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DIRAC FERMION IN A TIME-DEPENDENT SPHERICAL BOX

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Abstract. We consider a Dirac particle in a spherical box with a time-dependent radius. Analytical and numerical solutions of the time-dependent Dirac equation with time-dependent boundary (Dirichlet) conditions are obtained. Using the obtained solutions, physically observable characteristics of the dynamical confinement, such as the average kinetic energy (as a function of time) and average quantum force acting on the particle by the moving wall, are calculated. The trembling motion is analysed by computing the average coordinate of the Dirac particle as a function of time. The absence of the geometric phase is shown by direct calculation.

Keywords: Dirac equation, ball with time-dependent radius, dynamical confinement, Dirichlet boundary condition.

1. Introduction

Dirac fermions appear in various topics of the contemporary physics. In early stages of quantum mechanics, they have been considered in the context of high energy physics and quantum field theory, however, during past two decades, quasiparticles described in terms of the Dirac equation attracted significant attention in condensed matter physics [1–5]. This happened due to the fact that in materials, such as graphene, topological insulators and Weyl semimetals, quasiparticles (electrons and holes), can "mimic" relativistic behaviour being described in terms of Dirac equation. Therefore, these materials are unified under the general term, "Dirac materials" [1–5]. Besides the Dirac materials, relativistic quasiparticles described in terms of the Dirac equation appear in some optical systems [6, 7]. In most of the cases, Dirac fermions appear in confined form and the boundary of the confining domain is not always static, i.e. changes in time. Such changes/fluctuations of the confinement boundary can affect the physical properties of system under consideration, as they cause changes in the boundary conditions for the quantum mechanical wave (e.g. Schrödinger, Dirac, etc.) equations. Such an effect leads to the important mathematical problem, solving the quantum evolution equations with time-dependent boundary conditions. In physics, this problem is called "dynamical quantum confinement". Earlier dynamical confinement attracted much attention in the context of nonrelativistic quantum mechanics described by Schrödinger equation (see, for example, the [8, 9], for review). Despite the considerable progress made in the study of dynamical confinement in nonrelativistic quantum mechanics, relativistic cases, especially the case of relativistic spin-half particles, are still not the focus of many studies. We note that dynamical confinement for relativistic scalar particles have been studied in the context of dynamical Casimir effect [10]. In this paper, we address the problem of dynamical confinement for spin-half Dirac fermions by considering a spherical confining domain, where the time dependence of the boundary does not break the spherical symmetry, so that the system is described in terms of the radial Dirac equation. We note that recently, an one-dimensional counterpart of such a problem was considered in the [9].

2. Dirac particle in a spherical box

The Dirac equation on confined domains has been studied in [11–16]. [11] presents a pioneering study of the Dirac equation in a hard-wall box, where physically relevant self-adjoint general boundary conditions were derived. One-dimensional Dirac equation in a quantum box has been studied in details in [12, 13]. The one-dimensional Dirac equation in time-varying domains was studied in [14], by considering special cases of the dynamical confinement and deriving an analytical solution to the problem. Practical applications of the Dirac equation to graphene quantum dots were considered in [16]. The dynamics of a Dirac particle under an one-dimensional dynamical confinement was studied in a recent paper [9]. Here, we consider dynamical confinement caused by a spherical box with a time-varying radius, described in terms of the radial Dirac equation with time-dependent boundary conditions.

We choose standard representation of the Dirac matrices according to [17] and follow the conventions for the spherical spinors used there. For the spherically symmetric situations, the Hamiltonian can be decomposed into a direct sum of parts with a given value of the total momentum j, its third component j_3, and parity $(-1)^l$ ([17], see, e.g., [18] for more details). Let us remind here only the basic formula:

$$\Psi(t,\vec{x}) = \begin{pmatrix} rf(t,r)\Omega_{jlj_3}(\vec{n}) \\ (-1)^{\frac{1+l-l'}{2}} rg(t,r)\Omega_{jl'j_3}(\vec{n}) \end{pmatrix}, \quad r = |\vec{x}|, \quad \vec{n} = \frac{\vec{x}}{r}.$$

Further, notations $l = j + \frac{1}{2}\mu$, $l' = j - \frac{1}{2}\mu$, $\mu = \pm 1$, $\kappa = \mu(j + \frac{1}{2})$, $\nu = l + \frac{1}{2}$, $\nu' = l' + \frac{1}{2}$ are used. Notice, that $\kappa = \pm 1, \pm 2, \ldots$.

