

gl_3 ALGEBRA IN MIXED MATRIX REPRESENTATIONS

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ABSTRACT. It is demonstrated that the so-called mixed realization of the gl_3 algebra generators in terms of matrix differential operators in two variables, as presented by Smirnov-Turbiner (2013), can be “lifted” into the action in the Fock space associated with the five-dimensional Heisenberg algebra h_5 . A realization of the gl_3 generators in terms of matrix finite-difference (translation-invariant) operators, matrix discrete (dilatation-invariant) operators, matrix complex operators in (z, \bar{z}) , and their mixtures is presented.

KEYWORDS: Mixed representations in Fock space, matrix differential operators, matrix finite-difference (discrete) operators, matrix complex operators in (z, \bar{z}) .

To the memory of Miloslav Havlíček

symbolically, while:

$$[T_i^+, T_j^-] = E_{ii} - \delta_{ij} E_0.$$

1. INTRODUCTION

As a result of numerous discussions with Miloslav Havlíček (Havlicek)¹, which ran over many years about mixing the representations of Lie algebras [1], as discussed between Yu. F. Smirnov (1935–2008) and the present author, in the article [2] there were constructed the so-called mixed representations of the gl_{n+1} algebra realized by differential operators of the first order in n variables with matrix coefficients². In the particular case of the gl_3 algebra, these generators take the form:

$$\begin{aligned} E_{11} &= x_1 \partial_1 + M_{11}, & E_{22} &= x_2 \partial_2 + M_{22}, \\ E_{12} &= x_1 \partial_2 + M_{12}, & E_{21} &= x_2 \partial_1 + M_{21}, \\ E_0 &= k - x_1 \partial_1 - x_2 \partial_2, \\ T_1^- &= \partial_1, & T_2^- &= \partial_2, \end{aligned} \tag{1}$$

$$\begin{aligned} T_1^+ &= x_1(k - x_1 \partial_1 - x_2 \partial_2) - x_1 M_{11} - x_2 M_{12}, \\ T_2^+ &= x_2(k - x_1 \partial_1 - x_2 \partial_2) - x_1 M_{21} - x_2 M_{22}, \end{aligned}$$

see [2], Section 3, where $M_{11}, M_{12}, M_{21}, M_{22}$ are the generators of the gl_2 algebra realized by $n \times n$ matrices, here, the notations $\partial_1 \equiv \frac{\partial}{\partial x_1}, \partial_2 \equiv \frac{\partial}{\partial x_2}$ are used. For a non-negative integer k , this representation is finite-dimensional with marks/spins $[k, n]$, it is characterized by the Young tableau with two rows of length k and n , correspondingly. One can check explicitly that $T_i^-, E_{ij}, E_0, T_i^+$ span the algebra gl_3 . In particular:

$$[E, T^+] = T^+,$$

¹In many instances in the scientific literature the family name of Miloslav is written as *Havlicek*, to avoid confusions since now on we will use this name.

²For convenience, we will always assume the canonical commutation relations:

$$[\tilde{E}_{ij}, \tilde{E}_{kl}] = \delta_{jk} \tilde{E}_{il} - \delta_{il} \tilde{E}_{kj},$$

for the gl_{n+1} generators if they are not specified otherwise.

The generator E_0 is called the Euler-Cartan generator or number operator. It plays the role of a constant having the grading zero in whatever sense. The representation (1) acts in the space of n -tuples, with columns of a size n .

The Casimir operators of gl_3 algebra in this realization are given by:

$$\begin{aligned} C_1 &= E_{11} + E_{22} + E_0 = k + M_{11} + M_{22} \\ &\equiv k + C_1(M), \\ C_2 &= E_{12}E_{21} + E_{21}E_{12} + T_1^+ T_1^- + T_1^- T_1^+ \\ &\quad + T_2^+ T_2^- + T_2^- T_2^+ + E_{11}^2 + E_{22}^2 + E_0^2 \\ &= k(k+2) + M_{11}^2 + M_{22}^2 \\ &\quad + M_{12}M_{21} + M_{21}M_{12} - M_{11} - M_{22} \\ &\equiv k(k+2) + C_2(M) - C_1(M), \end{aligned}$$

