Acta Polytechnica **65**(5):554–561, 2025
https://doi.org/10.14311/AP.2025.65.0554

Published by the Czech Technical University in Prague

ON REALIZATIONS OF LIE ALGEBRAS

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Abstract. We discuss and compare the main methods for constructing of Lie vector fields from the given Lie algebra structure constants. Generic realizations of three conformal algebras are obtained by the algebraic method and all realizations of the three-dimensional complex special linear algebra are obtained by the general method and compared with the finite-dimensional weight representations.

Keywords: Realization, representation, local group.

1. Introduction

Despite the fact that Sophus Lie himself began constructing realizations, this problem still remains unsolved for many important cases. The description of Lie algebra representations by vector fields is of great interest and widely applicable, e.g. to the integration of ordinary differential equations. Some new trends in this area are indicated in [1, 2]. Realizations are also used in the group classification of partial differential equations and in the classification of gravity fields of a general form under the motion groups or groups of conformal transformations [3]. The construction of realizations of Lie algebras is also a necessary prerequisite for finding differential invariants and constructing mathematical models with nontrivial symmetry. An exhaustive description of nonequivalent realizations of given Lie algebras by vector fields is also an independent fundamental mathematical problem.

In this work, we review the main methods of realization construction and apply them to special linear and conformal Lie algebras. In particular, we present the realization of $\mathfrak{sl}(2,\mathbb{C})$ that is not equivalent to a finite-dimensional representation.

The paper is arranged as follows. We first review the basic definitions and notations in Section 2, then in Section 3, we describe and compare the main methods representing an abstract Lie algebra by vector fields. In Section 4, we compare realizations of $\mathfrak{sl}(2,\mathbb{C})$ constructed by the direct method and from the weight representations. And, in Section 5, we apply the Shirokov's method to three conformal Lie algebras, with the resulting generic realizations being presented in Appendix A.

2. Definitions and statement of the problem

Consider a Lie algebra $\mathfrak{g} = (V, [\cdot,\ \cdot])$, where V is an n-dimensional complex or real vector space with a bilinear antisymmetric operation $[\cdot,\ \cdot]\colon V\times V\to V$ that satisfies the Jacobi identity, and is usually called a Lie bracket or a commutator.

Fixing the basis $e_1,\dots,e_n$ of V, we can define the Lie algebra $\mathfrak{g}$ by its commutation relations:

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k,$$

or by the structure constant tensor c, with the components $c_{ij}^k\in\mathbb{C}$ or $c_{ij}^k\in\mathbb{R}$.

Hereafter, we assume that the indices $i, j, k, \tilde{i}, \tilde{j}$ and $\tilde{k}$ run from 1 to n, and we will imply the summation over the repeating indices.

The general linear group acts on the variety of n-dimensional Lie algebras as follows. Let $A\in\mathrm{GL_n}(V)$ and $B = A^{-1}$, then the components of the initial structure constant tensor c and the resulting tensor $\tilde{c}$ are connected by the formula:

$$\tilde{c}_{\tilde{i}\tilde{j}}^{\tilde{k}} = A_{\tilde{i}}^{i} A_{\tilde{j}}^{j} B_{k}^{\tilde{k}} c_{ij}^{k}.$$

Denote the whole automorphism group of $\mathfrak{g}$ by $\mathrm{Aut}(\mathfrak{g})\subseteq\mathrm{GL_n(V)}$ and the group of the inner automorphisms by $\mathrm{Inn}(\mathfrak{g})$.

In this work, we mostly follow the definitions proposed in [4] with some minor modern modifications and generalization to the case of complex field. Note that we work *only locally.* The definitions given below are similar for the field of real and complex numbers, but differ in some details.

Let $M\subset\mathbb{R}^m$, $m\in\mathbb{N}$ be an m-dimensional smooth manifold and Vect(M) denote the Lie algebra of smooth vector fields on M.

Definition 1. A realization *of Lie algebra* $\mathfrak{g}$ *in vector fields on* M *is a homomorphism* $R\colon\mathfrak{g}\to\mathrm{Vect}(M)$.

The realization is called faithful if $\ker R = \{0\}$, *and unfaithful otherwise.*

Loosely speaking, we consider Lie algebras of homogeneous first-order differential operators:

$$\xi^1(x)\frac{\partial}{\partial x_1}+\xi^2(x)\frac{\partial}{\partial x_2}+\cdots+\xi^m(x)\frac{\partial}{\partial x_m},$$

with the coefficients ξ^α that are smooth functions on M, hereafter indices α,β and γ run from 1 to m, $\partial_\alpha=\partial_{x_\alpha}=\frac{\partial}{\partial x_\alpha}$ and $x=(x_1,x_2,\ldots,x_m)$.

In the case of a complex Lie algebra, we denote a domain of $\mathbb{C}^m$ as M and $\mathrm{Vect}(M)$ are the vector fields on M with analytical coefficients. The formulation of the realization definition is the same.

The main problem considered in this work is the construction of all possible realizations of an abstract Lie algebra $\mathfrak{g}$ given by its commutation relations. Note that for the fixed Lie algebra $\mathfrak{g}$, it is reasonable to look for the faithful realizations only, since all unfaithful realizations can be investigated as faithful realizations of some lower-dimensional algebra.

As it follows from the definition, realization is a partial case of representations and we will discuss connections between realizations and weight representations in the next section.

To find exhaustive lists, we need to define which realizations of a given Lie algebra we will consider different, or, conversely, equivalent.

Definition 2 ($\mathbb{R}$). *The realizations $R_1\colon \mathfrak{g}\to\mathrm{Vect}(M_1)$ and $R_2\colon \mathfrak{g}\to\mathrm{Vect}(M_2)$ are called weakly equivalent if there exist $\varphi\in\mathrm{Aut}(\mathfrak{g})$ and a diffeomorphism $f\colon M_1\to M_2$ such that $R_2(v)=f_*\,R_1(\varphi(v))$ for all $v\in\mathfrak{g}$. Here, f_* is the pushforward from $\mathrm{Vect}(M_1)$ to $\mathrm{Vect}(M_2)$.*

If φ is the identical transformation, the realizations are called strongly equivalent.