We consider the Dirac operator on the three-dimensional sphere of time dependent radius $R_0(t)$, let us assume that R_0 is C^2-smooth. To assure a unitary evolution of the system, we replace the time derivative with the modified one given as (cf. [19]):

$$\nabla_t = \partial_t + \frac{\dot{R}_0(t)}{2R_0(t)}\left(r\partial_r + \partial_r r\right). \tag{1}$$

The modified radial Dirac equation now reads as follows:

$$i\nabla_t \Psi = \left(-i\alpha\partial_r + \alpha_1\frac{\kappa}{r} + \beta m\right)\Psi, \tag{2}$$

where:

$$\alpha = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to check that Equation (2) provides the conservation of probability:

$$\frac{d}{dt}\int_0^{R_0(t)} \Psi(r,t)^\dagger \Psi(r,t)dr = 0.$$

We use the following transformations of distance, time and the wave function:

$$y = \frac{r}{R_0(t)}, \quad \tau = \int_0^t \frac{1}{R_0(\xi)}\,d\xi, \qquad \psi(r,t) = \frac{1}{\sqrt{R_0(t)}}\varphi(y,\tau(t)), \tag{3}$$

with $\varphi = \binom{\varphi_1}{\varphi_2} \in L^2((0,1),\mathbb{C}^2)$. Then, from the Equation (2), we obtain:

$$i\partial_\tau\varphi(y,\tau) = \left(-i\alpha\partial_y + \alpha_1\tfrac{\kappa}{y} + mR_0(t(\tau))\beta\right)\varphi(y,\tau), \tag{4}$$

where $y \in (0,1)$ and $\varphi_1(0,\tau) = \varphi_2(0,\tau) = 0$.

Thus, we "mapped" the problem onto the Dirac equation on a fixed interval but with a time-dependent mass $M(\tau) = mR_0(t(\tau))$.

2.1. Radial operator domain

The self-adjointness and the basic spectral properties of the considered Dirac operator in three dimensions were established in [20]. A simple example given there shows that the operator domain is not contained in the Sobolev space H^1. However, the domains of the corresponding radial operators, and consequently those on the right hand side of Equation (4), are contained in H^1. This was observed already in [18] on the basis that the full Dirac operator has the domain inside $H^1_{loc}(\mathbb{R}^3,\mathbb{C}^4)$ and the radial operators contain only terms regular at the bag surface. We give here an independent proof based on the form of the radial operator itself only, without the reference to its origin.

In Equation (4), the formal differential operator:

$$T = -i\alpha\partial_y + \alpha_1\frac{\kappa}{y} + M\beta$$

appears. The maximal operator T_{max} domain consists of functions $\varphi \in L^2((0,1),\mathbb{C}^2)$ such that $T\varphi \in L^2((0,1),\mathbb{C}^2)$.

Proposition 1. *For $\kappa = \pm 1, \pm 2, \ldots$, the domain:*

$$\mathcal{D}(T_{max}) = \{\varphi \in H^1((0,1),\mathbb{C}^2) \,|\, \varphi(0) = 0\}, \tag{5}$$

where $\varphi(0)$ is the trace of φ at 0.

Proof. Let $\varphi \in \mathcal{D}(T_{max})$. Evidently, $\varphi \in H^1((\varepsilon,1))$ for any $0 < \varepsilon < 1$ as α is constant and invertible and $(\alpha_1\frac{\kappa}{y} + M\beta)\varphi \in L^2((\varepsilon,1))$. Let f be the upper/lower component of φ. Then:

$$f'(y) \pm \frac{\kappa}{y}f(y) = F(y), \tag{6}$$

with a function $F \in L^2((0,1))$ according to the form of T.

Consider first the case of upper sign and $\kappa \geq 1$. The above equation has the solution:

$$f(y) = y^{-\kappa}\int_0^y \xi^\kappa F(\xi)\,d\xi + cy^{-\kappa}, \tag{7}$$

with a constant c provided that the integral is convergent. However:

$$\left|\int_0^y \xi^\kappa F(\xi)\,d\xi\right| \leq \frac{1}{\sqrt{2\kappa+1}}y^{\kappa+\frac{1}{2}}\|F\|_{L^2},$$

by the Schwarz inequality, so $c = 0$, and $f(0) = 0$.

Let us now consider the case of lower sign and $\kappa \geq 1$. The solution is now:

$$f(y) = -y^\kappa\int_y^1 \xi^{-\kappa}F(\xi)\,d\xi + c_1 y^\kappa, \tag{8}$$

with a constant c_1. As:

$$\left|y^\kappa\int_y^1 \xi^{-\kappa}F(\xi)\,d\xi\right| \leq \frac{1}{\sqrt{2\kappa-1}}\sqrt{y - y^{2\kappa}}\|F\|_{L^2},$$

we obtain, again, $f(0) = 0$. The cases of $\kappa \leq -1$ are also included due to the different signs in the equation above.

