical Physics in Kiev (Ukraine), Raisa Moiseevna Asherova, Yuri Fedorovich Smirnov, and Valeriy Nikolaevich Tolstoy from the Skobeltsyn Institute of Nuclear Physics at the Lomonosov Moscow State University, as well as Wolfgang Lassner from the Leipzig University (at the times of the German Democratic Republic, Karl-Marx Universität) and Patrick J. Moylan from the University of Pennsylvania in Abington (USA).

These names by no means exhaust the collaboration of Havlíček with other scholars. He was always willing to help and counsel, and has motivated many people

¹Or whose principal affiliation was related to Prague when the publications appeared.

to pursue their researches and to clear up doubts or apparently unsurmountable difficulties. Anecdotally, I remember Miloslav’s astonishing ability to find rare or untraceable references, like a series of articles in hard-to-find Soviet journals that I had, after several desperate attempts in many places, asked him for during a visit back in 2006, and that he managed to procure within a few days.

2. THE INFINITE DIMENSIONAL LIE ALGEBRA $A(P, S)$

The topic of Miloslav’s CSc. thesis [2], defended in December 1968, benefitted from a productive collaboration (and lifetime friendship) with Jan Votruba, son of the eminent physicist and pioneer of modern theoretical physics in Czechoslovakia, Václav Votruba, with both Votrubas being undoubtedly very influential in the formation of Miloslav’s scientific interests. These works already show the deep interest that Miloslav took in combining the algebraic/geometric formalism with real physical phenomena and modelization, a subject that had begun to be considered systematically in the mid 1960’s (see e.g. [3]), based on the pioneering work by Giulio Racah in the 1950s [4]. From these initial articles referring to this subject, that we enumerate below, many new ideas and problems emerged, that would constitute one of the main activity areas of Miloslav for the rest of his scientific life:

- M. Havlíček, J. Votruba. On the physical representations of an infinite-dimensional lie algebra. *Czechoslovak Journal of Physics B* **16**(8):631–642, 1966. <https://doi.org/10.1007/BF01689564>
- M. Havlíček, J. Votruba. The tensor product of one-particle representations of an infinite-dimensional Lie algebra. *Czechoslovak Journal of Physics B* **17**(10):809–821, 1967. <https://doi.org/10.1007/BF01691631>
- J. Votruba, M. Havlíček. On the representation of infinite-dimensional Lie algebra $A(P, S)$. In *High Energy Physics and Theory of Elementary Particles*, pp. 330–335. International School of Theoretical Physics, Naukova Dumka, Kyiv, 1966
- J. Votruba, M. Havlíček. Mass formulas for meson nonets. *Czechoslovak Journal of Physics B* **19**(6):721–729, 1969. <https://doi.org/10.1007/BF01697128>
- M. Havlíček. Representations of an algebra of the Gell-Mann-Dashen type. *Communications in Mathematical Physics* **13**(1):73–80, 1969. <https://doi.org/10.1007/BF01645272>
- M. Havlíček. About one class of representations of the Lie algebra. *Communications in Mathematical Physics* **20**(2):130–142, 1971. <https://doi.org/10.1007/BF01646532>

These papers extended earlier work by Formánek on the nontrivial coupling $A(P, S)$ of internal and space-

time symmetries [11, 12], combining the Poincaré algebra with $SU(2)$ in new representations. Specifically, the Lie algebra $A(P, SU(2))$ proposed in this extension has the basis:

$$\begin{aligned} [L_{\mu\nu}, L_{\rho\lambda}] &= i(g_{\mu\rho}L_{\nu\lambda} - g_{\nu\rho}L_{\mu\lambda} \\ &\quad + g_{\mu\lambda}L_{\rho\nu} - g_{\nu\lambda}L_{\rho\mu}), \\ [P_\mu, L_{\rho\lambda}] &= i(g_{\rho\mu}P_\nu - g_{\nu\mu}P_\rho), \\ [P_\mu, P_\nu] &= 0, \\ [P_\mu, T_{\rho_1 \dots \rho_m}^{(1)}] &= -\alpha i T_{\mu\rho_1 \dots \rho_m}^{(2)}, \\ [P_\mu, T_{\rho_1 \dots \rho_m}^{(2)}] &= \alpha i T_{\mu\rho_1 \dots \rho_m}^{(1)}, \\ [P_\mu, T_{\rho_1 \dots \rho_m}^{(3)}] &= 0, \\ [L_{\mu\nu}, T_{\rho_1 \dots \rho_m}^{(i)}] &= i \sum_{j=1}^m \left(g_{\nu\rho_j} T_{\rho_1 \dots \rho_{j-1} \mu \dots \rho_m}^{(i)} \right. \\ &\quad \left. - g_{\mu\rho_j} T_{\rho_1 \dots \rho_{j-1} \nu \dots \rho_m}^{(i)} \right), \\ [T_{\rho_1 \dots \rho_m}^{(i)}, T_{\rho_1 \dots \rho_m}^{(k)}] &= i \epsilon_{ik\ell} T_{\mu_1 \dots \mu_n \rho_1 \dots \rho_m}^{(\ell)}, \end{aligned}$$

where $P_\mu, L_{\nu\rho}, \mu, \nu, \rho = 0 \dots 3; i, k, \ell = 1, 2, 3$.

It should be noted that structures of this type were later obtained and considered independently in the context of Kac–Moody algebras, their generalizations and extensions, as well as other infinite-dimensional algebraic structures.

The representations of this algebra, specifically the irreducible Hermitean representations (IHR in short), also considering the special case of one isomultiplet, were constructed and studied in detail. In addition, the mass formulae (depending on the third component of the isotopic spin) and the spin-spectrum were analysed, hence extending considerably an earlier symmetry scheme for particles proposed by Votruba and Havlíček [13]. In connection with the classification of one-particle states into S-multiplets (i.e. one-particle representations), the tensor product of two such representations and the corresponding reduction problem were considered. Here, an interesting phenomenon emerged: while in the case of irreducible Hermitean representations with a single value of $\mathbf{T}^i \mathbf{T}^i$, the analogy of $SU(2)$ and $A(P_2, SU(2))$ was almost complete (from the perspective of the pure $SU(2)$ classification), for the tensor products the behaviour showed important differences, and a new series of IHR appeared, that determined an action on Hilbert spaces decomposable as direct sums of a (finite) discrete number of irreducible spaces associated to the Poincaré algebra, leading to non-linear mass formulae.

In the context of infinite dimensional algebras, and motivated by a problem of Dashen and Gell-Mann arising in the application of current algebras to elementary particle physics, Havlíček proposed an alternative construction, starting from centreless Lie algebras of finite dimension, and studying a certain class (called F -representations) of this infinite-dimensional object in a Hilbert space, considering those operators

represented linearly and possessing a common dense domain in the Hilbert space, subjected to certain linear constraints. In a more general frame, operator representations of pairs $(\mathfrak{g}', \mathfrak{g})$ in Hilbert spaces were studied in connection with the integrability property with respect to a distinguished subalgebra, under certain conditions. A notion of irreducibility for such representations was introduced, for which a version of the Schur lemma was obtained; specifically, that a symmetric operator that commutes with the representation and leaves a dense space D in the domain invariant is necessarily a scalar.