Definition 3 ($\mathbb{C}$). *The realizations $R_1\colon \mathfrak{g}\to\mathrm{Vect}(M_1)$ and $R_2\colon \mathfrak{g}\to\mathrm{Vect}(M_2)$ are called weakly equivalent if there exist $\varphi\in\mathrm{Aut}(\mathfrak{g})$ and a biholomorphic mapping $f\colon M_1\to M_2$ such that $R_2(v)=f^*\,R_1(\varphi(v))$ for all $v\in\mathfrak{g}$. Here, f^* is the pullback from $\mathrm{Vect}(M_1)$ to $\mathrm{Vect}(M_2)$.*

If φ is the identical transformation, the realizations are called strongly equivalent.

The strong equivalence is verified in a simpler way than the weak one and it can be used for construction of realizations using realizations of subalgebras, but in general case, we should classify all realizations of the given Lie algebra with respect to the weak equivalence. However, the algebraic method of constructing realizations (Shirokov's method) gives grounds to consider the hypothesis that for two weakly equivalent realizations, it is possible to construct a non-degenerate change of variables that transforms them from one to another, at least this hypothesis may hold when the full automorphism group coincides with the group of inner automorphisms.

3. Construction methods

There are two main approaches to the construction and classification of realizations of Lie algebras:

(1.) Construction of basis vector fields that satisfy the given structure constants of the Lie algebra;

(2.) construction of finite-dimensional spaces of vector fields closed with respect to a multiplication (commutation) that satisfies the definition of a Lie bracket.

Regarding the second approach, the classification of realizations was started by Sophus Lie himself and was done for the case of one variable over real and complex fields and for two variables in the complex case. Within the framework of the second approach, there are several interesting methods for constructing realizations, in particular for nilpotent Lie algebras. Certain results have also been obtained for spaces of a small number of variables, but this approach is not relevant to the problem we are considering in this paper, so we will not provide a detailed overview of the methods and results.

3.1. The direct method

As for the first approach to constructing realizations of Lie algebras, the most obvious is *the direct method* (which is technically quite complex). First, it was formulated in [4], here we present it in detail since this is the only general method known to us today. The direct method consists of three main steps:

(1.) Take n linearly independent vector fields of the general form $e_i=\xi^{i\alpha}(x)\partial_\alpha$, and require them to satisfy the given commutation relations of $\mathfrak{g}$.

(2.) By comparing coefficients near different partial differentiation operators, we obtain a system of first-order PDEs for the coefficients ξ^{ia}. Integrate this system considering all the possible cases.

(3.) Transform the solution into the simplest form, using nondegenerate transformations of the coordinates on M and automorphism transformations of $\mathfrak{g}$.

The direct method is quite difficult to apply, primarily due to the need to solve a system of partial differential equations and the requirement to check a large number of branching cases. But we can propose several approaches that make the procedure less cumbersome. First, we formulate the ideas, then we introduce all the definitions and formulate the necessary statements:

- To transform one of basis elements to the shift operator, for example, $e_1=\frac{\partial}{\partial x_1}$. Note that the choice of the basis element that we transform to the shift operator significantly affects the subsequent complexity of the calculations and the form of the operators that we obtain as a result.
- To classify sequential realizations of a series of nested subalgebras of $\mathfrak{g}$, starting with a one-dimensional subalgebra and ending with $\mathfrak{g}$. Inequiva-

lence of the obtained realizations can be guaranteed if we consider a chain of *megaideals*.

- To split all possible cases into the groups with the different *realization ranks*.

Let us fix $x \in M$ and let R_x be a realization of $\mathfrak{g}$ in this point. Consider the linear map $R_x \colon \mathfrak{g} \to \mathrm{Vect}(M)(x)$. The matrix that corresponds to this linear map is the n by m matrix ξ formed by the coefficients of the realization:

$$\xi(x) = \begin{pmatrix} \xi^{11}(x) & \xi^{12}(x) & \dots & \xi^{1m}(x) \\ \xi^{21}(x) & \xi^{22}(x) & \dots & \xi^{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^{n1}(x) & \xi^{n2}(x) & \dots & \xi^{nm}(x) \end{pmatrix}.$$

Definition 4. *The general (maximal possible) rank of the linear map R_x is called a* rank of realization R *and is denoted as rank R.*

The realization rank value possesses the obvious inequality $0 \le rank\, R_x \le n$, where n is the dimension of the Lie algebra $\mathfrak{g}$. The second inequality is dictated by the number of rows in matrix ξ, which is equal to the number of basis vector fields of $\mathfrak{g}$.

If there exists a subset $\mathfrak{g}_0 \subset \mathfrak{g}$ such that $rank\, R_1(\mathfrak{g}_0) \ne rank\, R_2(\mathfrak{g}_0)$, then the realizations R_1 and R_2 are strongly inequivalent.

Definition 5. *A megaideal $\mathfrak{m}$ of $\mathfrak{g}$ is such a vector subspace $\mathfrak{m} \subset \mathfrak{g}$, that it is invariant under any transformation from* $\mathrm{Aut}(A)$.

It is clear that any megaideal is a subalgebra and, moreover, an ideal in $\mathfrak{g}$. But there exist ideals which are not megaideals. Moreover, any megaideal is invariant with respect to all the derivations, i.e. it is a characteristic subalgebra.

Let $\mathfrak{m}$ be a megaideal and R_1 and R_2 be realizations of the algebra $\mathfrak{g}$. If $R_1\big|_{\mathfrak{m}}$ and $R_2\big|_{\mathfrak{m}}$ are inequivalent, then R_1 and R_2 are inequivalent too.