and, finally:

$$C_3 = -\frac{1}{2}C_1^3 + \frac{3}{2}C_1C_2 + 3C_2 - 2C_1^2 - 2C_1,$$

where $C_1(M), C_2(M)$ are the Casimir operators of the gl_2 algebra. In this realization (1), the Casimir operator C_3 is algebraically dependent on C_1 and C_2 . In fact, C_1 and C_2 are nothing but the Casimir operators $C_1(M), C_2(M)$ of the gl_2 sub-algebra. Therefore, the center of the gl_3 universal enveloping algebra in realization (1) is generated by the Casimir operators of the gl_2 sub-algebra realized by M_{ij} . Thus, it seems natural that these reps should be irreducible.

In this short note we will show that the mixed representation (1) can be converted into a representation acting on a Fock space associated with the Heisenberg algebra h_5 . In turn, the Fock space can be realized by finite-difference (on the uniform lattice), discrete (on

the exponential lattice) operators, and by complex operators in (z, \bar{z}) . This leads to matrix finite-difference (translation-invariant) operators, to matrix discrete (dilatation-invariant) operators, or to matrix complex operators in (z, \bar{z}) and their mixtures as the generators of the gl_3 algebra.

2. gl_3 MIXED REPRESENTATION IN A FOCK SPACE

Let us take the 5-dimensional Heisenberg algebra h_5 spanned by the generators p_1, p_2, q_1, q_2, I , which obey the commutation relations:

$$\begin{aligned} [p_1, q_1] &= 1, & [p_2, q_2] &= 1, & [p_1, q_2] &= 0, \\ [p_2, q_1] &= 0, & [p_1, p_2] &= 0, & [q_1, q_2] &= 0, \\ [p_{1,2}, I] &= 0, & [q_{1,2}, I] &= 0. \end{aligned} \tag{2}$$

The universal enveloping algebra of the algebra h_5 : U_{h_5} , is spanned by all ordered monomials in p_1, p_2, q_1, q_2 . Introducing the vacuum $|0\rangle$ as an object annihilated by p -operators:

$$p_1 |0\rangle = 0, \quad p_2 |0\rangle = 0,$$

in addition to the universal enveloping algebra U_{h_5} , leads to the definition of a Fock space.

It can be easily shown by direct calculation that:

$$\begin{aligned} E_{11} &= q_1 p_1 + M_{11}, & E_{22} &= q_2 p_2 + M_{22}, \\ E_{12} &= q_1 p_2 + M_{12}, & E_{21} &= q_2 p_1 + M_{21}, \\ E_0 &= k - q_1 p_1 - q_2 p_2, \\ T_1^- &= p_1, & T_2^- &= p_2, \\ T_1^+ &= q_1(k - q_1 p_1 - q_2 p_2) - q_1 M_{11} - q_2 M_{12}, \\ T_2^+ &= q_2(k - q_1 p_1 - q_2 p_2) - q_1 M_{21} - q_2 M_{22}, \end{aligned} \tag{3}$$

span the gl_3 algebra for any $k \in \mathbf{R}$, here $M_{11}, M_{12}, M_{21}, M_{22}$ are again generators of the gl_2 algebra obeying canonical commutation relations, see Footnote 1. The representation (3) is the main result of the present paper. By taking the Heisenberg algebra (2) realized in the coordinate-momentum representation:

$$q_1 = x_1, \quad p_1 = \partial_1, \quad q_2 = x_2, \quad p_2 = \partial_2,$$

with $M_{11}, M_{12}, M_{21}, M_{22}$ given by $n \times n$ matrices which spans gl_2 -algebra the representation (3) is reduced to (1). It is also worth noting that in the one-dimensional representation of gl_2 , when:

$$M_{11} = M_{12} = M_{21} = M_{22} = 0,$$

the representation (3) realizes the hidden algebra of the $A_2/3$ -body Calogero rational/Tremblay-Turbiner-Winternitz (at index 3, degenerate, see [3]) model in the Fock space [4].