3. REALISATIONS OF LIE ALGEBRAS

During his stay at the JINR in Dubna (1974–1977), Miloslav met Wolfgang Lassner, a physicist from the Karl Marx Universität Leipzig (GDR), who had written his CSc. thesis on the connected subgroups of the Poincaré group [14], being simultaneously an expert in symbolic computation and algorithm programming². Sharing common interests, jointly with Pavel Exner, with whom Miloslav had already studied some aspects of canonical matrix realisations³, they combined their expertise for an ambitious problem, still offering new challenges even today: the realisations of Lie algebras by creation and annihilation operators.

The problem of realisations of Lie algebras arises naturally in many physical applications, such as solutions of nuclear models, collective motions, many-body problems, and dynamical algebras in the context of particle physics, where the generators are identified with differential operators with respect to quantum-mechanical variables q^i and p_i . Though several results were available in the mid 1970s, no systematic and profound analysis of the problem had been yet considered. The subject of Havlíček's DrSc. dissertation was exactly this problem: a methodic approach to classify and construct realisations of the classical Lie algebras in the Weyl algebra and some of its extensions. These works, carried out between 1974 and 1977 [1], also considered the problem of computing the eigenvalues of the corresponding (generalised) Casimir operators on these realisations, expanding on work by other authors on the eigenvalue spectra of Casimir operators on irreducible representations (see e.g. [16] and references therein)⁴. This work significantly extended the known knowledge about the properties of canonical realisations of important families of classical Lie algebras, and served as a natural basis for generalisations to other types of Lie algebras that have been widely considered in the literature. The dissertation

²Actually, Lassner presented his Dissertation B on a topic also deeply related to realisations of Lie algebras [15].

³One of the first publications on the subject was the $o(m, n)$ case, presented at an international symposium on high energy physics and elementary particles held in Varna (Bulgaria) in 1974.

⁴The length of the thesis is 112 pages, with 111 references that comprised most of the work on realisations published until 1979.

comprises the material of the main articles concerning the Lie algebra case and appeared in these years⁵:

- M. Havlíček, P. Exner. On the minimal canonical realizations of the Lie algebra $O_c(n)$. *Annales de l'Institut Henri Poincaré Section A* **23**(4):313–333, 1975
- M. Havlíček, P. Exner. Matrix canonical realizations of the Lie algebra $\mathfrak{o}(m, n)$. I. Basic formulae and classification. *Annales de l'Institut Henri Poincaré Section A* **23**(4):335–347, 1975
- M. Havlíček, W. Lassner. Canonical realizations of the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$. I. Formulae and classification. *Reports on Mathematical Physics* **8**(3):391–399, 1975. [https://doi.org/10.1016/0034-4877\(75\)90081-6](https://doi.org/10.1016/0034-4877(75)90081-6)
- M. Havlíček, W. Lassner. Canonical realizations of the Lie algebras $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{R})$. II. Casimir operators. *Reports on Mathematical Physics* **9**(2):177–185, 1976. [https://doi.org/10.1016/0034-4877\(76\)90053-7](https://doi.org/10.1016/0034-4877(76)90053-7)
- P. Exner, M. Havlíček, W. Lassner. Canonical realizations of classical Lie algebras. *Czechoslovak Journal of Physics B* **26**(11):1213–1228, 1976. <https://doi.org/10.1007/BF01589833>
- M. Havlíček, W. Lassner. Canonical realizations of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$. *International Journal of Theoretical Physics* **15**(11):867–876, 1976. <https://doi.org/10.1007/BF01807449>
- M. Havlíček, W. Lassner. On the “near to minimal” canonical realizations of the Lie algebra C_n . *International Journal of Theoretical Physics* **15**(11):877–884, 1976. <https://doi.org/10.1007/BF01807450>
- M. Havlíček, W. Lassner. Matrix canonical realizations of the Lie algebra $\mathfrak{u}(p, q)$. *Reports on Mathematical Physics* **12**(1):1–8, 1977. [https://doi.org/10.1016/0034-4877\(77\)90040-4](https://doi.org/10.1016/0034-4877(77)90040-4)
- M. Havlíček, P. Exner. Matrix canonical realizations of the Lie algebra $\mathfrak{o}(m, n)$. II. Casimir operators. *Czechoslovak Journal of Physics B* **28**(9):949–962, 1978. <https://doi.org/10.1007/BF01596007>
- P. Exner, M. Havlicek, W. Lassner. Boson representations of classical Lie algebras. In *International Conference on Operator Algebras, Ideals and their Applications in Theoretical Physics*, pp. 277–278. 1977
- Č. Burdík, M. Havlíček. Boson realizations of semi-simple Lie algebras. In P. Winternitz, J. Harnad, C. S. Lam, J. Patera (eds.), *Symmetry in Physics*, vol. 34 of *CRM Proceedings & Lecture Notes*, pp. 87–98. American Mathematical Society, 2004

⁵The corresponding preprints at JINR, as far as they could be found, are: E2-8089, E2-8533, E2-8700, E2-8646, E2-8842, E2-9160, E2-9617, E2-9161.

The three main aspects of the generalised approach and classification of realisations can be summarised as follows:

- (a) New recursion formulae determining realisations with special properties for all types of classical complex Lie algebras and their consequences.
- (b) Analogous recursion formulae for the following types of (noncompact) real forms: $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{su}(p, q)$, $\mathfrak{o}(p, q)$, $\mathfrak{sp}(2n, \mathbb{R})$.
- (c) Comparison with minimal realisations and realisations of Gel'fand–Kirillov type [28, 29].