Moreover, if there exists a megaideal $\mathfrak{m}$ of $\mathfrak{g}$ such that $rank\, R_1(\mathfrak{m}) \ne rank\, R_2(\mathfrak{m})$ then the realizations R_1 and R_2 are weakly inequivalent.

The notion of megaideal allows us to construct realizations starting from the known lists of realizations for the low-dimensional algebras, but only in the case when we can find a nested chain of megaideals. Such a chain of megaideals cannot be constructed for simple Lie algebras, so constructing realizations for them is a particularly difficult task.

For low-dimensional Lie algebras, the direct method is quite effective and has allowed us to describe realizations of real Lie algebras of dimensions no higher than four [4].

3.2. Blattner's method

Let us consider the method proposed by Blattner [5] in 1969. It constructs transitive realizations of Lie algebras starting from their structure constants. For further explanation, we need to define transitive realizations.

By the third Lie theorem, there exists a local n-parametric transformation group G corresponding to the vector fields e_i, i.e. e_i are infinitesimal generators of the action of the group G on M. Note that such a correspondence between the group and the Lie algebra is given by the tangent space at the unit of the group (the zero of the Lie algebra). A Lie group G can act on M in several different ways: primitively (only one orbit is formed when G acts on M) or imprimitively (M is split into several orbits); transitively (orbits may exist but there are no stationary points), and intransitively (G decomposes M into orbits that are invariant under the action of G). If the local group G corresponding to a realization acts transitively on M, then the realization is also called *transitive*.

Let $\{e_1, \dots, e_k, \epsilon_1, \dots, \epsilon_m\}$ be the basis of a finite-dimensional Lie algebra $\mathfrak{g}$, where $\mathfrak{s} = \langle e_1, \dots, e_k \rangle$ is a subalgebra of co-dimension m, and $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$. Then, using the set of vectors $\{\epsilon_1, \dots, \epsilon_m\}$ complementary to the subalgebra $\mathfrak{s}$, we can construct the transformation φ_ϵ, defined for $v \in \mathfrak{g}$, that gives a transitive realization of the algebra $\mathfrak{g}$ by the formula:

$$\varphi_\epsilon(v) = \sum_{i=1}^{m} \left(\sum_{l \in \mathbb{N}^m} \boldsymbol{\chi}_i(\epsilon^l v) \frac{x^l}{l!} \right) \partial_i,$$

where l is multi-index and $l! := \prod_{i=1}^{m} l_i!$. Here $\epsilon^l v \in U(\mathfrak{g})$, and $\boldsymbol{\chi}_i$ — the Poincaré–Birkhoff–Witt coefficient before the monomial ϵ_i. A Poincaré–Birkhoff–Witt coefficient is the scalar that appears when a product of Lie algebra generators is rewritten in terms of the ordered PBW basis of the universal enveloping algebra. It is the coefficients $\boldsymbol{\chi}_i$ that significantly complicate the application of the Blattner's formula, since no regular procedure has been found for their calculation.

3.3. Shirokov's method

Let's consider another approach that is algebraic and does not require solving differential equations. This method was first proposed in 1997 by I. Shirokov for left-invariant vector fields, and it was extended to the transitive case in 2013 [6]. The main ideas of this method are to construct vector fields as duals (inverses) to differential one-forms, and to construct differential one-forms using adjoint representations.

Let the local Lie group that corresponds to $\mathfrak{g}$ is parametrized by the canonical coordinates of the second kind, i.e., group elements are represented as ordered products of exponentials, each corresponding to a separate basis element of $\mathfrak{g}$. In this case, we can construct components $\omega_i^j(x)$ of differential one-forms by the formula:

$$\omega_i^j(x) = \left(\exp(-x_1 \mathrm{ad}_{e_1}) \cdots \exp(-x_i \mathrm{ad}_{e_i})\right)_i^j.$$

Now the problem of constructing left-invariant vector fields reduces to finding the inverse transformation $\xi_k^i(x) = \left((\omega_l^j(x))^{-1}\right)_k^i$. Note that this construction of realization corresponds to the primitive action of the local group, we will call such a realization *generic*. All other realizations that correspond to the transitive group action can be obtained as projections of the generic realizations to the spaces of variables complementary to subalgebras. In the case of a non-transitive action of the corresponding local group, the method becomes more complicated. It was discussed in the thesis [7] of D. Gromada, but, for practical application, it is necessary to find the classification of subalgebras that satisfies certain conditions. This subtask is rather complicated itself.

Nevertheless, this algorithm is extremely efficient in constructing generic realizations and can be successfully applied to physically interesting high-dimensional algebras. Appendix A contains generic realizations of three important conformal Lie algebras $\mathfrak{c}(3,1)$, $\mathfrak{c}(3,0)$ and $\mathfrak{c}(2,1)$, which are constructed using the algebraic method.

3.4. Representations and realizations with linear coefficients

For each Lie algebra, there are always two representations: trivial and adjoint. Trivial representation has no essential applications but adjoint representation can always give us the realization with linear coefficients. Indeed, if the Lie algebra $\mathfrak{g}$ has a representation φ with the corresponding $(m \times m)$-matrices Φ_i, then the set of operators:

$$e_i = \sum_{\alpha=1}^{m} \left(\sum_{\beta=1}^{m} (\Phi_i)_\alpha^\beta x_\beta \right) \partial_\alpha$$

generates the realization $\mathfrak{g}$ and vice versa: any realization with linear homogeneous coefficients generates the matrix representation of the Lie algebra.

This method of constructing realizations is particularly interesting for the case of simple and semisimple Lie algebras, since it is difficult to apply the direct method to them due to the lack of megaideals. It is also difficult to classify subalgebras of simple algebras for the application of the Shirokov method, but, at the same time, weighted representations are known for simple Lie algebras.