We need further to refine our estimates to prove that $f \in H^1$. Let us start with Equation (7) ($c = 0$). Here:

$$f'(y) = -\kappa y^{-\kappa-1}\int_0^y \xi^\kappa F(\xi)\,d\xi + F(y).$$

As $F \in L^2$, we have to prove the square integrability of the first term. Let us calculate:

$$\begin{aligned}
&\int_0^1 \left| y^{-\kappa-1} \int_0^y \xi^\kappa F(\xi)\, d\xi \right|^2 dy \\
&\quad \leq \int_0^1 y^{-2\kappa-2} \left(\int_0^y \xi^\kappa |F(\xi)| d\xi \right)^2 dy \\
&\quad = \int_0^1 y^{-2\kappa-2} \left(\int_0^y \xi^\kappa |F(\xi)| d\xi \right) \left(\int_0^y \eta^\kappa |F(\eta)| d\eta \right) dy \\
&\quad = 2 \int_{0<\xi<\eta<y<1} y^{-2\kappa-2} \xi^\kappa \eta^\kappa |F(\xi)|\, |F(\eta)| dy\, d\xi\, d\eta \\
&\quad = 2 \int_{0<\xi<\eta<1} \frac{1}{2\kappa+1} (\eta^{-2\kappa-1} - 1) \xi^\kappa \eta^\kappa |F(\xi)|\, |F(\eta)| d\xi\, d\eta \\
&\quad \leq \frac{2}{2\kappa+1} \int_{0<\xi<\eta<1} \xi^\kappa \eta^{-\kappa-1} |F(\xi)|\, |F(\eta)| d\xi\, d\eta \\
&\quad = \frac{2}{2\kappa+1} \int_0^1 \eta^{-\kappa-1} \left(\int_0^\eta \xi^\kappa |F(\xi)| d\xi \right) |F(\eta)| d\eta \\
&\quad \leq \frac{2}{2\kappa+1} \|F\|_{L^2((0,1))} \left(\int_0^1 \eta^{-2\kappa-2} \left(\int_0^\eta \xi^\kappa |F(\xi)|\, d\xi \right)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

If the last integral converges we can divide by its square root and obtain:

$$\left(\int_0^1 y^{-2\kappa-2} \left(\int_0^y \xi^\kappa |F(\xi)|\, d\xi \right)^2 \right)^{\frac{1}{2}} \leq \frac{2}{2\kappa+1} \|F\|_{L^2((0,1))}. \tag{9}$$

In the general case, $|F|$ is an almost everywhere non-negative, integrable and square-integrable function. So it can be approximated by an almost everywhere non-decreasing sequence of bounded functions $F_n \nearrow |F|$, e.g. $F_n = \min(n, |F|)$. The above derivation of the estimate (9) passes for F_n, and then we obtain the general case in the limit. So $f' \in L^2$ and $f \in H^1$ in the considered case (7).

Let us now turn to the case (8). Then:

$$f'(y) = -\kappa y^{\kappa-1} \int_y^1 \xi^{-\kappa} F(\xi)\, d\xi + F(y) + c_1 \kappa y^{\kappa-1},$$

where the last two terms are apparently in L^2. We treat the first term similarly as above:

$$\begin{aligned}
&\int_0^1 \left| y^{\kappa-1} \int_y^1 \xi^{-\kappa} F(\xi)\, d\xi \right|^2 dy \\
&\quad \leq \int_0^1 \left(y^{\kappa-1} \int_y^1 \xi^{-\kappa} |F(\xi)|\, d\xi \right)^2 dy \\
&\quad = 2 \int_{0<y<\xi<\eta<1} y^{2\kappa-2} \xi^{-\kappa} \eta^{-\kappa} |F(\xi)|\, |F(\eta)|\, dy\, d\xi\, d\eta \\
&\quad = \frac{2}{2\kappa-1} \int_{0<\xi<\eta<1} \xi^{\kappa-1} \eta^{-\kappa} |F(\xi)|\, |F(\eta)|\, d\xi\, d\eta \\
&\quad = \frac{2}{2\kappa-1} \int_0^1 \left(\xi^{\kappa-1} \int_\xi^1 \eta^{-\kappa} |F(\eta)|\, d\eta\, |F(\xi)| \right) d\xi \\
&\quad \leq \frac{2}{2\kappa-1} \|F\|_{L^2((0,1))} \left(\int_0^1 \left(y^{\kappa-1} \int_y^1 \xi^{-\kappa} |F(\xi)|\, d\xi \right)^2 dy \right)^{\frac{1}{2}}.
\end{aligned}$$

If the last integral converges, we obtain:

$$\left(\int_0^1 \left(y^{\kappa-1} \int_y^1 \xi^{-\kappa} |F(\xi)|\, d\xi \right)^2 dy \right)^{\frac{1}{2}} \leq \frac{2}{2\kappa-1} \|F\|_{L^2((0,1))}. \tag{10}$$

Approximating general $F \in L^2$ by bounded functions, we obtain Equation (10) in the limit. So $f' \in L^2$ and $f \in H^1$. All possibilities of signs in Equation (6) are covered.