The embeddings of the classical series A_n , B_n , C_n , and D_n in the Weyl algebra $W_{2N(d)}$ were revisited and expanded, taking into account the Schurean property, and the number of realisation parameters determined for each class of realisations. Of special interest are the minimal and maximal number values such that the resulting realisations are nontrivial, with special emphasis on the Gel'fand–Kirillov conjecture. Within the range $N(d) = 1, \dots, n$, the following numbers of pairs were found for the classical series:

| | A_n | B_n | C_n | D_n |
|--------|-----------------------|-----------|-----------|-------------|
| $N(d)$ | $\frac{(2n+1-d)d}{2}$ | $d(2n-d)$ | $d(2n-d)$ | $d(2n-d-1)$ |

The problem of realisations in the quotient division ring of the Weyl algebras was also satisfactorily completed, and by denoting n_0 the minimal number of canonical pairs leading to nontrivial realisations, and n_k the maximal integer such that all realisations in $W_{2(n_0+n_k)}$ are Schurean, the following values were computed:

| | A_n | $B_n(n > 1)$ | C_n | $D_n(n > 2)$ |
|-------|-------|--------------|-------|--------------|
| n_0 | n | $2n-2$ | n | $2n-3$ |
| n_k | 0 | 1 | $n-1$ | 1 |

The nucleus of the work, however, concerns studying the set of skew-Hermitian and Schurean⁶ realisations for the real Lie algebras in (b). To this extent, the usual approach is enriched by considering further suitable embeddings of the Weyl algebra into more general algebraic structures, that allow for a greater generality of the operators realizing the Lie algebras, and avoiding the degeneracy concerning the number of resulting independent Casimir operators. However, the removal of degeneracy does not necessarily imply the preservation of the Schurean property, and obtaining the criteria and explicit construction procedures for non-degenerate Schurean realisations was posed. In the first papers with Pavel Exner on this topic, the existence problem was positively answered for the class $\mathfrak{o}(n, m)$ of pseudo-orthogonal Lie algebras by means of canonical matrix realisations, i.e. by realisations in the matrix Weyl algebra $W_{2n, M} = W_{2n} \otimes \text{Mat}(M, \mathbb{C})$, also considering the case of minimal canonical realisations for the complexification $\mathfrak{o}(n, \mathbb{C})$. The analogous

⁶In other words, such that the Casimir operators are realised by scalar multiples of the identity.

construction for the pseudo-unitary group $u(r, s)$ was written in collaboration with Wolfgang Lassner.

Also in collaboration with Lassner, Miloslav considered the problem for the canonical embedding $\mathfrak{gl}(n, \mathbb{R}) \supset \mathfrak{sl}(n, \mathbb{R})$, but focusing on embeddings into a Weyl algebra with an adequate number of canonical pairs. A family of $(d + 1)$ -parametric classes with $d \leq n - 1$ in the Weyl algebra in dependence of $N(d) = \frac{(2n-d-1)d}{2}$ pairs was obtained, having the following salient properties:

- (1.) The realisations are skew-Hermitean.
- (2.) Casimir operators are realised as scalars (Schurean).
- (3.) Non-equivalence of realisations is determined up to endomorphisms in the Weyl algebra.
- (4.) The obtained realisations in the $(d + 1)$ -class admit Casimir operators whose eigenvalues can be expressed polynomially in terms of $(d+1)$ symmetric functions of the parameters.

The corresponding realisations for the special Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ were recovered obtained by restriction, still preserving the above-mentioned structural properties. In this context, it is worthy to recall how canonical realisations $F_{\mu\nu}$ of $\mathfrak{gl}(n - 1, \mathbb{R})$ in W_{2m} with commutation relations:

$$[F_{\mu\nu}, F_{\rho\sigma}] = \delta_{\nu\rho}F_{\rho\sigma} - \delta_{\mu\sigma}F_{\rho\nu}$$

were expanded to an α -parameterised family of realisations of $\mathfrak{gl}(n, \mathbb{R})$ in $W_{2n-2+2m}$, defining the new operators as:

$$\begin{aligned} E_{\mu\nu} &= q_\mu p_\nu + F_{\mu\nu} + \frac{1}{2}\delta_{\mu\nu}\mathbf{1}, \\ E_{n\mu} &= -p_\mu, \\ E_{\mu n} &= q_\mu \left(\sum_\nu q_\nu p_\nu + \frac{n}{2} - i\alpha \right) + \sum_\nu q_\nu F_{\mu\nu}, \\ E_{nn} &= -\sum_\nu q_\nu p_\nu - \left(\frac{n-1}{2} - i\alpha \right) \mathbf{1}, \quad \alpha \in \mathbb{C}. \end{aligned} \tag{1}$$

The real importance of these formulae, which have implications far beyond of what was explicitly stated in these papers, was only recognised much later in the context of quasi-exactly solvable models (see e.g. [30, 31]).

The symplectic algebra, appearing naturally in the context of canonical transformations as a subgroup of the corresponding inhomogeneous group, as well as in other physical contexts such as dynamical algebras in n -dimensional harmonic oscillators, prior to their application in nuclear collective motions [32, 33], was the last class to be considered to complete the discussion on the classical series. Here, it was observed that certain one-parameter sets of realisations of $\mathfrak{sp}(2n, \mathbb{R})$ could be obtained by means of the previously studied one-parameter set of minimal realisations for $\mathfrak{gl}(2n, \mathbb{R})$, leading to an extremely relevant

formula for constructing canonical realisations of arbitrary finite-dimensional Lie algebras. Several of these realisations were later used for studying the metaplectic group; specifically, the quantisation schemes invariant under metaplectic transformations, of relevance in the context of wave optics with aberration [34]. Other applications of these results were found in the algorithmic construction of invariants of inhomogeneous Lie algebras obtained by contraction of symplectic groups [35].

Some of the generic results concerning the realisations of the mentioned classes of Lie algebras can be summarised in table form as follows:

| | $\mathfrak{gl}(n, \mathbb{R})$ | $u(p, q)$ | $\mathfrak{o}(m, n)$ | $\mathfrak{sp}(2n, \mathbb{R})$ |
|--------|--------------------------------|------------------------------|----------------------|---------------------------------|
| Alg. | $W_{2N(d)}$ | $W_{2N(d), M(d)}^{N(d)}$ | $W_{2N(d), M(d)}$ | $W_{2N(d)}$ |
| d | $1, \dots, n - 1$ | $1, \dots, 2q - \delta_{pq}$ | $1, \dots, n$ | $1, \dots, n$ |
| $N(d)$ | $\frac{(2n-d-1)d}{2}$ | $\frac{(2p+2q-d-1)d}{2}$ | $d(m+n-d-1)$ | $d(2n-d)$ |
| $M(d)$ | — | ξ_0 | ξ_0 | — |

where $W_{2N(d), M(d)}^{N(d)}$ denotes a certain localisation of the Weyl algebra (i.e. rational functions of canonical pairs) and the precise value of ξ_0 depends on the value of d as well as those of (p, q) and (m, n) .