It is clear that some realizations with non-linear coefficients will be equivalent to those obtained from representations, but the more interesting question is whether all realizations can be obtained from the representations. To investigate this connections between realizations and representations, we considered the smallest simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ in Section 4. Note that this case has been studied by many authors in the past, for example [8].

4. Realizations and representations of $\mathfrak{sl}(2,\mathbb{C})$

Consider the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of 2×2 traceless complex matrices, the standard choice of basis is:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

$$[e,\ f] = h, \quad [h,\ e] = 2e, \quad [f,\ h] = 2f. \tag{2}$$

Let us also consider the basis e_1, e_2, e_3, where:

$$e = e_1, \quad h = -2e_2, \quad f = -e_3,$$

and commutation relations have the form:

$$[e_1,\ e_2] = e_1, \quad [e_2,\ e_3] = e_3, \quad [e_1,\ e_3] = 2e_2. \tag{3}$$

Irreducible finite-dimensional weight representations of $\mathfrak{sl}(2,\mathbb{C})$ in the basis $e,\ f,\ h$ are:

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & d \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad \Phi_2 = \frac{1}{2} \begin{pmatrix} -d & 0 & \cdots & 0 \\ 0 & -d+2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & d \end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -d & 0 & \cdots & 0 & 0 \\ 0 & -d+1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

The realizations that correspond to these representations are ($d \in \mathbb{N}$):

$$e_1 = \sum_{\alpha=2}^{d+1} (\alpha - 1) x_{\alpha-1} \partial_\alpha,$$

$$e_2 = \sum_{\alpha=1}^{d+1} \left(-\frac{d}{2} - 1 + \alpha \right) x_\alpha \partial_\alpha,$$

$$e_3 = \sum_{\alpha=1}^{d} (-d + \alpha - 1) x_{\alpha+1} \partial_\alpha.$$

Applying the direct method we obtain the exhaustive list of inequivalent realizations of $\mathfrak{sl}(2,\mathbb{C})$:

(1.) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2 + x_2\partial_3$,

(2.) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ (x_1^2 + x_2^2)\partial_1 + 2x_1x_2\partial_2$,

(3.) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2$,

(4.) $\partial_1,\ x_1\partial_1,\ x_1^2\partial_1$.

Note that this list contains one less realization than the similar list obtained for the case of real numbers [4].

To establish the correspondence between the realizations and representations, we consider different values of $d = 1$, $d = 2$, $d = 3$, … transform one of the operators to the form ∂_{x_1} and look for the locally invertible transformations for the rest of the variables. We also compare the ranks of the realizations. In this way we have shown that the case $d = 1$ is transformed to the realization (3.) by the change of variables $\tilde{x}_1 = \frac{x_1}{x_2}$, $\tilde{x}_2 = \frac{1}{x_1^2}$.

And the case $d = 2$ is transformed to the realization (2.) by the change of variables $\tilde{x}_1 = \frac{x_2+1}{x_1} - x_2^2 + x_1x_3$, $\tilde{x}_2 = x_1^{-\frac{1}{2}}\left(x_2^2 - x_1x_3\right)$, $\tilde{x}_3 = \frac{1}{x_1}$.

The cases $d \geq 3$ are equivalent to the realization (1.) since all of them are of the rank three.

The only realization that was not obtained from the irreducible weight representations is the realization of the rank one (4.).

To investigate whether it is possible to linearize this realization we look for non-degenerate transformations $\tilde{x}_\alpha = f_\alpha(x)$, $\alpha = 1, \ldots, m$, $x = (x_1, x_2, \ldots, x_m)$ such that for some complex matrices a, b and c:

$$\begin{aligned}
\tilde{e}_1 &= (a_{\alpha 1}\tilde{x}_1 + \cdots + a_{\alpha m}\tilde{x}_m)\partial_{\tilde{x}_\alpha},\\
\tilde{e}_2 &= (b_{\alpha 1}\tilde{x}_1 + \cdots + b_{\alpha m}\tilde{x}_m)\partial_{\tilde{x}_\alpha},\\
\tilde{e}_3 &= (c_{\alpha 1}\tilde{x}_1 + \cdots + c_{\alpha m}\tilde{x}_m)\partial_{\tilde{x}_\alpha}.
\end{aligned}$$

Solving the obtained system of PDEs, we come to the functional system that is linear and homogeneous with respect to the functions $f_\alpha(x)$, therefore, the Jacobian determinant is zero and non-degenerate transformations do not exist.

This simple example allows us to draw an important conclusion that irreducible representations do not allow us to obtain all differential equations that are invariant under a given group or Lie algebra.

5. Realizations of conformal algebras

Consider three important conformal groups: the standard conformal group $\mathrm{C}(3,1)$ and two conformal groups of pseudo-Euclidean spaces $\mathrm{C}(3,0)$ and $\mathrm{C}(2,1)$. Their Lie algebras are denoted $\mathfrak{c}(3,1)$, $\mathfrak{c}(3,0)$ and $\mathfrak{c}(2,1)$, respectively. Some covariant realizations of conformal algebras and de Sitter algebras are already known, but we constructed realizations in spaces of fifteen and ten variables, respectively. This is the highest possible dimensions of spaces of essential variables for these algebras. By essential variables we mean variables that cannot be eliminated through a nondegenerate change of variables, i.e., they cannot be replaced by invariants of vector fields. Realizations with fewer variables, in particular "classical" realizations, can be obtained from the presented realizations by a projection to the spaces of lower dimensions.

The 15-dimensional Lie algebra $\mathfrak{c}(3,1)$ of the conformal group is the Lie algebra of the maximal invariance group of the Maxwell's equations in flat space-time. This group, from many points of view, unites all physical groups (Lorentz, Poincaré, de Sitter, orthogonal, etc.). It is generated by ten Poincaré operators P_μ, $J_{\mu\nu}$, dilation operator D and special conformal transformation operators K_μ, where $\mu, \nu = 1, 2, 3, 4$.

