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CLASSICAL AND QUANTUM SUPERINTEGRABLE SYSTEMS ON THE SPHERE AND THE HYPERBOLIC 2-SPACE

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Abstract. We present two superintegrable Hamiltonian systems in two dimensions, defined on the sphere and on the hyperbolic plane. These systems are generalised à la Tremblay-Turbiner-Winternitz (TTW), involving the introduction of a real parameter $k > 0$, with the aim of extending superintegrable Hamiltonian systems to curved spaces in a way similar to the TTW system on the plane. We carry out both classical and quantum analyses of these new systems. We prove that the superintegrability of the initial systems (i.e. when $k = 1$) is preserved when k is rational, as in the TTW case. A detailed study of their classical counterparts and trajectories is also included.

Keywords: Superintegrable systems, factorisation of Hamiltonians, Tremblay-Turbiner-Winternitz Hamiltonan systems.

1. Introduction

In [1], a family of superintegrable systems defined on a homogeneous space of the pseudo-orthogonal Lie group $O(p,q)$ was introduced. This work increased the number of known superintegrable systems at the time [2]. This family of superintegrable Hamiltonian systems (SHSs) has since been studied from various perspectives [3–8].

In 2010, Tremblay, Turbiner, and Winternitz introduced the well-known TTW integrable system [9, 10], which generalises the Smorodinsky-Winternitz superintegrable system [11, 12]. Its introduction renewed the scientific community's interest in superintegrable systems, and the number of new SHSs has steadily grown since then [13–18].

The classical TTW system [9, 10] is characterised by the Hamiltonian:

$$\mathsf{H} = p_r^2 + \frac{1}{r^2}p_\phi^2 + \omega r^2 + \frac{k^2}{r^2}\left(\frac{a}{\cos^2 k\phi} + \frac{b}{\sin^2 k\phi}\right), \quad (1)$$

where $r \in (0,\infty)$, $0 < \phi < \frac{\pi}{2k}$; k, ω, a, b are real numbers with $k, \omega \neq 0$, and $a, b > 0$, respectively. The quantum version of this system is:

$$\begin{aligned} H = &-\partial_r^2 - \frac{1}{r}\partial_r + \omega r^2 \\ &+ \frac{k^2}{r^2}\left(-\frac{1}{k^2}\partial_\phi^2 + \frac{a}{\cos^2 k\phi} + \frac{b}{\sin^2 k\phi}\right). \end{aligned} \quad (2)$$

Now, by making the change of variables $k\,\phi = \theta$, and replacing a and b with $\alpha^2 - \frac{1}{4}$ and $\beta^2 - \frac{1}{4}$, respectively, (as in [19]), the Hamiltonian (2) becomes:

$$\begin{aligned} H = &-\partial_r^2 - \frac{1}{r}\partial_r + \omega r^2 \\ &+ \frac{k^2}{r^2}\left(-\partial_\theta^2 + \frac{\alpha^2 - \frac{1}{4}}{\cos^2\theta} + \frac{\beta^2 - \frac{1}{4}}{\sin^2\theta}\right), \end{aligned} \quad (3)$$

where $0 < \theta < \frac{\theta}{2}$ and $\alpha^2, \beta^2 > \frac{1}{4}$.

In this paper, we present a generalisation à la TTW of two superintegrable Hamiltonian systems defined in two-dimensional curved spaces, specifically the sphere S^2 and the hyperbolic plane $\mathbf{H}^2$. A direct extension of the original TTW system to two-dimensional spherical and hyperbolic spaces can be found in [20–23].

The structure of the paper is as follows: in Section 2, we introduce the original system defined on S^2, as well as its TTW-type generalisation. In addition, we present a mathematical overview of the factorisation method for Hamiltonians – originating in the work of Schrödinger – which allows us to construct generalised ladder and shift operators. We also present a theorem that establishes a systematic method for constructing two symmetries (or integrals of motion) that commute with the Hamiltonian, thereby demonstrating the superintegrability of the new system. Section 3 is devoted to the explicit construction of these symmetries, which are obtained by factorising two sub-Hamiltonians derived from the original system through the separation of variables method. In Section 4, we study the classical counterpart of this Hamiltonian and derive its classical trajectories in an algebraic way. The hyperbolic case is presented in Section 5, where we follow an analogous approach, as the Hamiltonian systems exhibit formal similarities. We conclude with some final remarks and an appendix, where we present a more general version of Theorem 1 from Section 2.

2. TTW $SO(3)$-Hamiltonian

Let us consider the Hamiltonian [6]:

$$H := -\sum_{i=0}^{2} J_i^2 + \frac{l_i^2 - \frac{1}{4}}{s_i^2}, \quad (4)$$

where $(s_i) \equiv (s_0, s_1, s_2) \in \mathbb{R}^3$ verifies $s_0^2 + s_1^2 + s_2^2 = 1$ and $l_i \in \mathbb{R}$. The differential operators $J_i := \varepsilon_{ijk}\, s_j\, \partial_{s_k}$ (ε_{ijk} is the Levi-Civita symbol) span the Lie algebra $so(3)$ since they are the infinitesimal generators in the vector field representation.

It is worth noting that $\sum_{i=0}^{2} J_i^2$ is the quadratic Casimir operator of $so(3)$, which is the Laplace-Beltrami operator on the sphere S^2, which is, in turn, the orbit (symmetric space) of $SO(3)$.