It remains to prove that functions satisfying conditions in Equation (5) belong to $\mathcal{D}(T_{max})$, i.e. that $y^{-1}\varphi \in L^2$. As now:

$$\varphi(y) = \int_0^y \varphi'(y)\, dy,$$

the required integrability follows from Equation (9), which holds for $\kappa = 0$ as well. In fact, Equation (9) reduces to the usual Hardy inequality for $\kappa = 0$. The Proposition is completely proved now. ■

Proposition 2. *Operator $H \subset T_{max}$ is self-adjoint, $H = H^*$, if and only if:*

$$\begin{aligned}
\mathcal{D}(H) = \{\varphi \in H^1((0,1), \mathbb{C}^2) \,|\, \varphi(0) = 0, \\
c\varphi_1(1) + d\varphi_2(1) = 0\},
\end{aligned} \tag{11}$$

where c, d are real numbers, which are not simultaneously zero ($c^2 + d^2 = 1$ may be required).

Proof. By Proposition 1, the only non-trivial boundary values are the two components of $\varphi(1)$. The boundary condition defining self-adjoint H reads that the boundary value $\varphi(1)$ lies in a linear subspace L of $\mathbb{C}^2$, which can be specified by its orthogonal projector R. So $R = R^+$ is a Hermitian matrix satisfying the relation $R^2 = R$. It has the form:

$$R = \begin{pmatrix} r_1 & r_2 \\ \overline{r_2} & r_4 \end{pmatrix},$$

with real r_1 and r_4. The symmetry condition for H:

$$0 = (\varphi, H\psi) - (H\varphi, \psi) = \varphi(1)^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \psi(1),$$

for every $\varphi, \psi \in \mathcal{D}(H)$, reads:

$$R^+ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} R = 0.$$

When analysing the obtained conditions on R, one can see that r_2 must be real and the boundary conditions leading to the maximal symmetric operators H are just (11). ■

Remark. The self-adjointness conditions (11) are just those found in [18] for the impenetrable spherical shell. The Dirichlet boundary condition considered by us corresponds to $c = 1$, $d = 0$, and the known MIT bag boundary condition to $c = d = 1$.

As $H^1((0,1),\mathbb{C}^2)$ is compactly embedded in $L^2((0,1),\mathbb{C}^2)$ by the Rellich-Kondrachov theorem, H has a compact resolvent, and therefore a discrete spectrum due to the Riesz-Schauder theorem. In particular, H has a countable complete set of eigenstates.

2.2. Solution of the Dirac equation on time-dependent spherical box

The solutions of Equation (4) can be written in terms of the complete set of the eigenfunctions of the massless Dirac equation given by ($n \in \mathbb{Z}$):

$$\begin{gathered}\left(-i\alpha\partial_y + \alpha_1\frac{\kappa}{y}\right)\psi_n = \varepsilon_n\psi_n,\\ \psi_{n,1}(0) = \psi_{n,2}(0) = \psi_{n,1}(1) = 0,\end{gathered} \tag{12}$$

The eigenvalues can be numbered in such a way that $\varepsilon_n > 0$ for $n > 0$ and $\varepsilon_n < 0$ for $n < 0$. Let us denote $k_n = |\varepsilon_n|$. The solution of Equation (12) can be written as:

$$\begin{gathered}\psi_n(y) = A_n\begin{pmatrix}\operatorname{sgn}(n)\sqrt{y}J_\nu(k_n y)\\ \mu\sqrt{y}J_{\nu'}(k_n y)\end{pmatrix},\\ (\psi_n,\psi_k) = \delta_{n,k},\end{gathered} \tag{13}$$

where $A_n = \left[\frac{1}{2}(J_{\nu'}(k_n)^2 - J_{\nu+1}(k_n)J_{\nu-1}(k_n))\right]^{-\frac{1}{2}}$. Using the boundary conditions given in Equation (12):

$$J_\nu(k_n) = 0, \tag{14}$$

and $k_n = k_{-n}$ is chosen as the $|n|^{\text{th}}$ positive root of J_ν. The Equation (13) holds for $n \neq 0$, the trivial case of $n = 0$, $\varepsilon_0 = 0$, which occurs for $\kappa > 0$, having the form $\psi_{01}(y) = 0$. $\psi_{02}(y) = A_0 y^\kappa$.