As commented before, the Equation (1) contains much more information than suspected, although this fact is not easily recognisable. Actually, to a certain extent, it seems natural to extrapolate the given realisations to the combined case of differential and matrix representations, hence leading naturally to the notion of mixed realisations or matrix differential operators. This was actually the ansatz considered by Yuri F. Smirnov and Alexander V. Turbiner in [30], where, starting from the usual vector field realisation $\hat{E}_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu}$ and a representation $M_{\rho\sigma}$ by (unspecified) operators that commute with the realisation, mixed realisations of $\mathfrak{gl}(n + 1, \mathbb{R})$ were obtained with a striking similarity to Equation (1)⁷:

$$\begin{aligned} E_{\mu\nu} &= \hat{E}_{\mu\nu} + M_{\mu\nu}, \\ E_{n\mu} &= \frac{\partial}{\partial x_\mu}, \\ E_{\mu n} &= x_\mu \left(k - \sum_\nu x_\nu \frac{\partial}{\partial x_\nu} \right) - \sum_\nu x_\nu F_{\mu\nu}, \\ E_{nn} &= k - \sum_\nu x_\nu \frac{\partial}{\partial x_\nu}, \quad k \in \mathbb{R}. \end{aligned} \tag{2}$$

To this day, it remains unknown to which extent Miloslav had already considered the possibility of mixed realisations, as no explicit hint could be found in either the Dubna preprints or the published versions. However, it was clear to many people, among them to Israel Moiseevich Gel'fand, that Miloslav knew more than he had shared; a suspicion that has been later confirmed through personal conversations with him⁸. Humble as he was, Miloslav never claimed any

⁷Up to an inessential change of the sign in the $E_{\mu n}$ generators.

⁸See the introduction of [30] for a first-hand account.

credit for the obtainment of the algebra $\mathfrak{gl}(n+1, \mathbb{R})$ of matrix differential operators, though strong hints in that direction that can be extracted after a detailed and quite laborious analysis of the papers lead us to suspect that this construction was to a certain extent familiar to him.

4. REALISATIONS OF LIE SUPERALGEBRAS

As a natural extension of the work already achieved for Lie algebras, relevant physical applications led Miloslav and collaborators to consider the analogous problem for the case of Lie superalgebras [36], also combined with aspects of the general representation theory, where several important structural results could be obtained. Among these works, from which some of the most relevant are listed below, the first were written in collaboration with Lassner and Exner, where they extended the techniques that they had developed for the case of Lie algebras, leading to a series of papers with Jiří Blank (1939–1990) as the main collaborator:

- J. Blank, M. Havlíček, M. Bednář, W. Lassner. Canonical representations of the Lie superalgebra $osp(1, 4)$. *Czechoslovak Journal of Physics B* **31**(11):1286–1301, 1981. <https://doi.org/10.1007/BF01603588>
- J. Blank, M. Havlíček, P. Exner, W. Lassner. Boson-fermion representations of Lie superalgebras: The example of $osp(1, 2)$. *Journal of Mathematical Physics* **23**(3):350–353, 1982. <https://doi.org/10.1063/1.525373>
- J. Blank, M. Havlíček. Irreducible $*$ -representations of Lie superalgebras $B(0, n)$ with finite-degenerated vacuum. General considerations. Tech. Rep. E2-85-112, JINR Dubna, 1985
- J. Blank, M. Havlíček. Irreducible $*$ -representations of Lie superalgebras $B(0, n)$ with finite-degenerated vacuum. Results for $B(0, 1)$. Tech. Rep. E2-85-160, JINR Dubna, 1985
- Y. F. Smirnov, V. N. Tolstoi, A. A. Sakharuk, et al. The Dyson type boson realizations for representations of the semisimple Lie algebras and superalgebras. In *Group Theoretical Methods in Physics*, vol. 1–3, pp. 67–76. Harwood Academic Publishers, 1982
- J. Blank, M. Havlíček. Irreducible $*$ -representations of Lie superalgebras $B(0, n)$ with finite-degenerated vacuum. *Journal of Mathematical Physics* **27**(12):2823–2831, 1986. <https://doi.org/10.1063/1.527257>
- J. Blank, M. Havlíček. Irreducible $*$ -representations of the Lie superalgebras $B(0, n)$ with finite-degenerated vacuum. II. *Journal of Mathematical Physics* **29**(3):546–559, 1988. <https://doi.org/10.1063/1.528048>

- J. Blank, M. Havlíček. On the tensor product of supersingleton representations of $osp(1, 2n)$. *Czechoslovak Journal of Physics B* **39**(11):1192–1207, 1989. <https://doi.org/10.1007/BF01605320>
- J. Blank, M. Havlíček. On the tensor product of supersingleton representations of Lie superalgebras $osp(1, 2n)$. In *Selected Topics in Quantum Field Theory and Mathematical Physics*, pp. 190–196. World Scientific, 1989

Among the important structural results obtained for $B(0; n)$ -type superalgebras, we emphasise a new method for constructing infinite-dimensional representations of superalgebras, as well as their description in terms of creation-annihilation operators of para-Bose systems with n degrees of freedom. A lot of attention was also devoted to the $*$ -representations of Lie superalgebras of type $B(0; n)$ and real form $\mathfrak{osp}(1; 2n)^9$, and the tensor product decomposition of metaplectic representation σ_n . In particular, a precise decomposition of the $*$ -representation $\hat{\sigma}_n$ of the orthosymplectic superalgebra $\mathfrak{osp}(1; 2n)$ related to the Kronecker product of two supersingleton representations was obtained. These results allowed a considerable simplification in the description of generators and relations in the odd sector.

Concerning the $*$ -representations of $\mathfrak{osp}(1; 2n)$, it was observed that the commutation relations of the odd generators of the algebra coincide with those derived by Y. Ohnuki and S. Kamefuchi in 1982, considering parastatistics of order 2, for constructing the reduction of the tensor product of generic star representations of $\mathfrak{osp}(1; 2n)$. Calling $P_M^{(n)}$ the representation of $\mathfrak{sp}(n, \mathbb{R})$ obtained by restriction to the even sector of the orthosymplectic superalgebra, several intriguing properties were deduced. In particular, it was shown that each representation $P_M^{(n)}$ generically admits two non-equivalent $*$ -extensions to the pseudounitary group $SU(n, n)$. For the special value $n = 2$, the procedure provided explicit forms for infinite-dimensional representations of $\mathfrak{osp}(1; 4)$, previously considered and classified by Heidenreich [46].

5. QUANTUM GROUPS

With the emergence of quantum groups in the late 1980s and the beginning of 1990s, Havlíček's attention was focused on this important topic, where he collaborated with several colleagues in Prague and abroad, mainly with Anatoly U. Klimyk in Kiev, a great specialist in structure and representation theory of Lie (super)algebras, as well as Edita Pelantová and Severin Pošta. These works were primarily devoted to a concise, systematic and deep structural study of

⁹ $\Omega(z^*) = \Omega(z)^*$ adjoint operation adjoint of linear differential operator.

