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UNITARY GENERATION OF GHZ STATES

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Abstract. Entangled states represent fascinating elements of quantum mechanics and quantum technology. Their generation is an interesting problem combining many techniques developed in quantum optics. Using algebraic properties of special transforms we derive the explicit form of the effective Hamiltonian that enables the generation of highly nonclassical states of the GHZ type. Such Hamiltonians belong to the higher order Hamiltonians often discussed in nonlinear quantum optics. We derive the form of the effective Hamiltonian that enables the generation of highly nonclassical states of the GHZ type.

Keywords: Quantum optics.

The basic idea of the manuscript was the result of the discussions with Dr. A. Bandilla. He passed away some time ago, however, his ideas and dedication to physics are still with us.

1. Introduction

GHZ states and their generalizations found their applications far beyond the initial intension proving nonclassicality without statistical tests. GHZ states are at present used for the design of multipartite communication across quantum networks with several parties involved, multipartite secret sharing. The GHZ states are also considered elementary building blocks for quantum computing where out of these states more complicated states are formed on which, or by means of, computations are carried out. There are multipartite (higher than three parties) generalizations of GHZ states. They are becoming indispensable for the latest developments in quantum computing and quantum technologies [1, 2]. Among other studies, the generation of GHZ states is a useful subject of theories.

The generation and manipulation of quantum states is one of the standard tasks for physics, particularly in quantum optics. There are several reasons for being interested in the problem of quantum state generation. Firstly, specially prepared quantum states are tools for performing experiments and measurements. Certain experiments can be performed only with the aid of certain classes of states. The sensitivity and precision of measurements is dependent on the "quality" of the used states. Quantum states can be by itself of interest. On the level of quantum states two of the crucial properties of quantum mechanics can be demonstrated [3, 4]. First of all the quantum theory enables the linear superposition of allowed states. This quite obvious statement gets an exotic flavour when we begin to be interested on states that are almost classical. It is not too difficult to prepare a (Glauber) coherent state with an almost arbitrary amplitude. However, it is an experimental challenge to prepare the (coherent) superposition of two coherent states with two different large amplitudes. Such a state was named Schrödinger cat in the recent literature. It exemplifies how difficult it is to preserve the quantum coherence on the "classical" level as such superpositions are extremely fragile towards dissipation and leads almost inevitably into the real of decoherence in quantum theory. The other important feature of quantum states is the entanglement [5]. When we deal with systems composed from several quantum objects the quantum state of the whole system can exhibit correlations that have no classical analogue. This feature is at the heart of quantum theory and was pointed out already by the founders of quantum theory. So since the early days of quantum mechanics considerable effort was invested into the generation of entangled pairs of particles (atoms, photons). This couples of objects were named Bell pairs and with their aid several experiments have been performed to test quantum mechanics, especially to discriminate between quantum mechanics and alternative local hidden variable theories. The Bell pairs are states having strong two-particle quantum correlations. It is an interesting task to search for generalizations involving more then two quantum objects [6]. This might be atoms, photons or their combinations. The higher order analogues of Bell pairs have been named GHZ state describing a special superposition of three particles. It enables a more sophisticated check of quantum mechanical predictions. Recently, several proposals have been put forward to solve the problem how to generate such states and some of them have been realized experimentally. The aim of the present paper is a proposal of one additional method to this list.

2. Quantum state generation

The various schemes of quantum state generation combines in principle two different building blocks. The first one is a unitary transformation between two given states (the source and the target), the other one is the projection of the desired state from an ensemble of states [7]. As a third case we could point out the shaping of a specially chosen potential in which the desired state (wave function) would be defined [8]. However, this implementation is technically quite difficult to realize and implicitly incorporates the two previous ones.

The first case is a unitary transformation of an initial state. Let us suppose we wish to generate the target state $|\psi_{out}\rangle$. To generate this particular state by unitary transformation we have to specify the initial state $|\psi_i\rangle$ and the unitary transformation $\hat{U}$. The two states are then related:

$$|\psi_{out}\rangle = \hat{U}|\psi_i\rangle. \tag{1}$$

The problem to get the final state is transferred to the choice of the initial state and the transformation operator. Let us, for the illustration, limit our consideration to optical field or to harmonic oscillators. As a natural choice we can choose the vacuum state of the harmonic oscillator $|0\rangle$ for the initial state $|\psi_i\rangle$. Then we have to specify the unitary transformation with respect to this choice. In quantum optics a large class of final states is easily related to the vacuum.