Considering spherical coordinates (ϕ_1, ϕ_2) such that:

$$\begin{aligned} s_0 &= \cos\phi_2\, \cos\phi_1, \\ s_1 &= \cos\phi_2\, \sin\phi_1, \\ s_2 &= \sin\phi_2, \end{aligned} \tag{5}$$

the Hamiltonian (4) becomes:

$$\begin{aligned} H = &- \partial^2_{\phi_2} + \tan\phi_2\, \partial_{\phi_2} + \frac{l_2^2 - \frac{1}{4}}{\sin^2\phi_2} \\ &+ \frac{1}{\cos^2\phi_2}\left[-\partial^2_{\phi_1} + \frac{l_0^2 - \frac{1}{4}}{\cos^2\phi_1} + \frac{l_1^2 - \frac{1}{4}}{\sin^2\phi_1}\right], \end{aligned} \tag{6}$$

with $0 < \phi_1 < \frac{\pi}{2}$, $0 < \phi_2 < \frac{\pi}{2}$. Now we modify this Hamiltonian following the TTW-Hamiltonians (2) and (3) obtaining a one-parameter family of Hamiltonians depending on $k \in \mathbb{R}^*$:

$$\begin{aligned} H_k = &- \partial^2_{\phi_2} + \tan\phi_2\, \partial_{\phi_2} + \frac{l_2^2 - \frac{1}{4}}{\sin^2\phi_2} \\ &+ \frac{k^2}{\cos^2\phi_2}\left[-\partial^2_{k\phi_1} + \frac{l_0^2 - \frac{1}{4}}{\cos^2 k\phi_1} + \frac{l_1^2 - \frac{1}{4}}{\sin^2 k\phi_1}\right], \end{aligned} \tag{7}$$

with now $0 < \phi_1 < \frac{\pi}{2k}$, $0 < \phi_2 < \frac{\pi}{2}$. To simplify the notation, we introduce the change of variables $(k\phi_1, \phi_2) \to (\theta, \phi)$ such that $0 < \theta, \phi < \frac{\pi}{2}$ [19]:

$$\begin{aligned} H_k = &- \partial^2_{\phi} + \tan\phi\, \partial_{\phi} + \frac{l_2^2 - \frac{1}{4}}{\sin^2\phi} \\ &+ \frac{k^2}{\cos^2\phi}\left[-\partial^2_{\theta} + \frac{l_0^2 - \frac{1}{4}}{\cos^2\theta} + \frac{l_1^2 - \frac{1}{4}}{\sin^2\theta}\right]. \end{aligned} \tag{8}$$

Note that we actually have a family of Hamiltonians depending on four real parameters (k, l_0, l_1, l_2).

Taking into account the variable separation, i.e. $\Psi(\theta, \phi) = \psi(\theta)\varphi(\phi)$, the well-known eigenvalue equation $H_k\, \Psi = E\, \Psi$ splits in two equations:

$$H^{\phi}_{M_k}\, \varphi(\phi) = E\, \varphi(\phi), \quad H^{\theta}\psi(\theta) = E'\, \psi(\theta), \tag{9}$$

where:

$$\begin{aligned} H^{\phi}_{M_k} &= -\partial^2_{\phi} + \tan\phi\, \partial_{\phi} + \frac{l_2^2 - \frac{1}{4}}{\sin^2\phi} + \frac{M_k^2}{\cos^2\phi} \\ &= H^{\phi} + \frac{M_k^2}{\cos^2\phi}, \end{aligned} \tag{10}$$

$$H^{\theta} = -\partial^2_{\theta} + \frac{l_0^2 - \frac{1}{4}}{\cos^2\theta} + \frac{l_1^2 - \frac{1}{4}}{\sin^2\theta}, \tag{11}$$

with $M_k := k\sqrt{E'}$ the separation constant. In the following we will use β instead E', such that $\beta^2 := E'$, hence $M_k := k\beta$. Thus the new Hamiltonian (8) becomes:

$$H_k = H^{\phi}_{M_k} + \frac{k^2(H^{\theta} - \beta^2)}{\cos^2\phi}. \tag{12}$$

2.1. Hamiltonian factorisation

As it is well know from the paper by Infeld and Hull [24] in order to construct the shift operators, we factorise the Hamiltonian [25, 26].

Let $\{H_m\}_{m\in\mathbb{Z}}$ be a family of Hamiltonians such that $\forall m \in \mathbb{Z}$:

$$H_m = A_m^+ A_m^- + \lambda_m = A_{m-1}^- A_{m-1}^+ + \lambda_{m-1}. \tag{13}$$

Then, it is straightforward to prove that $A_m^{\pm}$ are shift operators (13), verifying:

$$\begin{aligned} A_m^- &: \mathcal{H}_m \to \mathcal{H}_{m+1}, \\ A_m^+ &: \mathcal{H}_{m+1} \to \mathcal{H}_m, \end{aligned} \tag{14}$$

where by $\mathcal{H}_m$, we design the eigenfunction space of the Hamiltonian H_m (as differential operators).