Inserting the expansion:

$$\varphi(y,\tau) = \sum_{n=-\infty}^{+\infty} a_n(\tau)\psi_n(y) \tag{15}$$

into Equation (4) and projecting onto ψ_n, we obtain:

$$i\dot{a}_n(\tau) = \varepsilon_n a_n(\tau) - mR_0(t(\tau))a_{-n}(\tau), \quad n \in \mathbb{Z},$$

i.e. the system of ordinary differential equations for a_n, a_{-n}:

$$\begin{aligned}i\dot{a}_n(\tau) &= k_n a_n(\tau) - mR_0(t(\tau))a_{-n}(\tau),\\ i\dot{a}_{-n}(\tau) &= -k_n a_{-n}(\tau) - mR_0(t(\tau))a_n(\tau),\end{aligned} \quad n \in \mathbb{N}, \tag{16}$$

which can be written in a matrix form as follows:

$$\begin{gathered}i\dot{a}^{(n)}(\tau) = \begin{pmatrix}k_n & -mR_0(t(\tau))\\ -mR_0(t(\tau)) & -k_n\end{pmatrix}a^{(n)}(\tau),\\ a^{(n)}(\tau) = \begin{pmatrix}a_n(\tau)\\ a_{-n}(\tau)\end{pmatrix}, \quad n \in \mathbb{N}.\end{gathered} \tag{17}$$

In particular, coefficients $a_{\pm n}$ remain bounded in τ for every $n \in \mathbb{N}$.

For the special case of a massless Dirac particle, i.e. for $m = 0$, the system (16) splits into two uncoupled equations, whose solutions are given as:

$$a_{\pm n}(\tau) = a_{\pm n}(0)e^{\mp ik_n\tau}. \tag{18}$$

For a linearly moving wall of the sphere, given by $R_0(t) = A + Bt$, by taking into account relation between τ and t given in Equation (3), one can obtain the exact solution for a massless Dirac particle as:

$$a_{\pm n}(\tau) = a_{\pm n}(0)e^{\mp i\frac{k_n}{B}\ln\left(1+\frac{B}{A}t\right)}. \tag{19}$$

The solution of the original Dirac equation for time-dependent box given by Equation (2) can be found using the relation between ψ and φ in Equation (3) that yields:

$$\Psi(r,t) = \frac{1}{\sqrt{R_0(t)}}\sum_{n=-\infty}^{+\infty} a_n(0)e^{-i\frac{\varepsilon_n}{B}\ln\left(1+\frac{B}{A}t\right)}\psi_n\left(\frac{r}{R_0(t)}\right),$$

where ψ_n are given by Equation (13) and $a_{\pm n}$ fulfill the relation following from the norm conservation:

$$\sum_{n=-\infty}^{+\infty}|a_n(0)|^2 = 1.$$

For the general case, we are able to solve the system (17) only numerically, starting from the initial values:

$$a_n(0) = \frac{1}{\sqrt{R_0(0)}}\int_0^{R_0(0)}\psi_n^\dagger\frac{r}{R_0(0)}\Psi(r,0)dr, \tag{20}$$

where ψ_n are given by Equation (13).

For the illustrative purposes, we choose the initial condition, e.g. as a Gaussian wave packet given by:

$$\Psi(r,0) = \frac{f(r)}{\sqrt{s_1^2+s_2^2}}\begin{pmatrix}s_1\\ s_2\end{pmatrix},$$

with s_1 and s_2 being the initial spin polarisation and:

$$f(r) = \frac{1}{\sqrt{d\sqrt{2\pi}}}r(R_0(0)-r)\exp\left[-\frac{(r-r_0)^2}{4d^2} + iv_0 r\right], \tag{21}$$

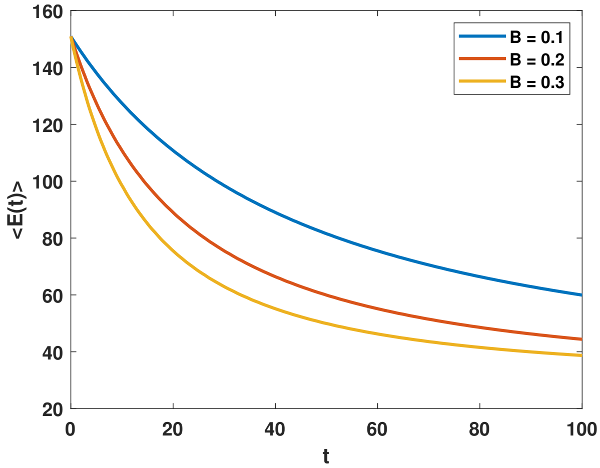
where, d, r_0 and v_0 are the packet's width, initial position of the centre of mass, and initial velocity, respectively.