These are the coherent and squeezed states [9]. The unitary operators in the first case is:

$$\hat{U} = \exp\{\alpha\hat{a}^\dagger - \alpha^*\hat{a}\}, \tag{2}$$

where $\hat{a}, \hat{a}^\dagger$ are the annihilation and creation operators of the harmonic oscillator. For the squeezed states the transformation is defined as:

$$\hat{U} = \exp\{\eta\hat{a}^{\dagger 2} - \eta^*\hat{a}^2\}. \tag{3}$$

Both transformations can be induced by a Hamiltonian that is at most quadratic in the annihilation and creation operators:

$$\hat{H} = \kappa_l(\hat{a} + \hat{a}^\dagger) + \kappa_n(\hat{a}^2 + \hat{a}^{\dagger 2}) \tag{4}$$

and describes the two basic (effective) models in quantum optics – the driven oscillator and the parametric down-conversion. In both cases we deal with parametric processes, i.e. a strong external field drives a medium that can be effectively described by the Hamiltonian just given. In general, the basic limitation of this method is the rather limited spectrum of interaction that can be used and so this limits also the possible states $|\psi_{out}\rangle$. The interactions are mainly bounded by the possibilities of atom field interaction. In addition, when a certain type of interaction is allowed, it needs not to be very efficient, i.e. it might be difficult to obtain (from the experimentalists point of view) a "strong enough output".

The second possibility is the use of projection techniques. This method is commonly used in cavity electrodynamics experiments. The first step is the preparation of an appropriate state. Let us assume we have a state $|\psi_i\rangle$ and we wish to prepare a state $|\phi\rangle$. The initial state can be in the number state basis expressed in the form:

$$|\psi_i\rangle = \sum_{n=0}^{\infty} c_n|n\rangle. \tag{5}$$

By applying a projection operator $|\phi\rangle\langle\phi|$ on the state $|\psi\rangle$ we can project out the desired number state. In the optical domain it is almost impossible to realize such general projectors. To manipulate state by projection techniques is an alternative used way. The field to be manipulated is coupled to another system (atom) and by projecting (manipulating) on the atoms we reshape the field state. The basic model of atom field interaction in a cavity is the Jaynes-Cumming model [10, 11]. It describes the interaction of a two-level atom with a single–mode cavity field. The (interaction) Hamiltonian has the form:

$$\hat{H} = \kappa(\hat{a}\sigma^+ + \hat{a}^\dagger\sigma^-). \tag{6}$$

For an initially excited atom (described by the state $|+\rangle$) and a cavity field with the number state expansion $|\psi_c\rangle = \sum c_n|n\rangle$ the state of the whole system at time t is given as:

$$\begin{aligned} |\psi(t)\rangle &= \exp(-i\hat{H}t)|\psi(0)\rangle \\ &= \sum_{n=0}^{\infty} c_n\{\cos(\kappa\sqrt{n+1}t)|n\rangle|+\rangle \\ &\quad - i\sin(\kappa\sqrt{n+1}t)|n+1\rangle|-\rangle\}. \end{aligned} \tag{7}$$

When me measure the atom in the excited state the field collapses into the corresponding state:

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n\cos(\kappa\sqrt{n+1}t)|n\rangle. \tag{8}$$

For a strong enough initial coherent state (choice of the expansion parameters c_n) and sufficient interaction time the just given state can be expressed as a superposition of two coherent states with identical absolute value of the amplitude but with different phases, i.e. it has the form of a "Schödinger cat". The just described procedure gives the heart of the basic of state engineering in a cavity. By sending a sequence of atoms into the cavity (not necessarily in the excited state but in a controlled superposition) we can force the cavity field into any prescribed superposition. The basic drawback of this form of engineering is that it depends crucially on the outcome of the conditional measurement. When the favourable outcome is not received we have to either start the whole procedure anew or we have to prepare the whole strategy with the inclusion of correction steps which naturally makes the procedure longer, because it contains more steps. This can make the method more fragile towards losses for instance.

In the next Section we will analyse the possibility to generate highly entangled states of more then two "particles" using methods of nonlinear optics. We will start from entangled pairs of particles and ask for unitary transformations that enable to generate chains of this elementary entangled units.

3. GHZ state generation

Progress in nonlinear optics enabled to generate in a rather efficient way entangled photons forming the Bell-pairs. The state is generated via spontaneous nondegenerate parametric down-conversion and can be used for the creation of higher order entangled states like for instance GHZ states. In the following we will identify the Hamiltonian that enables to couple entangled states to each other in an unitary fashion. The Bell-state emerging from a nonlinear crystal can be written as:

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}\{|\uparrow_a\rangle|\downarrow_b\rangle + |\downarrow_a\rangle|\uparrow_b\rangle\}, \quad (9)$$

where the states $|\uparrow_x\rangle$, $|\downarrow_x\rangle$ describe the polarization state of the entangled modes a and b. The Bell-pair state can be rewritten in the (scalar) harmonic oscillator bases. We use always two scalar modes for one mode with polarization. The Bell state can be written as:

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}\{|1_1\rangle|0_2\rangle|1_3\rangle|0_4\rangle + |0_1\rangle|1_2\rangle|0_3\rangle|1_4\rangle\}. \quad (10)$$

We assume that we are able to mix this field with an additional photon in the state:

$$|\psi_P\rangle = \frac{1}{\sqrt{2}}\{|1_5\rangle|0_6\rangle + |0_5\rangle|1_6\rangle\}. \quad (11)$$

We use the product state:

$$|\psi_{in}\rangle = |\psi_B\rangle|\psi_P\rangle \quad (12)$$

as the input state for an unitary transformation to generate the desired GHZ state. It can be a passive transformation as the initial state $|\psi_{in}\rangle$ has the sufficient number of excitations. The GHZ state is in our notation as:

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}\{|1_1\rangle|0_2\rangle|1_3\rangle|0_4\rangle|1_5\rangle|0_6\rangle + |0_1\rangle|1_2\rangle|0_3\rangle|1_4\rangle|1_5\rangle|0_6\rangle\}. \quad (13)$$

The input state $|\psi_{in}|\rangle$ can be written as the coherent superposition of two GHZ states, namely:

$$|\psi_{in}\rangle = |\psi_B\rangle|\psi_P\rangle = \frac{1}{\sqrt{2}}(|\psi_{GHZ}\rangle + |\psi_{GHZ1}\rangle), \quad (14)$$

where the second GHZ state takes the form:

$$|\psi_{GHZ1}\rangle = \frac{1}{\sqrt{2}}\{|1_1\rangle|0_2\rangle|1_3\rangle|0_4\rangle|0_5\rangle|1_6\rangle + |0_1\rangle|1_2\rangle|1_3\rangle|0_4\rangle|1_5\rangle|0_6\rangle\}. \quad (15)$$

It is easily checked that the two states are mutually orthogonal, i.e. $\langle\psi_{GHZ}|\psi_{GHZ1}\rangle = 0$. Due to this fact we have to search for an orthogonal transformation $\hat{U}$ which realizes the transformation:

$$|\psi_{GHZ}\rangle = \hat{U}\frac{1}{\sqrt{2}}(|\psi_{GHZ}\rangle + |\psi_{GHZ1}\rangle) \quad (16)$$

in the basis of the two GHZ states this transformation is an rotation, i.e. an $SU(2)$ transformation. In matrix notation this transformation can be written in the form:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (17)$$

Because this transformation is cyclic, i.e. $U^2 = 1$ we can formally associate a Hamiltonian with the transformation in the form:

$$H = \frac{\pi}{2}(1 - U). \quad (18)$$

This is still written in the basis of the GHZ states. We need to rewrite the transformation operator in the basis of creation and annihilation operators of the participating modes $\hat{a}_i, \hat{a}_i^\dagger$. A brief inspection reveals that the transformation can be expressed in terms of one photon projection operators in the Bell state $|\psi_B\rangle$, and the transformation which interchange the two photon state of the additional entangle state $|\psi_P\rangle$:

$$U = \frac{1}{\sqrt{2}}[1 + (\hat{a}_5^\dagger\hat{a}_6 - \hat{a}_5\hat{a}_6^\dagger)(\hat{a}_i^\dagger\hat{a}_i - \hat{a}_j^\dagger\hat{a}_j)]. \quad (19)$$

The index i can take values 1 or 3 and the index j the values 2 or 4. It guarantees the projection of Bell state $|\psi_B\rangle$ on the first or second GHZ states, whose superposition the Bell state is. These relations express the Hamiltonian in terms of mode of creation and annihilation operators. The Hamiltonian belongs to the class of third order nonlinear optical Hamiltonians. It could be realized either within the frame of interacting optical modes in an nonlinear medium or in systems with trapped ions. The interaction naturally expresses also effective all the processes that generate entangled states from nonentangled states. Note that this strategy is possible to use for quite a general class of states. We use the observation that only four modes out of six are involved in the transformation. We can consider generally, the initial state is the state that has the form:

$$|\psi_{in}\rangle = |\psi_1\rangle|\psi_2\rangle, \quad (20)$$

where:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|\Phi_{10}\rangle|1_3\rangle|0_4\rangle + (|\Phi_{01}\rangle|0_3\rangle|1_4\rangle), \quad (21)$$

and:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|1_5\rangle|0_6\rangle + (|0_5\rangle|1_6\rangle). \quad (22)$$