In addition, the operators $A_m^{\pm}$ are intertwining operators, i.e.:

$$\begin{aligned} A_m^+\, H_{m+1} &= H_m\, A_m^+, \\ A_m^-\, H_m &= H_{m+1}\, A_m^-. \end{aligned} \tag{15}$$

An important consequence of this result is that we can obtain the eigenvectors and the eigenvalues of the discrete spectrum of the original Hamiltonian knowing the ground states of the Hamiltonians H_m of the family. Effectively, let us suppose that $\psi^0_{(m)}$ are the ground states of the Hamiltonians H_m of the hierarchy. They are determined by the condition:

$$A_m^- \psi^0_{(m)} = 0, \tag{16}$$

hence from Equation (13):

$$H_m\, \psi^0_{(m)} = \left(A_m^+\, A_m^- + \lambda_m\right) \psi^0_{(m)} = \lambda_m \psi^0_{(m)}. \tag{17}$$

Thus, the scalars λ_m appearing in the factorisation are the energies of the grounds states. Then, defining:

$$\prod_{i=0}^{m} A_i^+ := A_0^+ \cdots A_m^+, \quad m \in \mathbb{N}, \tag{18}$$

and taking into account (15), by induction, we obtain:

$$H_0 \prod_{i=0}^{m} A_i^+ = \left(\prod_{i=0}^{m} A_i^+\right) H_{m+1}. \tag{19}$$

Therefore, we conclude that the m^{th} excited eigenfunction $\psi^m_{(0)}$ of the original Hamiltonian H_0 is obtained by the consecutive application of the operators A^+ over the ground state $\psi^0_{(m)}$ of H_m:

$$\psi^m_{(0)} = \prod_{i=0}^{m-1} A_i^+ \psi^0_{(m)}. \tag{20}$$

Hence, the eigenvalues of the Hamiltonian H_0 are obtained in an algebraic way.

2.2. Higher rank ladder/shift operators

Let us begin by considering a family of Hamiltonians $\{H_m\}_{m\in\mathcal{I}}$, and introduce some general definitions concerning two types of operators:

- Ladder operators, which connect eigenvectors of a fixed Hamiltonian H_k (with $k \in \mathcal{I}$ fixed), corresponding to different eigenvalues.
- Shift operators, which connect eigenvectors of different Hamiltonians within the family $\{H_m\}_{m\in\mathcal{I}}$, corresponding to the same eigenvalue.

Let H_m be a Hamiltonian, $\mathcal{H}_m$ the Hilbert space spanned by its eigenvectors and ψ^m_λ an eigenvector of H_m with eigenvalue λ. An operator $L_{\pm n}$ (with $n \in \mathbb{N}$) satisfying:

$$\begin{aligned} L_{\pm n}: \quad & \mathcal{H}_m \longrightarrow \mathcal{H}_m \\ & \psi^m_\lambda \longmapsto \psi^m_{\lambda\pm n} = L_{\pm n}\psi^m_\lambda \end{aligned} \tag{21}$$

is called a *generalised ladder* operator. The standard ladder operator corresponds to the case $n = 1$.

An operator $S_{\pm n}$ (with $n \in \mathbb{N}$) satisfying:

$$\begin{aligned} S_{\pm n}: \mathcal{H}_m & \longrightarrow \mathcal{H}_{m\pm n} \\ \psi^m_\lambda & \longmapsto \psi^{m\pm n}_\lambda = S_\pm \psi^m_\lambda \end{aligned} \tag{22}$$

is called a *generalised shift* operador.

To summarise:

Ladder operators connect eigenstates of a single Hamiltonian with different eigenvalues. Shift operators connect eigenstates of different Hamiltonians within the family $\{H_m\}$ with the same eigenvalue.

2.3. Superintegrability of the TTW $SO(3)$-Hamiltonian

By combining both types of operators in a suitable way, we can construct a symmetry of the Hamiltonian system – namely, a finite differential operator X that commutes with the Hamiltonian [27–29]. At the classical level, these symmetries correspond to integrals of motion, also known as constants of motion.

Theorem 1. *Let H_k be the Hamiltonian given in Equation* (12)*. Suppose there exist ladder and shift operators as defined in Equations* (21) *and* (22)*, respectively:*

$$\begin{aligned} \psi_\beta \in \mathcal{H}^\theta & \xrightarrow[n\in\mathbb{N}^*]{L_{\pm 2n}} \psi_{\beta\pm 2n} \in \mathcal{H}^\theta, \\ \varphi_{M_k} \in \mathcal{H}^\phi_{M_k} & \xrightarrow[m\in\mathbb{N}^*]{S_{\pm 2m}} \varphi_{M_k\pm 2m} \in \mathcal{H}^\phi_{M_k\pm 2m}, \end{aligned} \tag{23}$$

where $\psi_\beta \in \mathcal{H}^\theta$ is an eigenvector of H^θ with eigenvalue β^2 and $\varphi_{M_k} \in \mathcal{H}^\phi_{M_k}$ is an eigenvector of $H^\phi_{M_k}$ as given in (10)*, such that $k\,\beta = M_k$. Then, if $k = \frac{m}{n}$, there are two symmetry operators ($X^\pm$) of the Hamiltonian H_k, (i.e. operators that commute with H_k), defined as:*

$$X^\pm := L_{\pm 2n}\, S_{\pm 2m}, \quad m, n \in \mathbb{N}^* \equiv \mathbb{N} - \{0\}. \tag{24}$$

Since there are $2\times 2 - 1$ independent symmetries, namely $X^\pm$ and H_k, the system is maximally superintegrable.

Proof. We only need to prove that the two operators $X^\pm$ defined in Equation (24) commute with the Hamiltonian, i.e.:

$$[H_k, X^\pm] = 0, \quad \forall\, M_k,\, \beta,\, E, \tag{25}$$

as shown in [28].