3. Quantum dynamics under the dynamical confinement

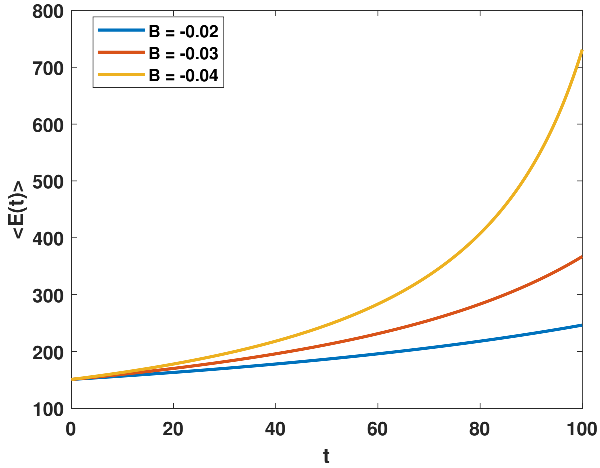
3.1. Fermi acceleration

One of the features of the time-dependent box is the so-called Fermi acceleration, which implies an increase of the particle velocity caused by its interaction with the periodically oscillating wall of the box. An important physically observable characteristics of the Fermi acceleration in quantum regime is the average kinetic energy of the particle, which is determined as:

$$\langle E(t)\rangle = \int_0^{R_0(t)}\Psi(r,t)^\dagger H\Psi(r,t)\,dr.$$

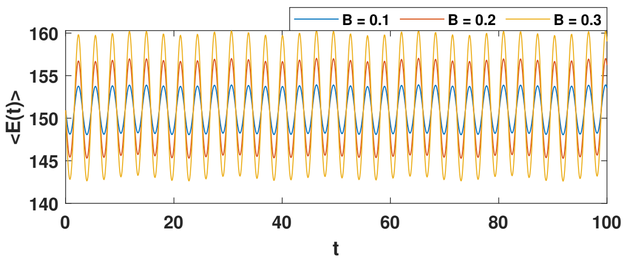


(a). The average kinetic energy of a Dirac particle confined in an expanding spherical box ($R_0(t) = A + Bt$, $A = 5$) as a function of time at different values of B for $m = 1$. The wave packet's width, initial position of the center of mass, initial velocity and the initial spin polarization are chosen as $d = 0.1$, $r_0 = \frac{R_0(0)}{2}$, $v_0 = 0$, $s_1 = 1$ and $s_2 = 0$, respectively.

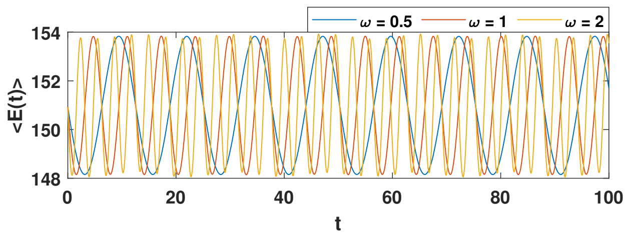


(b). The average kinetic energy of a Dirac particle confined in a contracting box ($R_0(t) = A + Bt$, $A = 5$) as a function of time at different values of B. The other parameters are as in Figure 1a.

Figure 1. The average kinetic energy of a Dirac particle confined in an expanding and a contracting box.



(a). Time-dependence of the average kinetic energy of a Dirac particle in a spherical box ($R_0(t) = A + B\sin\omega t$) at different values of amplitude for fixed $A = 5$, $\omega = 2$ and $m = 1$. The wave packet's width, initial position of the center of mass, initial velocity and the initial spin polarization are chosen as $d = 0.1$, $r_0 = \frac{R_0(0)}{2}$, $v_0 = 0$, $s_1 = 1$ and $s_2 = 0$, respectively.



(b). Time-dependence of the average kinetic energy of a Dirac particle in a spherical box ($R_0(t) = A + B\sin\omega t$) at different values of frequency for fixed $B = 0.1$. The other parameters are as in Figure 2a.

Figure 2. Time-dependence of the average kinetic energy of a Dirac particle in a spherical box at different values of amplitude and frequency.