$U_q(\mathfrak{gl}(n, \mathbb{C}))$ and $U_q(\mathfrak{so}_m)$, although other topics concerning the general structure of quantum groups were considered as well:

- Č. Burdík, M. Havlíček, A. Vančura. Irreducible highest weight representations of quantum groups $U_q(\mathfrak{gl}(n, \mathbb{C}))$. *Communications in Mathematical Physics* **148**(2):417–423, 1992. <https://doi.org/10.1007/BF02100869>
- M. Havlíček, E. Pelantová, A. Klimyk. Nonstandard $U_q(\mathfrak{so}_3)$ and $U_q(\mathfrak{so}_4)$: Tensor products of representations, oscillator realizations and roots of unity. *Czechoslovak Journal of Physics* **47**(1):13–16, 1997. <https://doi.org/10.1023/A:1021431709238>
- M. Havlíček, E. Pelantová. Santilli-Fairlie algebra $U_q(\mathfrak{so}_3)$: Tensor products of representations, oscillator realizations and roots of unity. *Hadronic Journal* **20**(6):603–614, 1997
- M. Havlíček, S. Pošta, A. U. Klimyk. Representations of the cyclically symmetric q -deformed algebra $U_q(\mathfrak{so}_3)$. *Czechoslovak Journal of Physics* **48**(11):1347–1353, 1998. <https://doi.org/10.1023/A:1021692803323>
- M. Havlíček, A. U. Klimyk, E. Pelantová. Representations of the q -deformed algebra $U_q(\mathfrak{so}_4)$ for q a root of unity. *Methods in Functional Analysis and Topology* **4**(3):39–44, 1998
- M. Havlíček, A. U. Klimyk, S. Pošta. Representations of the cyclically symmetric q -deformed algebra $\mathfrak{so}_q(3)$. *Journal of Mathematical Physics* **40**(4):2135–2161, 1999. <https://doi.org/10.1063/1.532856>
- M. Havlíček, A. U. Klimyk, S. Pošta. Representations of the q -deformed algebra $U_q(\mathfrak{iso}_2)$. *Journal of Physics A: Mathematical and General* **32**(25):4681, 1999. <https://doi.org/10.1088/0305-4470/32/25/310>
- M. Havlíček, S. Pošta, A. U. Klimyk. Representations of the q -deformed algebra $\mathfrak{so}_q(2, 1)$. In *Symmetry in Nonlinear Mathematical Physics*, vol. 30 of *Proceedings of Institute of Mathematics of NAS of Ukraine*, pp. 280–287. 2000
- M. Havlíček, A. U. Klimyk, S. Pošta. Central elements of the algebras $U'_q(\mathfrak{so}_m)$ and $U_q(\mathfrak{iso}_m)$. *Czechoslovak Journal of Physics* **50**(1):79–84, 2000. <https://doi.org/10.1023/A:1022825031633>
- M. Havlíček, A. U. Klimyk, S. Pošta. Classification of representations of the algebra $U_q(\mathfrak{so}_3)$ through examples. *Czechoslovak Journal of Physics* **50**(11):1235–1238, 2000. <https://doi.org/10.1023/A:1022804806462>
- M. Havlíček, S. Pošta, A. U. Klimyk. Some basic properties of nonstandard deformations $U'_q(\mathfrak{so}_3)$, $U'_q(\mathfrak{so}_4)$. In S. T. Ali, H.-D. Doebner, M. Keyl, R. Werner (eds.), *Trends in Quantum Mechanics: Proceedings of the International Symposium Goslar, Germany*, pp. 10–17. 1999
- M. Havlíček, S. Pošta. On the classification of irreducible finite-dimensional representations of $U'_q(\mathfrak{so}_3)$ algebra. *Journal of Mathematical Physics* **42**(1):472–500, 2001. <https://doi.org/10.1063/1.1328078>
- M. Havlíček, A. U. Klimyk, S. Pošta. Representations of the q -deformed algebra $U'_q(\mathfrak{so}_4)$. *Journal of Mathematical Physics* **42**(11):5389–5416, 2001. <https://doi.org/10.1063/1.1402631>
- M. Havlíček, S. Pošta, A. U. Klimyk. Classification of representations of the algebra $U_q(\mathfrak{so}_3)$ through examples II. *Physics of Atomic Nuclei* **64**(12):2151–2155, 2001. <https://doi.org/10.1134/1.1432917>
- Č. Burdík, M. Havlíček, O. Navrátil, S. Pošta. Ideals of the enveloping algebra $U(\mathfrak{osp}(1, 2))$. *Journal of Generalized Lie Theory and Applications* **2**(3):132–136, 2008
- R. M. Asherova, Č. Burdík, M. Havlíček, et al. q -analog of Gel'fand-Graev basis for the noncompact quantum algebra $U_q(\mathfrak{u}(n, 1))$. *Symmetry, Integrability and Geometry: Methods and Applications* **6**:10, 2010. <https://doi.org/10.3842/SIGMA.2010.010>
- M. Havlíček, S. Pošta. Central elements of quantum deformations. *AIP Conference Proceedings* **1307**(1):125–130, 2010. <https://doi.org/10.1063/1.3527408>
- M. Havlíček, S. Pošta. Center of quantum algebra $U'_q(\mathfrak{so}_3)$. *Journal of Mathematical Physics* **52**(4):043521, 2011. <https://doi.org/10.1063/1.3579992>

These articles, of great technical complexity, expanded upon and generalised various results, and clarified many questions that have, until then, been only partially studied or understood. An exhaustive study of the standard $U_q(\mathfrak{so}(n))$ and nonstandard $U'_q(\mathfrak{so}(n))$ quantum algebras for $n = 2, 3, 4$ and their representations began. Being rather difficult to condense the depth of these works to a few lines, we merely enumerate some of the salient structural results. One remarkable achievement was the construction of tensor products in the Santilli-Fairlie algebra $U_q(\mathfrak{so}(3))$, which does not possess a Hopf algebra structure, a property that is central for the description of tensor products in the context of quantum groups. Nevertheless, a procedure for constructing a tensor product was given, together with an explicit method for the product of two irreducible representations of $U_q(\mathfrak{so}(3))$ for positive q . Moreover, for the case, when q is a root of unity, irreducible representations were constructed. In order to cover the remaining cases, an algebra homomorphism Ψ from the nonstandard algebra $U_q(\mathfrak{so}(3))$ to the extension $\widehat{U}_q(\mathfrak{sl}(2))$ of the Hopf algebra $U_q(\mathfrak{sl}(2))$ was introduced. The composition of Ψ with irreducible representations of $\widehat{U}_q(\mathfrak{sl}(2))$ led to (not necessarily irreducible) representations of

$U_q(\mathfrak{so}(3))$. Analysing in detail the decomposition into irreducible components, all IR of $U_q(\mathfrak{so}(3))$ when q is not a root of unity could be determined.

Another difficult problem addressed in these articles was the analysis of the centre of the nonstandard quantum algebra $U'_q(\mathfrak{so}(3))$ (earlier proposed by Gavrilik and Klimyk in [65] in order to construct representations by operators acting according to the Gel'fand-Tsetlin pattern). For the case of roots of unity, it was shown that there are four generators of the center, which depend on the order of q and whose polynomial dependence was explicitly indicated, while in the non-root of unity case, the centre was shown to be a polynomial algebra in one variable. These results were relevant for the general case of the central elements of $U'_q(\mathfrak{so}_m)$ and $U'_q(\mathfrak{iso}_m)$, allowing also to find relations with the case of $U'_q(\mathfrak{so}_3)$. In this context, Miloslav and S. Pošta showed how the so-called Diamond lemma, a quite unknown result in ring theory, could be successfully applied to the Poincaré–Birkhoff–Witt property in quantum algebras, allowing to derive, in simple and elegant way, certain facts concerning their centre.