Let us consider the eigenfunction $\varphi_{M_k}\psi_\beta$ of H_k with eigenvalue E. Then:

$$[H_k, X^\pm]\varphi_{M_k}\psi_\beta = (H_k - E)X^\pm\varphi_{M_k}\psi_\beta. \tag{26}$$

On the other hand, from Equations (23) and (24), we have:

$$X^\pm\varphi_{M_k}\psi_\beta = \psi_{\beta\pm 2n}\varphi_{M_k\pm 2m}. \tag{27}$$

The Hamiltonian H_k, using Equations (10) and (12), can be rewritten as:

$$H_k = H^\phi + \frac{{M_k}^2}{\cos^2\phi} + \frac{k^2(H^\theta - \beta^2)}{\cos^2\phi}. \tag{28}$$

Since ${M_k}^2 = (M_k \pm 2m)^2 - (2m)^2 \mp 4M_k\, m$, we obtain:

$$\begin{aligned} H_k = H^\phi_{M_k\pm 2m} - \frac{(2m)^2}{\cos^2\phi} \mp \frac{4M_k m}{\cos^2\phi} \\ + \frac{k^2(H^\theta - \beta^2)}{\cos^2\phi}. \end{aligned} \tag{29}$$

From Equations (27) and (29), it follows that:

$$H_k\, X^\pm\varphi_{M_k}\psi_\beta = E\varphi_{M_k\pm 2m}\psi_{\beta\pm 2n}, \tag{30}$$

where we used the relation $M_k = k\beta$ together with the rationality condition $k = \frac{m}{n}$. By substituting Equations (27) and (30) into (26), we find that $[H_k, X^\pm] = 0$, as claimed. ∎

3. Analysis of the TTW $SO(3)$-Hamiltonian

We will use the theory of Hamiltonian factorisation [24] to study the TTW $SO(3)$-Hamiltonian given in Equation (8).

3.1. Factorisation of H^θ

Let us start with the Hamiltonian H^θ (11), which is a trigonometric Pöschl-Teller Hamiltonian [30]:

$$H^\theta = -\partial^2_\theta + \frac{l_0^2 - \frac{1}{4}}{\cos^2\theta} + \frac{l_1^2 - \frac{1}{4}}{\sin^2\theta}. \tag{31}$$

The corresponding operators $A^\pm_n$ and λ_n (13), relative to H^θ are [6]:

$$\begin{aligned} A^\pm_n = \pm\,\partial_\theta - (l_0 + n + \frac{1}{2})\tan\theta \\ + (l_1 + n + \frac{1}{2})\cot\theta, \\ \lambda_n \equiv E'_n \equiv \beta^2 = (l_0 + l_1 + 2n + 1)^2. \end{aligned} \tag{32}$$

The hierarchy of Hamiltonians H_n^θ obtained by using Equation (13) have the same expression of H^θ (Equation (11)), that it is the Hamiltonian with $n = 0$ of the hierarchy, after the changes $l_0 \to l_0 + n$ and $l_1 \to l_1 + n$.

The hierarchy of Hamiltonians H_n^θ, defined via Equation (13), all share the same functional form as H^θ in Equation (11) – which corresponds to the case $n = 0$, up to the parameter shifts $l_0 \to l_0 + n$ and $l_1 \to l_1 + n$.

The fundamental states of H_n^θ are obtained by Equation (16) and they are:

$$\psi^0_{(n)}(\theta) = \cos^{l_0+n+\frac{1}{2}}\theta \sin^{l_1+n+\frac{1}{2}}\theta, \tag{33}$$

and the excited states of the initial Hamiltonian $H^\theta \equiv H_0^\theta$ are given by:

$$\psi^n_{(0)}(\theta) = N\cos^{l_0+\frac{1}{2}}\theta \sin^{l_1+\frac{1}{2}}\theta\, P_n^{(l_0,l_1)}(\cos 2\theta), \tag{34}$$

where N is the normalisation constant and $P_n^{(l_0,l_1)}$ are Jacobi polynomials. The energy of these states is $E'_n = \beta^2 = \lambda_n$ as given in Equation (32).

From [8], the ladder operators are:

$$\begin{aligned} L_\beta^\pm &:= \pm(\beta \pm 1)\sin 2\theta\,\partial_\theta \\ &\quad + \beta(\beta \pm 1)\cos 2\theta - l_0^2 + l_1^2. \end{aligned} \tag{35}$$

They act as:

$$\begin{aligned} \mathcal{H}^\theta \ni \psi_\beta &\xrightarrow{L_\beta^+} \psi_{\beta+2} \in \mathcal{H}^\theta, \\ \mathcal{H}^\theta \ni \psi_{\beta+2} &\xrightarrow{L_\beta^-} \psi_\beta \in \mathcal{H}^\theta. \end{aligned} \tag{36}$$

We can define generalised ladder operators as:

$$\begin{aligned} L^+_{\beta\to\beta+2n} &:= \prod_{i=2(n-1)}^{0} L^+_{\beta+i}, \\ L^-_{\beta\to\beta-2n} &:= \prod_{i=2n}^{2} L^+_{\beta-i}, \end{aligned} \tag{37}$$

that act as:

$$\begin{aligned} \mathcal{H}^\theta \ni \psi_\beta &\xrightarrow{L^+_{\beta\to\beta+2n}} \psi_{\beta+2n} \in \mathcal{H}^\theta, \\ \mathcal{H}^\theta \ni \psi_\beta &\xrightarrow{L^-_{\beta\to\beta-2n}} \psi_{\beta-2n} \in \mathcal{H}^\theta. \end{aligned} \tag{38}$$

We now consider index-free operators $L^\pm$ and $(L^\pm)^n$, defined by removing the subscript β from the operators $L_\beta^\pm$ defined in Equation (36), as follows:

$$L^+\psi_\beta := L_\beta^+\psi_\beta, \qquad L^-\psi_\beta := L^-_{\beta-2}\psi_\beta, \quad \forall\beta. \tag{39}$$