3. THREE CANONICAL PAIRS

Two operators a, b form a *canonical pair* if their commutator:

$$[a, b] = 1,$$

where $[a, b] = ab - ba$. The simplest example of a canonical pair is given by the coordinate-momentum representation:

$$[\partial_x, x] = 1.$$

3.1. TRANSLATION-INVARIANT CANONICAL PAIR

Let us take the shift operator:

$$T_\delta f(x) = f(x + \delta), \quad T_\delta = e^{\delta \partial_x},$$

where $\delta \in \mathbf{C}$ is a parameter, which is called the *spacing*, and construct the pair of shift operators (see e.g. [5]):

$$D_\delta = \frac{T_\delta - 1}{\delta}, \quad X_\delta = xT_\delta = x(1 - \delta D_\delta), \tag{4}$$

where the operator D_δ is defined as:

$$D_\delta f(x) = \frac{f(x + \delta) - f(x)}{\delta},$$

sometimes, it is called the Norlund derivative. The operators D_δ, X_δ are translation-invariant. The vacuum is chosen to be one, $|0\rangle = 1$. This canonical pair acts naturally on the space of polynomials (in x). In the limit $\delta \rightarrow 0$, this degenerates into the coordinate-momentum representation, $D_\delta \rightarrow \partial_x$ and $X_\delta \rightarrow x$. It is easy to check that the commutator $[D_\delta, X_\delta] = 1$, hence, D_δ, X_δ form a canonical pair.

3.2. DILATATION-INVARIANT CANONICAL PAIR

Let us introduce the dilatation operator:

$$T_q f(x) = f(qx), \quad T_q = q^A, \quad A \equiv x \partial_x,$$

where $q \in \mathbf{C}$, and construct a canonical pair of dilatation-invariant operators:

$$D_q = x^{-1} \frac{T_q - 1}{q - 1}, \quad X_q = \frac{A(q - 1)}{T_q - 1} x, \tag{5}$$

see [6]. It can be easily checked that $[D_q, X_q] = 1$ for any q , thus, D_q, X_q form a canonical pair. Usually, the operator D_q is called the Jackson symbol (or the Jackson derivative). The vacuum is equal to one, $|0\rangle = 1$. This canonical pair acts naturally on the space of polynomials (in x). Both operators X_q, D_q are pseudodifferential operators which action on monomials as follows:

$$D_q x^n = \{n\}_q x^{n-1}, \quad X_q x^n = \frac{n + 1}{\{n + 1\}_q} x^{n+1},$$

where $\{n\}_q = \frac{1 - q^n}{1 - q}$ is the so called q -number n .

3.3. COMPLEX (z, \bar{z}) CANONICAL PAIR

Take the space $L_2(\mathbb{C}, d\mu)$ of square-integrable functions on \mathbb{C} with the Gaussian measure:

$$d\mu(z) = \pi^{-1} e^{-z\bar{z}} dv(z),$$

where $dv(z) = dx dy$ is the Euclidean volume element on $\mathbb{C} = \mathbb{R}^2$. Let us consider the following lowering and raising operators [7], for discussion see [8]:

$$\mathbf{a} = \frac{\partial}{\partial \bar{z}}, \quad \mathbf{a}^\dagger = -\frac{\partial}{\partial z} + \bar{z}. \quad (6)$$

They are unitary-conjugated (adjoint). The vacuum vector $|0\rangle$, defined by:

$$\mathbf{a}|0\rangle = 0,$$

is any analytic function. It is easy to check that:

$$[\mathbf{a}, \mathbf{a}^\dagger] = I,$$

where I is the unit operator, thus, a canonical pair is formed.

4. gl_3 MIXED REPRESENTATIONS IN MATRIX OPERATORS

It is evident that the representation (3) acting in the Fock space of columns allows us to construct the gl_3 mixed representations in matrix operators.