Besides these exhaustive results, a quantum analogue of Gel'fand–Graev bases for the noncompact quantum algebra $U_q(\mathfrak{u}(n, 1))$ was also obtained, and the Hermitean irreducible representations corresponding to a discrete series were computed. These results were shown to be crucial for the computation of pairings on Mickelsson algebras, and hence for the computation of inverse Shapovalov forms [66].

6. LIE GRADINGS

The notion of gradation in Lie algebras is an old one, and besides the well known Cartan decomposition, other gradings have already been used in the literature (e.g. in the superalgebra context), although the problem had not been considered generically. In 1989, Hans Zassenhaus and Jiří Patera began such a systematic classification of gradings [67], introducing the notion of fine grading and giving a criterion for this property. The classification problem, however, remained mainly untouched for some years due to the death of Zassenhaus in 1991, and it was not until the mid 1990s, when Edita Pelantová and Miloslav joined Patera to complete the task, with Jiří Tolar joining the group later. These works not only finished the program established by Patera and Zassenhaus, but also led to new techniques that have found wide applications, both in the Lie theory and physical problems. We enumerate Miloslav's principal articles concerning this subject:

- M. Havlíček, J. Patera, E. Pelantová. On the maximal Abelian subgroups of the diagonalizable automorphisms of simple classical Lie algebras. In H. D. Doebner, P. Nattermann, W. Scherer (eds.), *Group Theoretical Methods in Physics*, Proceedings of XXI International Colloquium on Group The-

oretical Methods in Physics, pp. 116–120. World Scientific, Singapore, 1997

- M. Havlíček, J. Patera, E. Pelantová. On the fine gradings of simple classical Lie algebras. *International Journal of Modern Physics A* **12**(1):189–194, 1997. <https://doi.org/10.1142/S0217751X97000268>
- M. Havlíček, J. Patera, E. Pelantová. On Lie gradings II. *Linear Algebra and its Applications* **277**(1–3):97–125, 1998. [https://doi.org/10.1016/S0024-3795\(97\)10039-8](https://doi.org/10.1016/S0024-3795(97)10039-8)
- M. Havlíček, J. Patera, E. Pelantová. On Lie gradings III. Gradings of the real forms of classical Lie algebras. *Linear Algebra and its Applications* **314**(1–3):1–47, 2000. [https://doi.org/10.1016/S0024-3795\(00\)00099-9](https://doi.org/10.1016/S0024-3795(00)00099-9)
- J. Patera, M. Havlíček, E. Pelantová, J. Tolar. On fine gradings and their symmetries. *Czechoslovak Journal of Physics* **51**(4):383–391, 2001. <https://doi.org/10.1023/A:1017501925328>
- M. Havlíček, J. Patera, E. Pelantová, J. Tolar. Automorphisms of the fine grading of $\mathfrak{sl}(n, \mathbb{C})$ associated with the generalized Pauli matrices. *Journal of Mathematical Physics* **43**(2):1083–1094, 2002. <https://doi.org/10.1063/1.1430046>
- M. Havlíček, E. Pelantová, J. Patera, J. Tolar. Distinguished bases of $\mathfrak{sl}(n, \mathbb{C})$ and their symmetries. In *Quantum Theory and Symmetries*, pp. 366–370. World Scientific, 2002. https://doi.org/10.1142/9789812777850_0043
- M. Havlíček, J. Patera, E. Pelantová, J. Tolar. On Pauli graded contractions of $\mathfrak{sl}(3, \mathbb{C})$. *Journal of Nonlinear Mathematical Physics* **11**(1):37–42, 2004. <https://doi.org/10.2991/jnmp.2004.11.s1.4>
- M. Havlíček, E. Pelantová, J. Tolar. On representations of $\mathfrak{sl}(n, \mathbb{C})$ compatible with a \mathbb{Z}_2 -grading. *Acta Polytechnica* **50**(5):30–39, 2010. <https://doi.org/10.14311/1261>

These works in particular use the classification of the (maximal) commutative subgroups in the automorphism group of $\mathfrak{gl}(n, \mathbb{C})$, with special emphasis on the inner and outer automorphisms, as fine gradings can be classified, up to equivalence, by the maximal Abelian groups of diagonalisable automorphisms of the Lie algebra (called MAD-groups)¹⁰. As a first decisive advance towards the classification, a detailed procedure to determine fine gradings for the class of classical Lie algebras was given, and an equivalence criterion for fine gradings based on labeled graphs was proposed. It was soon recognised that fine gradings constitute a powerful tool for the structural study of solvable (respectively nilpotent) Lie algebras [78], and that the contraction and deformation theories of

¹⁰From a different point of view, the problem of Abelian subgroups had also been considered by Suprunenko and Tyshkevich in 1966 [77].

Lie algebras could be approached alternatively with these techniques, leading to new and interesting applications [79, 80]. In this context, the $\mathfrak{gl}(n, \mathbb{C})$ -fine gradings were analysed in detail in connection with $\mathfrak{sl}(n, \mathbb{C})$ -contractions. In particular, for the fine gradings of the latter algebra, it was observed that the group $SL(2, \mathbb{Z}_n)$ plays an analogue rôle to the Weyl group for the canonical Cartan grading.

Once the classical case systematised, the next difficult step consisted in extending these results to the real forms of the classical Lie algebras. To this extent, the maximal abelian diagonalisable groups of automorphisms for the real forms were studied, and four types of matrix subgroups of $GL(n, \mathbb{R})$ defined. This allowed them to transfer the classification problem of non-conjugate maximal abelian diagonalisable groups for real forms to the description of equivalence classes within the four types of matrix subgroups, hence enabling them to establish a precise classification.

Several other questions related to fine gradings were considered, such as the symmetries associated with the gradings and the class of fine gradings related to the n -dimensional Pauli matrices, which had found several intriguing applications. In this context, particular attention was given to the Pauli grading of $\mathfrak{sl}(3, \mathbb{C})$, where it was observed that the symmetry transformations of a given grading are in a one-to-one correspondence with a certain quotient group related to automorphisms that permute the grading blocks. For the case of $\mathfrak{sl}(3, \mathbb{C})$, the Jacobi identity associated with a graded contraction therefore led to a system of quadratic equations for the contraction parameters, the symmetries of which were related to $\mathbb{Z}_2 \times SL(2, \mathbb{Z}_3)$. These nonlinear equations were also analysed for their solutions. It should be mentioned that the compatibility of representations with \mathbb{Z}_2 -gradings, which have many potential applications to the case of Lie superalgebras, was also studied in detail for several Lie algebras.