This notation also extends to the generalised ladder operators $L^\pm_{\beta\to\beta\pm2n}$ defined in Equation (37):

$$\begin{aligned} \left(L^+\right)^n\psi_\beta &:= L^+_{\beta\to\beta+2n}\psi_\beta, \\ \left(L^-\right)^n\psi_\beta &:= L^-_{\beta\to\beta-2n}\psi_\beta, \end{aligned} \quad \forall\beta. \tag{40}$$

We state the following interesting result and leave its proof to the reader:

$$\left[\sqrt{H^\theta}, \left(L^\pm\right)^n\right] = \pm 2n\left(L^\pm\right)^n. \tag{41}$$

3.2. Factorisation of $H^\phi_{M_k}$

Our task now is to find intertwining operators $M^\pm$ that factorise the Hamiltonian $H^\phi_{M_k}$ given in Equation (10), that is:

$$H^\phi_{M_k} = M^+M^- + \mu. \tag{42}$$

We have identified four families of ladder operators, denoted $M^{\pm,i}$ with $i = 1, 2, 3, 4$, which yield the same factorisation of the Hamiltonian, but satisfy different intertwining relations [29]. These families are:

Solution 1:

$$\begin{aligned} M^{+,1} &= \partial_\phi + (k\beta - 1)\tan\phi + \frac{-2l_2+1}{2}\cot\phi, \\ M^{-,1} &= -\partial_\phi + k\beta\tan\phi + \frac{-2l_2+1}{2}\cot\phi, \\ \mu_1 &= \left(k\beta + l_2 - \frac{3}{2}\right)\left(k\beta + l_2 - \frac{1}{2}\right). \end{aligned} \tag{43}$$

By computing $M^{-,1}M^{+,1} + \mu_1$, we get that it is equal to:

$$-\partial_\phi^2 + \tan\phi\,\partial_\phi + \frac{(k\beta-1)^2}{\cos^2\phi} + \frac{(l_2-1)^2 - \frac{1}{4}}{\sin^2\phi}. \tag{44}$$

Thus, the Hamiltonian given in Equation (44) has the same form as $H^\phi_{M_k}$ in Equation (10), but with the parameters $k\beta$ and l_2 replaced by $k\beta - 1$ and $l_2 - 1$, respectively.

Solution 2:

$$\begin{aligned} M^{+,2} &= \partial_\phi + (k\beta - 1)\tan\phi + \frac{2l_2+1}{2}\cot\phi, \\ M^{-,2} &= -\partial_\phi + k\beta\tan\phi + \frac{2l_2+1}{2}\cot\phi, \\ \mu_2 &= \left(-k\beta + l_2 + \frac{1}{2}\right)\left(-k\beta + l_2 + \frac{3}{2}\right). \end{aligned} \tag{45}$$

Here, $M^{-,2}M^{+,2} + \mu_2$ gives:

$$-\partial_\phi^2 + \tan\phi\,\partial_\phi + \frac{(k\beta-1)^2}{\cos^2\phi} + \frac{(l_2+1)^2 - \frac{1}{4})}{\sin^2\phi}. \tag{46}$$

In this case, the Hamiltonian (45) has the same form as $H^\phi_{M_k}$ in Equation (10), but with the parameters $k\beta$ and l_2 replaced by $k\beta - 1$ and $l_2 + 1$, respectively.

Solution 3:

$$\begin{aligned} M^{+,3} &= \partial_\phi - (k\beta + 1)\tan\phi + \frac{2l_2+1}{2}\cot\phi, \\ M^{-,3} &= -\partial_\phi - k\beta\tan\phi + \frac{2l_2+1}{2}\cot\phi, \\ \mu_3 &= \left(k\beta + l_2 + \frac{1}{2}\right)\left(k\beta + l_2 + \frac{3}{2}\right). \end{aligned} \tag{47}$$

In this case, from $M^{-,3}M^{+,3} + \mu_3$, we get:

$$-\partial_\phi^2 + \tan\phi\,\partial_\phi + \frac{(k\beta+1)^2}{\cos^2\phi} + \frac{(l_2+1)^2 - \frac{1}{4})}{\sin^2\phi}, \tag{48}$$

where the parameters $k\beta$ and l_2 are replaced by $k\beta+1$ and $l_2 + 1$, respectively.

Solution 4:

$$
\begin{aligned}
M^{+,4} &= \partial_\phi - (k\beta+1)\tan\phi + \frac{-2l_2+1}{2}\cot\phi,\\
M^{-,4} &= -\partial_\phi - k\beta\tan\phi + \frac{-2l_2+1}{2}\cot\phi,\\
\mu_4 &= \left(-k\beta + l_2 - \frac{3}{2}\right)\left(-k\beta + l_2 - \frac{1}{2}\right).
\end{aligned} \tag{49}
$$

Now, $M^{-,4}M^{+,4} + \mu_4$ gives:

$$
-\partial_\phi^2 + \tan\phi\,\partial_\phi + \frac{(k\beta+1)^2}{\cos^2\phi} + \frac{(l_2-1)^2-\frac{1}{4})}{\sin^2\phi}, \tag{50}
$$

where the parameters $k\beta$ and l_2 are replaced by $k\beta+1$ and l_2-1, respectively.

We have obtained eight operators that modify the parameters $\beta\,k$ and l_2 by ±1, allowing movement in both directions along the $k\beta$ and l_2 axes.