The nonzero commutation relations of the Lie algebra have the form:

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= g_{\mu\rho}J_{\nu\sigma} - g_{\nu\rho}J_{\mu\sigma}\\
&\quad + g_{\mu\sigma}J_{\rho\nu} - g_{\nu\sigma}J_{\rho\mu}, &(4)\\
[J_{\mu\nu}, P_\rho] &= g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu, &(5)\\
[J_{\mu\nu}, K_\rho] &= g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu, &(6)\\
[P_\mu, K_\nu] &= 2(g_{\mu\nu}D + J_{\mu\nu}), &(7)\\
[P_\mu, D] &= P_\mu, &(8)\\
[K_\mu, D] &= -K_\mu. &(9)
\end{aligned}$$

here $g_{\mu\nu}$ is the metric tensor of Minkowski space $g_{11} = g_{22} = g_{33} = -g_{44} = 1$.

If we consider the well-known realization (10) of the conformal Lie algebra, we can see that it is the projection of the generic realization that corresponds to the subalgebra $\operatorname{span}\{J_{\mu\nu}, D, K_\mu\}$ with the complementary part $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$:

$$\begin{aligned}
P_\mu &= \partial_\mu, & J_{\mu\nu} &= x_\nu\partial_\mu - x_\mu\partial_\nu,\\
D &= x_\nu\partial_\nu, & K_\mu &= 2x_\mu x_\nu\partial_\nu - x^2\partial_\mu;
\end{aligned} \qquad (10)$$

where $x^2 = x_1^2 + \cdots + x_n^2$.

We also consider de Sitter algebras that correspond to the transformation groups of isometry of pseudo-Euclidean spaces with the metric forms $x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2$ and $x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2$. They are the groups of motion of 4-dimensional Riemannian spaces of constant curvature (de Sitter spaces). Both de Sitter spaces describe an expanding universe where the radial velocities of galaxies are proportional to the distances to points in space. For de Sitter algebras, we can use isomorphisms $\mathfrak{c}(3,0) \sim \mathfrak{so}(4,1)$ and $\mathfrak{c}(2,1) \sim \mathfrak{so}(3,2)$ with conformal commutation relations (4)–(9) for $\mu, \nu = 1, 2, 3$ and metric tensors $g_{11} = g_{22} = g_{33} = 1$ and $g_{11} = g_{22} = -g_{33} = 1$, respectively. Below, we present all their generic realizations that we obtained by the Shirokov's method with the complementary part $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$ in other words, this means that the basis elements are ordered exactly in the same way as they are presented in the appendices. The theory allows us to construct one realization for the both de Sitter algebras by parametrizing the structure constants with a parameter that changes the sign, but this significantly complicates the calculations and the appearance of the realizations.

In the future, we plan to construct projections of the obtained generating realizations onto spaces of lower dimensions, find differential invariants for them, and write invariant partial differential equations.

Acknowledgements

This work was supported by a grant from the Simons Foundation (SFI-PD-Ukraine-00014586, M.N., M.S.).

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Appendix A.

In this appendix we present generic realizations of Lie algebras of three important conformal groups: the standard conformal group C(3, 1) and two conformal groups of pseudo-Euclidean spaces C(3, 0) and C(2, 1).

A.1. Generic realization of the conformal algebra $\mathfrak{c}(3,1)$

To simplify the form of the formulas, we introduce the notations:

$$\sin x_i = \mathrm{s}_i, \quad \cos x_i = \mathrm{c}_i, \quad \tan x_i = \mathrm{t}_i, \quad \sinh x_i = \mathrm{sh}_i, \quad \cosh x_i = \mathrm{ch}_i, \quad \tanh x_i = \mathrm{th}_i, \quad i = 1, \ldots, 10.$$

$R_{\text{generic}}(\mathfrak{c}(3,1))$:

$$\begin{aligned}
P_1 &= \partial_1, \\
P_2 &= \partial_2, \\
P_3 &= \partial_3, \\
P_4 &= \partial_4, \\
J_{12} &= x_2\partial_1 - x_1\partial_{x_2} + \partial_5, \\
J_{13} &= x_3\partial_1 - x_1\partial_3 - \mathrm{th}_6\mathrm{s}_5\partial_5 + \mathrm{c}_5\partial_6 + 2\frac{\mathrm{s}_5}{\mathrm{ch}_6}\partial_8, \\
J_{14} &= -x_4\partial_1 - x_1\partial_4 + 2\frac{\mathrm{t}_7\mathrm{s}_5}{\mathrm{ch}_6}\partial_5 + \mathrm{t}_7\mathrm{s}_6\mathrm{c}_5\partial_6 + \mathrm{c}_5\mathrm{c}_6\partial_7 + \frac{\mathrm{s}_9\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8 + \mathrm{th}_6\mathrm{s}_7\mathrm{c}_9\mathrm{s}_5 + \mathrm{s}_9\mathrm{s}_8\mathrm{s}_5}{\mathrm{c}_7\mathrm{c}_9}\partial_8 \\
&\quad - \frac{\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 - \mathrm{s}_5\mathrm{c}_8}{\mathrm{c}_7}\partial_9 + \frac{\mathrm{s}_5\mathrm{s}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{c}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}, \\
J_{23} &= x_3\partial_2 - x_2\partial_3 + -\mathrm{th}_6\mathrm{c}_5\partial_5 - \mathrm{s}_5\partial_6 + \frac{\mathrm{c}_5}{\mathrm{ch}_6}\partial_8, \\
J_{24} &= -x_4\partial_2 - x_2\partial_4 + \frac{\mathrm{t}_7\mathrm{c}_5}{\mathrm{c}_7\mathrm{ch}_6}\partial_5 - \mathrm{t}_7\mathrm{s}_5\mathrm{s}_6\partial_6 - \mathrm{s}_5\mathrm{c}_6\partial_7 + \frac{\mathrm{th}_6\mathrm{s}_7\mathrm{c}_9\mathrm{c}_5 - \mathrm{s}_9\mathrm{s}_6\mathrm{c}_8\mathrm{s}_5 + \mathrm{s}_9\mathrm{c}_5\mathrm{s}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_8 \\
&\quad + \frac{\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8}{\mathrm{c}_7}\partial_9 - \frac{\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}, \\
J_{34} &= -x_4\partial_3 - x_3\partial_4 + \mathrm{c}_6\mathrm{t}_7\partial_6 - \mathrm{s}_6\partial_7 + \frac{\mathrm{t}_9\mathrm{c}_8\mathrm{c}_6}{\mathrm{c}_7}\partial_8 - \frac{\mathrm{s}_8\mathrm{c}_6}{\mathrm{c}_7}\partial_9 + \frac{\mathrm{c}_6\mathrm{c}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}, \\
K_1 &= \left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 + 2x_1x_4\partial_4 \\
&\quad - 2\left(x_2 + \frac{x_4\mathrm{t}_7\mathrm{s}_5}{\mathrm{ch}_6} - x_3\mathrm{th}_6\mathrm{s}_5\right)\partial_5 - 2(x_4\mathrm{t}_7\mathrm{s}_6 + x_3)\mathrm{c}_5\partial_6 - 2x_4\mathrm{c}_5\mathrm{c}_6\partial_7 \\
&\quad - 2\left(\frac{x_4\mathrm{t}_9\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8}{\mathrm{c}_7} + x_4\mathrm{t}_7\mathrm{th}_6\mathrm{s}_5 + \frac{x_3\mathrm{s}_5}{\mathrm{ch}_6} + \frac{x_4\mathrm{t}_9\mathrm{s}_5\mathrm{s}_8}{\mathrm{c}_7}\right)\partial_8 + 2\frac{(\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 - \mathrm{s}_5\mathrm{c}_8)x_4}{\mathrm{c}_7}\partial_9 \\
&\quad - 2\frac{(\mathrm{s}_5\mathrm{s}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{c}_8)x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} - (2x_1x_{11} - \mathrm{ch}_7\mathrm{c}_5\mathrm{c}_6)\partial_{11} + (\mathrm{sh}_7\mathrm{sh}_9\mathrm{c}_5\mathrm{c}_6 + \mathrm{ch}_9(\mathrm{s}_5\mathrm{c}_8 - \mathrm{s}_6\mathrm{s}_8\mathrm{c}_5) - 2x_1x_{12})\partial_{12} \\
&\quad + (\mathrm{sh}_9\mathrm{sh}_{10}(\mathrm{s}_5\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_6\mathrm{s}_8) + \mathrm{ch}_{10}(\mathrm{s}_6\mathrm{c}_8\mathrm{c}_5 + \mathrm{s}_5\mathrm{s}_8) + \mathrm{sh}_7\mathrm{ch}_9\mathrm{sh}_{10}\mathrm{c}_5\mathrm{c}_6 - 2x_1x_{13})\partial_{13} \\
&\quad + (\mathrm{sh}_9\mathrm{ch}_{10}(\mathrm{s}_5\mathrm{c}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{s}_8) + \mathrm{sh}_{10}(\mathrm{s}_6\mathrm{c}_8\mathrm{c}_5 + \mathrm{s}_5\mathrm{s}_8) + \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{c}_5\mathrm{c}_6 - 2x_1x_{14})\partial_{14} + 2x_1\partial_{15}, \\
K_2 &= 2x_1x_2\partial_1 + \left(-x_1{}^2 + x_2{}^2 - x_3{}^2 + x_4{}^2\right)\partial_{x_2} + 2x_2x_3\partial_3 \\
&\quad + 2x_2x_4\partial_4 + 2\left(\frac{x_4\mathrm{c}_5\mathrm{t}_7}{\mathrm{ch}_6} - x_1 - x_3\mathrm{c}_5\mathrm{th}_6\right)\partial_5 + 2(x_3 + x_4\mathrm{t}_7\mathrm{s}_6)\mathrm{s}_5\partial_6 + 2x_4\mathrm{s}_5\mathrm{c}_6\partial_7 \\
&\quad + 2\left(\frac{x_4\mathrm{t}_9}{\mathrm{c}_7}(\mathrm{s}_6\mathrm{c}_8\mathrm{s}_5 - \mathrm{c}_5\mathrm{s}_8) - x_4\mathrm{th}_7\mathrm{c}_5 - \frac{x_3\mathrm{c}_5}{\mathrm{ch}_6}\right)\partial_8 - 2\frac{(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8)x_4}{\mathrm{c}_7}\partial_9 + 2\frac{(\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_8)x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} \\
&\quad - (2x_2x_{11} + \mathrm{ch}_7\mathrm{s}_5\mathrm{c}_6)\partial_{11} + \big(\mathrm{ch}_9(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8) - \mathrm{sh}_7\mathrm{sh}_9\mathrm{s}_5\mathrm{c}_6 - 2x_2x_{12}\big)\partial_{12} \\
&\quad + \big(\mathrm{sh}_9\mathrm{ch}_{10}\mathrm{c}_5\mathrm{c}_8 + \mathrm{ch}_{10}\mathrm{c}_5\mathrm{s}_8 - \mathrm{sh}_7\mathrm{sh}_{10}\mathrm{ch}_9\mathrm{s}_5\mathrm{c}_6 - \mathrm{ch}_{10}\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 + \mathrm{sh}_9\mathrm{sh}_{10}\mathrm{s}_5\mathrm{s}_6\mathrm{s}_8 - 2x_2x_{13}\big)\partial_{13} \\
&\quad + \big(\mathrm{sh}_9\mathrm{ch}_{10}(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{c}_8) + \mathrm{sh}_{10}(\mathrm{s}_8\mathrm{c}_5 - \mathrm{s}_5\mathrm{s}_6\mathrm{c}_8) - \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{s}_5\mathrm{c}_6 - 2x_2x_{14}\big)\partial_{14} + 2x_2\partial_{15}, \\
K_3 &= 2x_1x_3\partial_1 + 2x_2x_3\partial_{x_2} + \left(-x_1^2 - x_2^2 + x_3^2 + x_4^2\right)\partial_3 + 2x_3x_4\partial_4 \\
&\quad - 2\mathrm{th}_6(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)\partial_5 - 2(x_4\mathrm{t}_7\mathrm{c}_6 - x_1\mathrm{c}_5 + x_2\mathrm{s}_5)\partial_6 \\
&\quad + 2x_4\mathrm{s}_6\partial_7 + \left(\frac{x_1\mathrm{s}_5 + x_2\mathrm{c}_5}{\mathrm{ch}_6} - 2\frac{x_4\mathrm{t}_9\mathrm{c}_6\mathrm{c}_8}{\mathrm{c}_7}\right)\partial_8 + 2\frac{\mathrm{s}_8\mathrm{c}_6x_4}{\mathrm{c}_7}\partial_9 - 2\frac{\mathrm{c}_6\mathrm{c}_8x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} - (\mathrm{ch}_7\mathrm{s}_6 + 2x_3x_{11})\partial_{11} \\
&\quad - (\mathrm{ch}_9\mathrm{c}_6\mathrm{s}_8 + 2x_3x_{12} + \mathrm{sh}_7\mathrm{sh}_9\mathrm{s}_6)\partial_{12} + \big(\mathrm{ch}_{10}\mathrm{c}_6\mathrm{c}_8 - \mathrm{sh}_9\mathrm{sh}_{10}\mathrm{c}_6\mathrm{s}_8 - \mathrm{sh}_7\mathrm{ch}_9\mathrm{sh}_{10}\mathrm{s}_6 - 2x_3x_{13}\big)\partial_{13} \\
&\quad + \big(\mathrm{ch}_{10}\mathrm{c}_6\mathrm{c}_8 - \mathrm{sh}_9\mathrm{ch}_{10}\mathrm{c}_6\mathrm{s}_8 - \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{s}_6 - 2x_3x_{14}\big)\partial_{14} + 2x_3\partial_{15},
\end{aligned}$$