4.1. MATRIX REPRESENTATION OF THE gl_3 ALGEBRA OF FINITE-DIFFERENCE OPERATORS

Let us take the translation-invariant canonical pair (4) and construct a realization of the Heisenberg generators of h_5 : p_1, p_2, q_1, q_2, I in the following way:

$$\begin{aligned} p_1 &= D_{\delta_1}(x), & p_2 &= D_{\delta_2}(y), \\ q_1 &= X_{\delta_1}(x), & q_2 &= X_{\delta_2}(y). \end{aligned}$$

Substituting these into (3), we arrive at a representation of the gl_3 algebra in the form of matrix finite-difference operators acting in the (x, y) space of columns/ n -tuples on the rectangular lattice with spacings δ_1, δ_2 .

4.2. MATRIX REPRESENTATION OF THE gl_3 ALGEBRA OF DISCRETE OPERATORS

Let us take the dilatation-invariant canonical pair (5) and construct a realization of the Heisenberg generators of h_5 : p_1, p_2, q_1, q_2, I in the following way:

$$\begin{aligned} p_1 &= D_{q_1}(x), & p_2 &= D_{q_2}(y), \\ q_1 &= X_{q_1}(x), & q_2 &= X_{q_2}(y). \end{aligned}$$

Substituting these into (3), we arrive at a representation of the gl_3 algebra in the form of discrete operators with matrix coefficients acting in the (x, y) space of columns/ n -tuples on rectangular lattice with dilations q_1, q_2 in the x, y directions, respectively.

4.3. MATRIX REPRESENTATION OF THE gl_3 ALGEBRA OF MIXED FINITE-DIFFERENCE/DISCRETE OPERATORS

Let us take the translation-invariant canonical pair (4) acting in the x -direction and the dilatation-invariant canonical pair (5) acting in the y -direction and construct a realization of the Heisenberg generators of h_5 : p_1, p_2, q_1, q_2, I in the following way:

$$\begin{aligned} p_1 &= D_\delta(x), & p_2 &= D_q(y), \\ q_1 &= X_\delta(x), & q_2 &= X_q(y). \end{aligned}$$

Substituting this realization into (3) we arrive at a representation of the gl_3 algebra in the form of matrix finite-difference/discrete operators acting in the (x, y) space of columns/ n -tuples on a rectangular uniform/exponential lattice with spacings δ in the x direction and dilation q in the y direction, respectively.

4.4. MATRIX REPRESENTATION OF COMPLEX (z, \bar{z}) GENERATORS

By taking two $(z_{1,2}, \bar{z}_{1,2})$ representations (6) acting on $\mathbb{C}^2(z_1, z_2)$ complex space in the form:

$$\begin{aligned} p_1 &= \mathbf{a}_1(z_1, \bar{z}_1), & q_1 &= \mathbf{a}_1^\dagger(z_1, \bar{z}_1), \\ p_2 &= \mathbf{a}_2(z_2, \bar{z}_2), & q_2 &= \mathbf{a}_2^\dagger(z_2, \bar{z}_2), \end{aligned} \quad (7)$$

and the unit generator I we construct a realization of the five-dimensional Heisenberg algebra h_5 in a 2D complex space. By taking (7) and substituting it into (3), we arrive at the matrix representation of the gl_3 algebra in the $\mathbb{C}^2(z_1, z_2)$ complex space of columns/ n -tuples.

5. CONCLUSION

In this paper, we were able to construct the Fock space representation $[n, k]$ of the gl_3 algebra (3) acting in the Fock space of columns/ n -tuples. This representation becomes finite-dimensional when k is a non-negative integer and integer n is related to the n -dimensional representation of the gl_2 algebra. This representation can be converted to the representations in terms of the first-order differential, finite-difference, discrete, and complex (z, \bar{z}) operators with matrix coefficients. All these representations are alternative to the standard gl_3 algebra representations acting in three-dimensional space (on flag manifold).

ACKNOWLEDGEMENTS

The author (AVT) partially supported by DGAPA grant IN104125 (Mexico).

This work is dedicated to the memory of Miloslav Havlíček – an exemplary scientist and citizen. The present author thinks that the representation (1) should be called the *Havlíček representation*.

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