This enables us to construct a hierarchy of Hamiltonians, denoted by $\{H^\phi_{M_k;n,m}\}_{n,m\in\mathbb{Z}}$, associated with the initial Hamiltonian $H^\phi_{M_k}$ in Equation (10), through the repeated application of the intertwining operators $M^{\pm,3}$, as described in Equation (13). The elements of this hierarchy are explicitly given by:

$$
\begin{aligned}
H^\phi_{M_k;n,m} = &-\partial_\phi^2 + \tan\phi\,\partial_\phi + \frac{(M_k+n)^2}{\cos^2(\phi)}\\
&+ \frac{(l_2+m)^2-\frac{1}{4}}{\sin^2(\phi)},
\end{aligned} \tag{51}
$$

where we have taken into account that $M_k = k\,\beta$. Note that for $n=0$ we recover the initial Hamiltonian $H^\phi_{M_k}$ given in Equation (10).

The fundamental states of the Hamiltonians $H^{k\beta}_{\phi,0,m}$ are obtained using Equation (16), and are given by [6]:

$$
\varphi^0_{(m)}(\phi) = \cos^{k\beta}\phi, \sin^{l_2+m+\frac{1}{2}}\phi, \tag{52}
$$

with $\beta = l_0 + l_1 + n' + \frac{1}{2}$, as in Equation (32), where we fix an arbitrary value $n = n' \in \mathbb{N}$ (see Subsection 3.1). The excited states of the initial Hamiltonian $H^\phi_{M_k;0,0}$, obtained by applying Equation (20), are:

$$
\begin{aligned}
\varphi^m_{(0)}(\phi) &= N\cos^{\beta\,k}\phi\,\sin^{l_2+\frac{1}{2}}\phi\\
&\times P_m^{(l_2+\frac{1}{2},\beta\,k)}(\cos 2\phi),
\end{aligned} \tag{53}
$$

where N is the normalisation constant and $P_m^{(l_0,l_1)}$ are Jacobi polynomials. The energy associated with these states is:

$$
E^{\beta\,k,m}_{(0)} = \left(\beta + l_2 + 2m + \frac{1}{2}\right)\left(\beta + l_2 + 2m + \frac{3}{2}\right). \tag{54}
$$

3.3. Factorisation of multi-parametric Hamiltonian

Given that the Hamiltonians now depend on multiple indices, we shall establish a generalisation of Subsection 2.1, since the original statement does not apply to this extended setting.

Proposition 2. *Let $\{H_{\mathbf{m}}\}_{\mathbf{m}\in\mathbb{Z}^{|\mathcal{I}|}}$ be a family of operators depending on a multi-index $\mathbf{m} = (m_i)_{i\in\mathcal{I}}$, where $|\mathcal{I}|$ denotes the cardinality of the index set $\mathcal{I}$. Assume that for all $\mathbf{m}\in\mathbb{Z}^{|\mathcal{I}|}$:*

$$
H_{\mathbf{m}} = A^+_{\mathbf{m}}A^-_{\mathbf{m}} + \lambda_{\mathbf{m}} = A^-_{f(\mathbf{m})}A^+_{f(\mathbf{m})} + \lambda_{f(\mathbf{m})}, \tag{55}
$$

where $f : \mathbb{Z}^{|\mathcal{I}|} \to \mathbb{Z}^{|\mathcal{I}|}$ acts, component-wise, as $f(\mathbf{m}) = (f_i(m_i))_{i\in\mathcal{I}}$ with each $f_i : \mathbb{Z}\to\mathbb{Z}$ invertible on its domain. Then $A^\pm_{\mathbf{m}}$ are shift operators (cf. Equation (14)) satisfying:

$$
\begin{aligned}
A^-_{\mathbf{m}} &: \mathcal{H}_{\mathbf{m}} \to \mathcal{H}_{f^{-1}(\mathbf{m})},\\
A^+_{\mathbf{m}} &: \mathcal{H}_{f^{-1}(\mathbf{m})} \to \mathcal{H}_{\mathbf{m}},
\end{aligned} \tag{56}
$$

where $\mathcal{H}_{\mathbf{m}}$ is the the eigenfunction space of the Hamiltonian $H_{\mathbf{m}}$.

Proof. Effectively, let $\psi^E_{\mathbf{m}} \in \mathcal{H}_{\mathbf{m}}$ be an eigenvector of $H_{\mathbf{m}}$ with eigenvalue E, i.e. $H_{\mathbf{m}}\,\psi^E_{\mathbf{m}} = E\,\psi^E_{\mathbf{m}}$. From Equation (55):

$$
\begin{aligned}
H_{f_i^{-1}(\mathbf{m})}\,A^-_{\mathbf{m}}\,\psi^E_{\mathbf{m}} &= A^-_{\mathbf{m}}\,(A^+_{\mathbf{m}}A^-_{\mathbf{m}} + \lambda_{\mathbf{m}})\,\psi^E_{\mathbf{m}}\\
&= A^-_{\mathbf{m}}\,H_{\mathbf{m}}\,\psi^E_{\mathbf{m}} = E\,(A^-_{\mathbf{m}}\,\psi^E_{\mathbf{m}}),
\end{aligned} \tag{57}
$$

since $f(f^{-1}(\mathbf{m})) = \mathbf{m}$ by definition of f^{-1}. Thus, $A^-_{\mathbf{m}}$ maps $\psi^E_{\mathbf{m}}$ to an eigenvector of $H_{f^{-1}(\mathbf{m})}$ with the same eigenvalue.