$$
\begin{aligned}
K_4 = &- 2x_1x_4\partial_1 - 2x_2x_4\partial_{x_2} - 2x_4x_3\partial_3 - \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)\partial_4 \\
&+ 2\frac{\mathrm{t}_7(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)}{\mathrm{ch}_6}\partial_5 + 2\mathrm{t}_7(x_1\mathrm{s}_6\mathrm{c}_5 - x_2\mathrm{s}_6\mathrm{s}_5 + x_3\mathrm{c}_6)\partial_6 \\
&- 2(x_2\mathrm{c}_6\mathrm{s}_5 - x_1\mathrm{c}_6\mathrm{c}_5 + x_3\mathrm{s}_6)\partial_7 \\
&+ 2\left(\frac{\mathrm{t}_9}{\mathrm{c}_7}(x_2\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - x_1\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8 - x_3\mathrm{c}_6\mathrm{c}_8 - x_1\mathrm{s}_5\mathrm{s}_8 - x_2\mathrm{s}_8\mathrm{c}_5) - \mathrm{th}_6\mathrm{t}_7(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)\right)\partial_8 \\
&- \frac{2}{\mathrm{c}_7}(x_2 - \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8 + x_1\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 + x_3\mathrm{c}_6\mathrm{s}_8 - x_1\mathrm{s}_5\mathrm{c}_8 - x_2\mathrm{c}_5\mathrm{c}_8)\partial_9 \\
&+ \frac{2}{\mathrm{c}_7\mathrm{c}_9}(x_1\mathrm{c}_5\mathrm{s}_6\mathrm{c}_8 - x_2\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 + x_3\mathrm{c}_6\mathrm{c}_8 + x_1\mathrm{s}_5\mathrm{s}_8 + x_2\mathrm{c}_5\mathrm{s}_8)\partial_{10} + (2x_4x_{11} + \mathrm{sh}_7)\partial_{11} \\
&+ (2x_4x_{12} + \mathrm{ch}_7\mathrm{sh}_9)\partial_{12} + (2x_4x_{13} + \mathrm{ch}_7\mathrm{ch}_9\mathrm{sh}_{10})\partial_{13} + (2x_4x_{14} + \mathrm{ch}_7\mathrm{ch}_9\mathrm{ch}_{10})\partial_{14} - 2x_4\partial_{15}, \\
D = &\; x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - x_{11}\partial_{11} - x_{12}\partial_{12} - x_{13}\partial_{13} - x_{14}\partial_{14} + \partial_{15}.
\end{aligned}
$$

A.2. Generic realizations of de Sitter algebras

$R_{\text{generic}}(\mathfrak{c}(3,0))$:

$$
\begin{aligned}
P_1 &= \partial_1, \\
P_2 &= \partial_2, \\
P_3 &= \partial_3, \\
J_{12} &= x_2\partial_1 - x_1\partial_2 + \partial_4, \\
J_{13} &= x_3\partial_1 - x_1\partial_3 - \mathrm{th}_5\mathrm{s}_4\partial_4 + \mathrm{c}_4\partial_5 + \frac{\mathrm{s}_4}{\mathrm{ch}_5}\partial_6, \\
J_{23} &= x_3\partial_2 - x_2\partial_3 - \mathrm{th}_5\mathrm{c}_4\partial_4 - \mathrm{s}_4\partial_5 + \frac{\mathrm{c}_4}{\mathrm{ch}_5}\partial_6, \\
K_1 &= \left(x_1^2 - x_2^2 - x_3^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 + 2(x_3\mathrm{s}_4\mathrm{th}_5 - x_2)\partial_4 - 2x_3\mathrm{c}_4\partial_5 - 2\frac{x_3\mathrm{s}_4}{\mathrm{ch}_5}\partial_6 \\
&\quad + (\mathrm{c}_4\mathrm{c}_5 - 2x_1x_7)\partial_7 + (\mathrm{s}_4\mathrm{ch}_6 - \mathrm{c}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_1x_8)\partial_8 + (\mathrm{c}_4\mathrm{s}_5\mathrm{c}_6 + \mathrm{s}_4\mathrm{s}_6 - 2x_1x_9)\partial_9 + 2x_1\partial_{10}, \\
K_2 &= 2x_1x_2\partial_1 + \left(x_2^2 - x_1^2 - x_3^2\right)\partial_2 + 2x_2x_3\partial_3 + 2(x_1 + x_3\mathrm{c}_4\mathrm{th}_5)\partial_4 + 2x_3\mathrm{s}_4\partial_5 - 2\frac{x_3\mathrm{c}_4}{\mathrm{ch}_5}\partial_6 \\
&\quad - (\mathrm{s}_4\mathrm{c}_5 + 2x_2x_7)\partial_7 + (\mathrm{s}_4\mathrm{s}_5\mathrm{s}_6 + \mathrm{c}_4\mathrm{c}_6 - 2x_8x_2)\partial_8 + (\mathrm{c}_4\mathrm{s}_6 - \mathrm{s}_4\mathrm{s}_5\mathrm{c}_6 - 2x_2x_9)\partial_9 + 2x_2\partial_{10}, \\
K_3 &= 2x_1x_3\partial_1 + 2x_2x_3\partial_2 + \left(x_3^2 - x_1^2 - x_2^2\right)\partial_3 - 2(x_1\mathrm{s}_4 + x_2\mathrm{c}_4)\mathrm{th}_5\partial_4 + 2(x_1\mathrm{c}_4 - x_2\mathrm{s}_4)\partial_5 \\
&\quad + 2\frac{x_1\mathrm{s}_4 + x_2\mathrm{c}_4}{\mathrm{ch}_5}\partial_6 - (\mathrm{s}_5 + 2x_3x_7)\partial_7 - (\mathrm{c}_5\mathrm{s}_6 + 2x_8x_3)\partial_8 + (\mathrm{c}_5\mathrm{c}_6 - 2x_9x_3)\partial_9 + 2x_3\partial_{10}, \\
D &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_7\partial_7 - x_8\partial_8 - x_9\partial_9 + \partial_{10}.
\end{aligned}
$$

$R_{\text{generic}}(\mathfrak{c}(2,1))$:

$$
\begin{aligned}
P_1 &= \partial_1, \\
P_2 &= \partial_2, \\
P_3 &= \partial_3, \\
J_{12} &= x_2\partial_1 - x_1\partial_2 + \partial_4, \\
J_{13} &= - x_3\partial_1 - x_1\partial_3 + \mathrm{s}_4\mathrm{t}_5\partial_4 + \mathrm{c}_4\partial_5 + \frac{\mathrm{s}_4}{\mathrm{c}_5}\partial_6, \\
J_{23} &= - x_3\partial_2 - x_2\partial_3 + \mathrm{c}_4\mathrm{t}_5\partial_4 - \mathrm{s}_4\partial_5 + \frac{\mathrm{c}_4}{\mathrm{c}_5}\partial_6, \\
K_1 &= \left(x_1^2 - x_2^2 + x_3^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 - 2(x_2 + x_3\mathrm{s}_4\mathrm{t}_5)\partial_4 - 2x_3\mathrm{c}_4\partial_5 - 2\frac{x_3\mathrm{s}_4}{\mathrm{c}_5}\partial_6 \\
&\quad + (\mathrm{c}_4\mathrm{ch}_5 - 2x_1x_7)\partial_7 + (\mathrm{s}_4\mathrm{ch}_6 + \mathrm{c}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_1x_8)\partial_8 + (\mathrm{s}_4\mathrm{sh}_6 + \mathrm{c}_4\mathrm{sh}_5\mathrm{ch}_6 - 2x_1x_9)\partial_9 + 2x_1\partial_{10}, \\
K_2 &= 2x_1x_2\partial_1 + \left(x_2^2 + x_3^2 - x_1^2\right)\partial_2 + 2x_2x_3\partial_3 - 2(x_3\mathrm{c}_4\mathrm{t}_5 - x_1)\partial_4 + 2x_3\mathrm{s}_4\partial_5 - 2\frac{x_3\mathrm{c}_4}{\mathrm{c}_5}\partial_6 \\
&\quad - (2x_2x_7 + \mathrm{s}_4\mathrm{ch}_5)\partial_7 + (\mathrm{c}_4\mathrm{ch}_6 - \mathrm{s}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_2x_8)\partial_8 + (\mathrm{c}_4\mathrm{sh}_6 - \mathrm{s}_4\mathrm{sh}_5\mathrm{ch}_6 - 2x_2x_9)\partial_9 + 2x_2\partial_{10}, \\
K_3 &= - 2x_1x_3\partial_1 - 2x_2x_3\partial_2 - \left(x_1^2 + x_2^2 + x_3^2\right)\partial_3 + 2(\mathrm{t}_5(x_1\mathrm{s}_4 + x_2\mathrm{c}_4)\partial_4 + 2(x_1\mathrm{c}_4 - x_2\mathrm{s}_4)\partial_5 \\
&\quad + 2\frac{x_1\mathrm{s}_4 + x_2\mathrm{c}_4}{\mathrm{c}_5}\partial_6 + (2x_3x_7 + \mathrm{sh}_5)\partial_7 + (\mathrm{ch}_5\mathrm{sh}_6 + 2x_3x_8)\partial_8 + (\mathrm{ch}_5\mathrm{ch}_6 + 2x_3x_9)\partial_9 - 2x_3\partial_{10}, \\
D &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_7\partial_7 - x_8\partial_8 - x_9\partial_9 + \partial_{10}.
\end{aligned}
$$