7. DIFFERENTIAL EQUATIONS

Miloslav's work on differential equations can be divided into two separate subjects: on the one hand, the computation of spectra and criteria for determining the eigenvalues of the Schrödinger equation, and, on the other hand, the problem of existence and obtaining nonlinear superposition principles for (systems) of differential equations. The works on spectra were done in collaboration with Ivan Úlehla and Jiří Hořejší, reputed specialists in quantum theory, while those on superposition rules were written with Severin Pošta and Pavel Winternitz, expanding previous work on the topic [81, 82]:

- I. Úlehla, M. Havlíček. New method for computation of discrete spectrum of radical Schrödinger operator. *Aplikace matematiky* **25**(5):358–372, 1980
- I. Úlehla, M. Havlíček, J. Hořejší. Eigenvalues of the Schrödinger operator via the Prüfer transformation.

Physics Letters A **82**(2):64–66, 1981. [https://doi.org/10.1016/0375-9601\(81\)90938-5](https://doi.org/10.1016/0375-9601(81)90938-5)

- M. Havlíček, S. Pošta, P. Winternitz. Nonlinear superposition formulas based on imprimitive group action. *Journal of Mathematical Physics* **40**(6):3104–3122, 1999. <https://doi.org/10.1063/1.532749>
- M. Havlíček, S. Pošta, P. Winternitz. Superposition formulas based on nonprimitive group action. In A. Coley, D. Levi, R. Milson, et al. (eds.), *Bäcklund and Darboux Transformations. The Geometry of Solitons*, vol. 29 of *CRM Proceedings & Lecture Notes*, pp. 225–231. American Mathematical Society, 2001

In the first of the articles with Úlehla, a new method for computing the discrete spectrum of a one-particle Schrödinger operator under spherical symmetry and appropriate potentials such that the spectrum is finite was proposed, by means of a transformation of the radial Schrödinger equation into a first-order ODE for the function $z(x, \chi)$:

$$\frac{dz}{dx} = (\ell + 1) \cos^2 z - \frac{1}{\ell + 1} (v(x, \chi) + \chi^2) \sin^2 z,$$

with the number of eigenvalues being determined by the condition $z(\infty, 0) = (2k + 1)\frac{\pi}{2}$, such that the discontinuity points coincide with the eigenvalues (hence constituting a kind of analogue of the Levinson theorem for this function). The procedure contemplated the replacement of the usual Ritz variational method by direct integration techniques. These discontinuities were obtained by successive integration, with accuracy increased for truncations at suitable values depending on the potential.

In a following paper, jointly with Jiří Hořejší, the eigenvalue problem for central potential was considered from the perspective of a modified Prüfer transformation, giving two theorems that established a theoretical framework for numerical eigenvalue computation for the case of rapidly decreasing and confining central potential, considered unified for the first time. This approach was later extended in [87] to long-range potentials, including the Coulomb case.

The papers on superposition, which is very deeply connected with the classification of realisations of Lie algebras by vector fields, continue the extensive work of Pavel Winternitz on the subject, focusing on imprimitive group actions on manifolds, specifically for the special linear groups $SL(n, \mathbb{C})$, making use of the realisations obtained before by Havlíček and Lassner. It was shown that such decomposable systems can be studied with indecomposable systems serving as building blocks. The cases for $n = 3, 4$ were computed explicitly, including the superposition rules for the corresponding ODE systems, providing also a description for the general case of $SL(n, \mathbb{C})$, including some relations to soliton equations and Bäcklund transformations. More specifically, when compared to the well-known situation with $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$, it

was concluded that the equations derived from the imprimitive action of $SL(n, \mathbb{C})$ should appear as Bäcklund transformations for integrable systems defined on flag manifolds.

8. REPRESENTATION THEORY

Havlíček's extensive work on the realisation problem of Lie algebras and superalgebras also had many important implications for their representation theory, which led to a first series of papers on highest weight representations, written in collaboration with Pavel Exner and Čestmír Burdík, and published at the beginning 1980s, as well as some applications to the Poincaré group, in collaboration with Jan Kotrbatý, Severin Pošta and Patrick Moylan:

- Č. Burdík, M. Havlíček, P. Exner. Highest-weight representations of the $\mathfrak{sl}(n+1, \mathbb{C})$ algebras: Maximal representations. *Journal of Physics A: Mathematical and General* **14**(5):1039, 1981. <https://doi.org/10.1088/0305-4470/14/5/023>
- Č. Burdík, P. Exner, M. Havlíček. Highest-weight representations of $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$ via canonical realizations. *Czechoslovak Journal of Physics B* **31**(5):459–469, 1981. <https://doi.org/10.1007/BF01596411>
- Č. Burdík, P. Exner, M. Havlíček. A complete set of irreducible highest-weight representations for $\mathfrak{sl}(3, \mathbb{C})$. *Czechoslovak Journal of Physics B* **31**(11):1201–1206, 1981. <https://doi.org/10.1007/BF01603579>
- M. Havlíček, P. Moylan. An embedding of the Poincaré Lie algebra into an extension of the Lie field of $SO_0(1, 4)$. *Journal of Mathematical Physics* **34**(11):5320–5332, 1993. <https://doi.org/10.1063/1.530307>
- S. Pošta, M. Havlíček. Note on Verma bases for representations of simple Lie algebras. *Acta Polytechnica* **53**(5):450–456, 2013. <https://doi.org/10.14311/AP.2013.53.0450>
- M. Havlíček, J. Kotrbatý, P. Moylan, S. Pošta. Construction of representations of Poincaré group using Lie fields. *Journal of Mathematical Physics* **59**(2):021702, 2018. <https://doi.org/10.1063/1.4993153>
- M. Havlíček, J. Kotrbatý, P. Moylan, S. Pošta. (Heisenberg-)Weyl algebras, Segal-Bargmann transform and representations of Poincaré groups. *Journal of Physics: Conference Series* **1194**(1):012043, 2019. <https://doi.org/10.1088/1742-6596/1194/1/012043>