Similarly, for $A^+_{\mathbf{m}}$:

$$
\begin{aligned}
&H_{\mathbf{m}}\,A^+_{\mathbf{m}}\,\psi^E_{f^{-1}(\mathbf{m})}\\
&\quad= A^+_{\mathbf{m}}(A^-_{\mathbf{m}}A^+_{\mathbf{m}} + \lambda_{\mathbf{m}})\psi^E_{f^{-1}(\mathbf{m})}\\
&\quad= A^+_{\mathbf{m}}\,H_{f^{-1}(\mathbf{m})}\,\psi^E_{f^{-1}(\mathbf{m})} = E(A^+_{\mathbf{m}}\,\psi^E_{f^{-1}(\mathbf{m})}).
\end{aligned} \tag{58}
$$

This completes the proof. It is worth noting that $A^\pm_{\mathbf{m}}$ preserve the eigenvalues. ∎

3.4. On the factorisation of the multi-indexed Hamiltonian $H^\phi_{M_k}$

In the following section, we analyse the solutions arising from the factorisation of the Hamiltonian $H^\phi_{M_k}$, in the context of the results of the preceding subsection, with the objective of constructing shift operators that facilitate the determination of the symmetries $X^\pm$.

Let us define the generalised operators $M^{\pm,i}_{n,m}$ in terms of the operators $M^\pm_i$ given in Equations (43), (45), (47), and (49) by performing the replacements $k\beta \to k\beta + n$ and $l_2 \to l_2 + m$ with $n, m \in \mathbb{Z}$, that is:

$$
M^{\pm,i}_{n,m} := M^\pm_i(k\beta \to k\beta + n, l_2 \to l_2 + m), \tag{59}
$$

where $i = 1, \dots, 4$.

From their definition (59) and from the substitutions $\beta\,k \to \beta\,k + n$ and $l_2 \to l_2 + m$ in the factorisation Equations (43)–(50), we can see that for each $i \in \{1,2,3,4\}$, the operators $M^{\pm,i}_{n,m}$ act on the hierarchy of Hamiltonians $\{H^\phi_{M_k;n,m}\}_{n,m\in\mathbb{Z}}$ as follows:

$$
\begin{aligned}
M^{+,i}_{n,m}\,M^{-,i}_{n,m} + \mu^i_{n,m} &= H^\phi_{M_k;n,m},\\
M^{-,i}_{n,m}\,M^{+,i}_{n,m} + \mu^i_{n,m} &= H^\phi_{M_k;f^{-1}_{1,i}(n),f^{-1}_{2,i}(m)},
\end{aligned} \tag{60}
$$

with $f_{1,i}^{-1}, f_{2,i}^{-1} : \mathbb{Z} \to \mathbb{Z}$ invertible index maps such that $f_{1,i}^{-1}(r), f_{2,i}^{-1}(r) : r \to r \pm 1$ depending on $i = 1, 2, 3, 4$. In other words, their definitions are determined by the index changes dictated by the intertwining Equations (43)–(50). For instance, $f_{1,4}^{-1}(n) = n + 1$ and $f_{2,4}^{-1}(m) = m - 1$. Moreover, the action on the spaces of eigenfunctions $\mathcal{H}^{\phi}_{M_k;n,m}$ of the hierarchy Hamiltonians is as follows:

$$\begin{aligned} M^{-,i}_{n,m} &: \mathcal{H}^{\phi}_{M_k;n,m} \to \mathcal{H}^{\phi}_{M_k;f_{1,i}^{-1}(n),f_{2,i}^{-1}(m)}, \\ M^{+,i}_{n,m} &: \mathcal{H}^{\phi}_{M_k;f_{1,i}^{-1}(n),f_{2,i}^{-1}(m)} \to \mathcal{H}^{\phi}_{M_k;n,m}. \end{aligned} \tag{61}$$

We have identified eight operators $M^{\pm,i}_{n,m}$ that transform the eigenstates according to:

$$M^{\pm,i}_{n,m} : \mathcal{H}^{\phi}_{M_k;n,m} \to \mathcal{H}^{\phi}_{M_k;n+a,m+b}, \tag{62}$$

where $a, b \in \{-1, +1\}$.

In Figures 1 and 2, we illustrate the action of the operators $M^{\pm,i}_{0,0} \equiv M^{\pm,i}$, which coincide with those defined in Equations (43), (45), (47), and (49). Note that, for our purposes, it is sufficient to consider only one set, either the M^+ or the M^- operators, as they act in a similar manner.

We can define index-free operators $M^{a,b}$ in terms of, for instance, the operators $M^{-;i}_{m,n}$ (for all $n, m \in \mathbb{Z}$) as follows:

$$M^{a,b}\,\psi_{m,n} := M^{-;i}_{m,n}\,\psi_{m,n}, \quad a, b = \pm 1. \tag{63}$$

Thus, from Figure 1 and Figure 2 it can be seen that $M^{1,1} = M^{-,3}_{m,n}$, $M^{1,-1} = M^{-,2}_{m,n}$, $M^{-1,1} = M^{-,4}_{m,n}$, and $M^{-1,-1} = M^{-,1}_{m,n}$. By composing two such operators, we can construct the shift operators S defined in Equation (22), which move only in one direction. In our case, we choose the direction along $k\beta$ (i.e. direction n).