Because all involved states are mutually orthogonal as those in the original GHZ state generation the same procedure can be applied and the result for the unknown Hamiltonian will be the same. The results after the action of the unitary transformation:

$$U = \frac{1}{\sqrt{2}}[1 + (\hat{a}_5^\dagger \hat{a}_6 - \hat{a}_5 \hat{a}_6^\dagger)(\hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4)], \quad (23)$$

on the initial state will be:

$$|\psi_{out}\rangle = \frac{1}{\sqrt{2}}(|\Psi_{10}\rangle|1_3\rangle|0_4\rangle|1_5\rangle|0_6\rangle + |\Psi_{01}\rangle|0_3\rangle|1_4\rangle|0_5\rangle|1_6\rangle). \quad (24)$$

This means also that the original Bell states can be generated in this manner. However, the order of the nonlinearity is in such a case higher. The reason is that in this case we start initially with a vacuum and we need a parametric process to pump energy into the manipulated system.

In connection with the more general additional state:

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|\Psi_{20}\rangle|1_5\rangle|0_6\rangle + |\Psi_{02}\rangle|0_5\rangle|1_6\rangle), \quad (25)$$

we might ask for the transformation that will decompose the initial state:

$$|\psi_{in3}\rangle = |\psi_1\rangle|\psi_3\rangle \quad (26)$$

into the form:

$$|\psi_{out1}\rangle = \frac{1}{2}(|\Psi_{10}\rangle|\Psi_{20}\rangle|1_3\rangle|0_4\rangle|1_5\rangle|0_6\rangle + |\Psi_{01}\rangle|\Psi_{02}\rangle|0_3\rangle|1_4\rangle|0_5\rangle|1_6\rangle). \quad (27)$$

We can proceed along the same type of argumentation as we done before. However, to get an explicit form we would have to specify in more detail the form of the states $|\Psi_{ij}\rangle$ to find the transformation operator between $|\Psi_{02}\rangle$ and $|\Psi_{20}\rangle$ and to express it in the form the annihilation and creation operators. If we have such transformation:

$$T(|\Psi_{02}\rangle) = |\Psi_{20}\rangle, \quad (28)$$

then the asking operator is:

$$U = \frac{1}{\sqrt{2}}[1 + (T\,\hat{a}_5^\dagger \hat{a}_6 - T^{-1}\,\hat{a}_5 \hat{a}_6^\dagger)(\hat{a}_3^\dagger \hat{a}_3 - \hat{a}_4^\dagger \hat{a}_4)], \quad (29)$$

so:

$$U\,(|\psi_{in3}\rangle) = |\psi_{out1}\rangle. \quad (30)$$

Applying the transformation operator:

$$U = \frac{1}{\sqrt{2}}(1 - T^{-1}\,\hat{a}_3^\dagger \hat{a}_3 \hat{a}_6^\dagger \hat{a}_5 - T\,\hat{a}_4^\dagger \hat{a}_4 \hat{a}_5^\dagger \hat{a}_6 + T^{-1}\,\hat{a}_5^\dagger \hat{a}_5 \hat{a}_3^\dagger \hat{a}_4 + T\,\hat{a}_6^\dagger \hat{a}_6 \hat{a}_4^\dagger \hat{a}_3) \quad (31)$$

on the initial state $|\psi_{in3}\rangle$, we get the decomposition:

$$|\psi_{out2}\rangle = \frac{1}{2}(|\Psi_{01}\rangle|\Psi_{02}\rangle)(|1_3\rangle|0_4\rangle|1_5\rangle|0_6\rangle + |\Psi_{10}\rangle|\Psi_{20}\rangle + |0_3\rangle|1_4\rangle|0_5\rangle|1_6\rangle). \quad (32)$$

For Bell-type states the already derived Hamiltonian would work well. However, for different states it will lead to Hamiltonians of higher order and will involve nonlinearities of higher order.

Entanglement presents one of the most fascinating parts of quantum mechanics. It is currently and very close inspection as it is not only of theoretical but also of practical interest. Entanglement is the essence of quantum computing and secret information transmission (quantum cryptography). It can be expected, that at least some of the effects in the realm of entanglement will become used in non quantum life in the future.

4. Conclusion

Entanglement presents one of the most fascinating parts of quantum mechanics. It is currently and very close inspection as it is not only of theoretical but also of practical interest. Entanglement is the essence of quantum computing and secret information transmission (quantum cryptography). Entanglement is rather commonly associated with the microscopic world. It can be expected, that at least some of the effects in the realm of entanglement will become used in non-quantum life in the future. In the present manuscript we reported on a way how the enigmatic entangled GHZ state can be generated. We derived an explicit form for generating optical GHZ states and discussed some details. The corresponding Hamiltonians (or generating operators) have the form of sums of higher order products of annihilation and creation (harmonic oscillator) operations indicating situation when such transform arise.

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