Using Equations (3) and (15) for the average kinetic energy we have:

$$
\begin{aligned}
\langle E(t)\rangle &= \sum_{n=0}^{\infty} E_n(t),\\
E_0(t) &= -\,m|a_0(\tau(t))|^2,\\
E_n(t) &= \frac{\varepsilon_n}{R_0(t)}\left(|a_n(\tau(t))|^2 - |a_{-n}(\tau(t))|^2\right)\\
&\quad - 2m\Re\left(\overline{a_n(\tau(t))}a_{-n}(\tau(t))\right), \quad \text{for } n \in \mathbb{N}.
\end{aligned} \tag{22}
$$

For the massless particle, Equations (22) and (18) lead to:

$$
\begin{aligned}
E_n(t) &= \frac{k_n}{R_0(t)}(|a_n(\tau(t))|^2 - |a_{-n}(\tau(t))|^2)\\
&= \frac{k_n}{R_0(t)}(|a_n(0)|^2 - |a_{-n}(0)|^2), \quad \text{for } n \in \mathbb{N}.
\end{aligned}
$$

For the massive case ($m \neq 0$), the average energy should be computed numerically. In Figure 1 the average kinetic energy is plotted as a function of time for linearly expanding (Figure 1a) and contracting (Figure 1b) sphere at different values of the wall's velocity. Figure 2 compares $\langle E(t)\rangle$ for the Dirac particle confined in a spherical box with harmonically oscillating radius at different values of the oscillation amplitude (Figure 2a) and frequency (Figure 2b). In both cases, $\langle E(t)\rangle$ is periodic in time and the larger the oscillation amplitude of the wall is, the larger the amplitude of $\langle E(t)\rangle$ is. As the wall's oscillation frequency increses, the period of $\langle E(t)\rangle$ decreases. Thus, no monotonic growth of average kinetic energy is possible for a Dirac particle in a harmonically oscillating box, i.e. no Fermi acceleration can be observed.

3.2. Zitterbewegung

One of the unusual effects in the Dirac particle's dynamics is the so-called "Zitterbewegung", i.e. the manifestation of the trembling motion of the particle. It was found, by considering the wave packet motion described by the time-dependent Dirac equation, that the average coordinate oscillates as a function of time. Here, we consider this phenomenon for a Dirac particle under dynamical confinement provided by a time-dependent box. The main characteristics of Zitterbewegung, the average position, is determined as:

$$\langle r(t)\rangle = \langle \Psi(r,t)|r|\Psi(r,t)\rangle. \tag{23}$$

Using Equation (15) $\langle r(t)\rangle$ can be written as follows:

$$\langle r(t)\rangle = R_0(t)\sum_{n,m} a_n^*(\tau(t))a_m(\tau(t))V_{nm}, \tag{24}$$

where the matrix elements:

$$V_{nm} = \int_0^1 \psi_n(y)^\dagger y \psi_m(y)\, dy$$

were computed numerically.

For a massless Dirac particle, i.e. for $m = 0$, the average coordinate can be defined using (18) for $R_0(t) = A + Bt$ as follows:

$$\langle r(t)\rangle = (A+Bt)\sum_{n,m} a_n^*(0)a_m(0) e^{i\frac{\pi(n-m)}{B}\ln\left(1+\frac{B}{A}t\right)}V_{nm}. \tag{25}$$

In Figure 3, the average position is plotted as a function of time for three regimes of the wall's motion, for linearly expanding, contracting, and harmonically oscillating walls. One can clearly see that Zitterbewegung can be observed for all three regimes.

3.3. Quantum force

The expectation value of the force, $\langle F(t)\rangle$ acting on a Dirac particle can be determined as:

$$\langle F(t)\rangle = -\left\langle \Psi \left| \frac{\partial H}{\partial R_0(t)} \right| \Psi \right\rangle = -\frac{\partial}{\partial R_0(t)}\langle E(t)\rangle, \tag{26}$$

where $\langle E(t)\rangle$ is the average kinetic energy of the particle. Using the expansion (22) and Equations (16):

$$\langle F(t)\rangle = -\frac{1}{\dot{R}_0}\frac{\partial}{\partial t}\langle E(t)\rangle = \sum_{n=0}^{\infty} F_n(t), \tag{27}$$

where:

$$F_n(t) = \frac{1}{R_0(t)^2}\left(|a_n(\tau(t))|^2 - |a_{-n}(\tau(t))|^2\right) \tag{28}$$

$n \in \mathbb{N}$. Apparently, $F_0(t) = 0$. The signs in Equation (28) suggest that an increase in the positive energy components of the particle state requires a positive force and its work, whereas an increase in the negative energy components requires a negative force.

Figure 4 presents the plots of the average quantum force, $\langle F(t)\rangle$ as a function of time for three regimes of time-dependence of the radius, linearly expanding, contracting, and harmonically oscillating ones. The force grows as a function of time for the contracting box, while it is observed to decay for the expanding box. It oscillates for the oscillating wall.