Using previous results concerning the canonical boson representations of $\mathfrak{sl}(n+1, \mathbb{C})$, for every given weight m , so-called maximal representations D_m were constructed, which contained an irreducible representation with m as the highest weight. Conditions for the irreducibility of the maximal representations were

analysed, and a comparison with the standard construction of representations performed. An advantage of the new procedure was the explicitness of the construction, which proved valuable for applications. In this context, the cases of $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$ were analysed in further detail, and irreducibility criteria for some types of infinite-dimensional representations were obtained by means of the canonical boson realisations. This allowed to derive a complete description of the irreducible highest-weight representations of $\mathfrak{sl}(3, \mathbb{C})$, which were shown to be partitioned into five disjoint classes, one corresponding to finite-dimensional representations, and the four remaining to infinite-dimensional representations:

$$\begin{aligned}\Omega_F &= \{\Lambda : \Lambda_i \in \mathbb{N}, i = 1, 2\}, \\ \Omega_1 &= \{\Lambda : \Lambda_1 \in \mathbb{N}, \Lambda_2 \notin \mathbb{N}\}, \\ \Omega_2 &= \{\Lambda : \Lambda_1 \notin \mathbb{N}, \Lambda_2 \in \mathbb{N}\}, \\ \Omega_{12} &= \{\Lambda : \Lambda_i \notin \mathbb{N}, i = 1, 2; 1 + \Lambda_1 + \Lambda_2 \in \mathbb{N}\}, \\ \Omega_{\max} &= \{\Lambda : \Lambda_i \notin \mathbb{N}, i = 1, 2; 1 + \Lambda_1 + \Lambda_2 \notin \mathbb{N}\},\end{aligned}$$

where Ω_F is well known from the work of I. M. Gel'fand and M. L. Tsetlin, and Ω_1 , Ω_2 , and Ω_{\max} have already been obtained in previous work. The remaining class Ω_{12} was obtained, completing the classification of infinite-dimensional highest weight representations. Beyond the explicit results, these papers contain precise indications for higher ranks, also indicating the great complexity in deriving explicit expressions for the corresponding classes.

Another remarkable contribution to representation theory was made in collaboration with Severin Pošta. This work provided an alternative construction of the Verma basis of the enveloping algebra and finite-dimensional representations of the A_n Lie algebras, as well as providing a more compact proof of the so-called Verma inequalities [95].

In the context of specific applications of the Poincaré group, Miloslav wrote three papers with Patrick Moylan that extended previous (and apparently unpublished [96]) work. The first one, from 1993, showed that the principal series of unitary ray representations of $SO_0(1, 4)$ is related, via the *-isomorphism between algebraic extensions of the Lie fields of $SO_0(1, 4)$ and the Poincaré group, to real mass, and positive and negative energy representations of the Poincaré Lie algebra with an arbitrary spin. This approach is deeply related to the embedding problem of Lie algebras into their universal enveloping algebra, leading to representations by higher order differential operators, with important implications concerning quasi-exactly solvable systems in quantum mechanics [97].

In the second one (jointly with Jan Kotrbatý and Severin Pošta), which is one of the last papers of Havlíček, the representations of the Poincaré group in $n = 2, 3, 4$ dimensions were realised on Hilbert spaces associated with the space of the Schrödinger representation of Weyl algebras. The localisation of universal

enveloping algebras is used as the main technique, determining first the algebraic relations between generators of the Poincaré algebra and Heisenberg pairs, in analogy with the Gel'fand–Kirillov conjecture. From these, the realisation of the representations is deduced. It is shown that the action of the basis elements can be integrated to strongly continuous unitary one-parameter subgroups on the Hilbert space, from which a further integration procedure leads to a representation of the corresponding Poincaré group. The equivalence of these unitary irreducible representations with those arising from the standard approach using the Wigner-Mackey theory was also proved.

9. A FUNDAMENTAL REFERENCE FOR QUANTUM MECHANICS

In 1993, the textbook “Linear Operators in Quantum Physics” by J. Blank, P. Exner and Miloslav finally appeared, after an adventurous and long genesis [98]. The idea of preparing a book on the fundamental aspects of Quantum Mechanics can be traced back to 1973, when the authors, based on their lectures on the topic at the Charles University and the Czech Technical University, prepared a series of lecture notes, that were revised, compiled and given a textbook form in the late 1980s. The publication, initially scheduled for 1989, was delayed by several years, due to the political and economic changes that shook the country. Unfortunately, prior to its publication, Jiří Blank died in 1990, and could not see the great success that the text would have, still being one of the central references in courses on Quantum Mechanics even today.

The book explained in detail the theory of linear operators on Hilbert spaces and their application in quantum theory. With an initial survey on fundamentals of linear algebra, topology and functional analysis, as well as measure theory, the main facts on Hilbert spaces and (bounded and unbounded) operators were presented. Physical applications of quantum theory with special emphasis on non-relativistic quantum mechanics, the second quantisation, and the scattering theory were developed in detail, enriched with many supplementary material and numerous exercises.

The relevance of this text was, however, not appreciated abroad until its first English edition appeared in 1994 at the American Institute of Physics [99]. Although extensive modifications were required due to technical and editorial reasons, the spirit of the first edition remained untouched, and both versions, albeit their differences, can be seen as uniform. The reception was very favourable, as can be seen in the different reviews of the book at *Zentralblatt der Mathematik* and *Mathematical Reviews*, among others¹¹, and the authors were awarded the Rector’s Prize in 1995 for their important contribution. Some years

¹¹See e.g. M. Gorzelańczyk Zbl 0873.46038 or H. Baumgärtel MR1275370.

later, a second edition with the addition of new topics was published in 2008, also being welcomed by the physical community, receiving hundreds of citations in other textbooks and research articles.

10. CONCLUSION

These brief summaries of extensive subjects do not exhaust Havlíček’s activity, as he also wrote other papers on quite a number of specific questions, such as the integrability properties of representations of semisimple Lie algebras in Hilbert spaces [100], the description of unstable systems [101], or about the notion of quantum-mechanical pseudo-Hamiltonians [102], to which a large number of contributions to national and international conferences must be added.

To sum it up, Miloslav’s work has covered almost any of the physical problems that uses group theory and algebraic formalism as a fundamental tool, providing new insights and motivating new notions and techniques that have since been proved to be very fruitful. Certainly, there are still a lot of new problems that arise from his work, as the completion of the classification of canonical realisations for the exceptional complex and compact Lie algebras of rank $d > 2$ (with the case of G_2 having been partially classified in [103]), which is certainly a difficult question, specially for the case of E_8 , due to the intricate structural particularities of these algebras. Another relevant problem concerns the mixed realisations, that so far have only been obtained for the (special) general linear algebra $\mathfrak{sl}(n+1, \mathbb{R})$ and $\mathfrak{gl}(n+1, \mathbb{R})$, but that are likely to be extended to other algebras of the classical series by subtle adaptation of Miloslav’s methods, and even considered for the class of inhomogeneous and other distinguished subalgebras by means of the contraction formalism, with many conceivable applications [104]. There still remains a lot of material to be properly understood in the series of papers on realisations of Lie algebras, that could potentially be also generalised to the superalgebra case and even beyond.

There is no doubt that Havlíček’s work has deeply influenced the younger generations of physicists in the Czech Republic and abroad, as well as many specialists working on group theory from the physical perspective, such as the writer of these lines, inspiring further investigations in several of the problems where he proposed an elegant and effective ansatz.

We would like to finish this short reminiscence with a sentence that, to the author’s understanding, perfectly summarises the influence of Havlíček’s work and personality in Prague and abroad: “Miloslavův odkaz žije dál”.

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