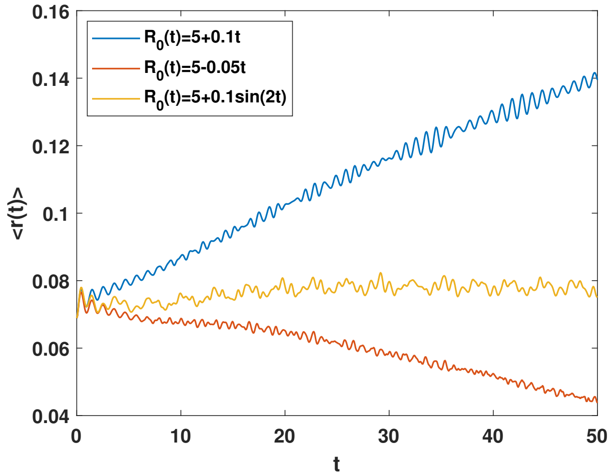


Figure 3. Time-dependence of the average position of a Dirac particle in a box with three regimes of wall's motion ($R_0(t) = 5 + 0.1t$, $R_0(t) = 5 - 0.05t$ and $R_0(t) = 5 + 0.1\sin 2t$) for $m = 1$. The wave packet's width, initial position of the centre of mass, initial velocity and the initial spin polarisation are chosen as $d = 0.1$, $r_0 = \frac{R_0(0)}{2}$, $v_0 = 0$, $s_1 = 1$ and $s_2 = 0$, respectively.

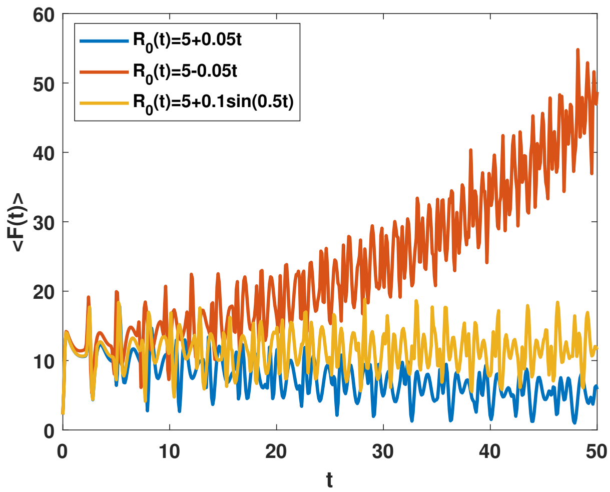


Figure 4. Time-dependence of the average force of a Dirac particle in a box with three regimes of the wall's motion ($R_0(t) = 5 + 0.05t$, $R_0(t) = 5 - 0.05t$ and $R_0(t) = 5 + 0.1\sin 0.5t$) for $m = 1$. The wave packet's width, initial position of the centre of mass, initial velocity, and the initial spin polarisation are chosen as $d = 0.1$, $r_0 = \frac{R_0(0)}{2}$, $v_0 = 0$, $s_1 = 1$ and $s_2 = 0$, respectively.

3.4. Berry phase

During an adiabatic change of the radius R_0, and consequently the effective mass M, along a closed curve C, the eigenfunctions can aquire a geometric (Berry) phase [21]. Self-adjoint Hamiltonians corresponding to T form a trivial analytic family of the type A (e.g. [22, par. XII.2]) with respect to the parameter M. Therefore, the eigenfunctions ϕ_n are analytic in M by the Kato-Rellich theorem, which justifies the interchange of differentiation and integration in the formula for the Berry phase [21]. Taking into account that the components of eigenfunctions ϕ_n have constant phases and their normalisation, we obtain:

$$\begin{aligned}\gamma_n(C) &= i\oint_C\left(\int_0^1 \phi_n(M,y)^\dagger\partial_M\phi_n(M,z)\,dy\right)dM\\ &= \frac{i}{2}\oint_C\left(\partial_M\int_0^1 \phi_n(M,y)^t\phi_n(M,z)\,dy\right)dM\\ &= 0,\end{aligned}$$

i.e. there is no Berry phase for a Dirac particle with time-dependent mass in a spherical box.

4. Conclusion

We studied the dynamics of a Dirac particle under dynamical confinement. The latter is assumed to be caused by a hard-wall sphere with a time-varying radius that ensures the spherical symmetry of the system. The system is modelled in terms of the radial Dirac equation with time-dependent boundary conditions. Analytical solutions of the problem are obtained for some special cases of the time-dependence of the sphere radius, while the for general case, the problem is solved numerically. Physically observable characteristics, such as average kinetic energy, average coordinate, and the average quantum force, are computed as functions of time for different regimes of the wall's motion. It is shown that for a harmonically breathing sphere, the average kinetic energy oscillates in time without a monotonic growth, which implies the absence of the Fermi acceleration. The absence of the Berry phase is shown by the direct calculation of the geometric phase.

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