Considering the composition of operators acting as $M^{a,b}\psi_{m,n} = \psi_{m+a,n+b}$, we find that:

$$M^{a,b}M^{-a,b}\psi_{m,n} = M^{a,b}\psi_{m-a,n+b} = \psi_{m,n+2b}, \tag{64}$$

where $a, b = \pm 1$. Equation (63) allows us to define the shift operators (22) as follows:

$$S^{\pm} := M^{a,\pm 1}\,M^{-a,\pm 1}, \tag{65}$$

that act as:

$$S^{\pm} : \mathcal{H}^{\phi}_{M_k;m,n} \to \mathcal{H}^{\phi}_{M_k;m,n\pm 2}. \tag{66}$$

In Figure 3 we show how the shift operator S^+ can be expressed in terms of the operators $M^{-1,1}$ and $M^{1,1}$. Although S^- is not depicted, the reader can infer that it is defined analogously in terms of the operators $M^{-1,-1}$ and $M^{1,-1}$.

Taking into account Equation (40), we can then obtain the shift operators $S_{\pm 2m}$ defined in Equation (23) by:

$$S_{\pm 2m} = \left(S^{\pm}\right)^m. \tag{67}$$

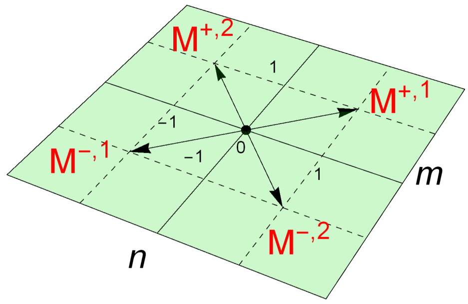


Figure 1. Action of the operators $M^{\pm,1}$ (Equation (43)) and $M^{\pm,2}$ (Equation (45))

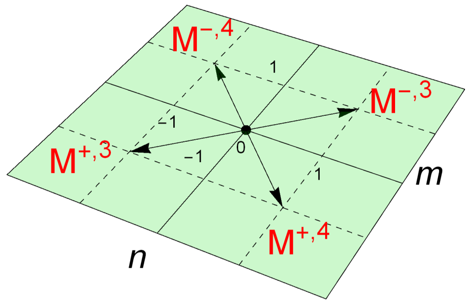


Figure 2. Action of the operators $M^{\pm,3}$ (Equation (47)) and $M^{\pm,4}$ (Equation (49)).

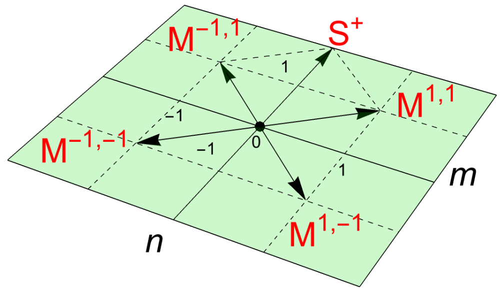


Figure 3. Shift operator (65) $S^+ = M^{1,1}\,M^{-1,1}$.

Finally, we have constructed the operators $L^{\pm}_{\pm 2n}$ (40) and $S_{\pm 2m}$ (67), which enable us to build the symmetries $X^{\pm} = L_{\pm 2n}\,S_{\pm 2m}$ (24) that commute with the Hamiltonian. This construction allows us to prove that the Hamiltonian system H_k (8) is superintegrable.

It is worth noting that the operators $X^{\pm}$ do not commute, since $[X^+, X^-] \neq 0$, which is a fact that can be directly verified.

Recall that $k = \frac{m}{n}$, with m and n integers, which can, without loss of generality, be taken an irreducible form, i.e. with m and $n > 0$ coprime. Although k can equivalently be written as $k = \frac{\gamma m}{\gamma n}$ for any $\gamma \neq 0$, the operators $L_{\pm 2n}$ and $S_{\pm 2m}$, depending on θ and ϕ respectively, commute. Using $L_{\pm 2n} = (L_{\pm})^n$ and $S_{\pm 2m} = (S\pm)^m$, we obtain $L_{\pm 2n\gamma}S_{\pm 2m\gamma} = (X^{\pm})^{\gamma}$, implying that the expression depends only on $X^{\pm}$. Thus, it is natural and sufficient to restrict k to its irreducible form.

4. Associated classical system

The classical version of the quantum Hamiltonian (8) is:

$$\mathsf{H}_k = p_\phi^2 + \frac{c^2}{\sin^2\phi} + \frac{k^2}{\cos^2\phi} \times \left(p_\theta^2 + \frac{a^2}{\cos^2\theta} + \frac{b^2}{\sin^2\theta}\right). \tag{68}$$

We can group the terms as:

$$\mathsf{H}^\theta = p_\theta^2 + \frac{a^2}{\cos^2\theta} + \frac{b^2}{\sin^2\theta}, \tag{69}$$

$$\mathsf{H}^\phi_{\mathsf{M}_k} = p_\phi^2 + \frac{\mathsf{M}_k^2}{\cos^2\phi} + \frac{c^2}{\sin^2\phi}, \tag{70}$$

where $\mathsf{M}_k := k\sqrt{\mathsf{H}^\theta}$.

Note that $\mathsf{H}_k = \mathsf{H}_k(p_\phi, \phi, p_\theta, \theta)$ and $\mathsf{H}^\theta = \mathsf{H}^\theta(p_\theta, \theta)$.

4.1. Superintegrability

In analogy with the quantum case [31], we can consider the ladder functions $\mathsf{L}^\pm(\theta, p_\theta)$:

$$\mathsf{L}^\pm = \left(b^2 - a^2\right)\frac{1}{\sqrt{\mathsf{H}^\theta}} + \cos 2\theta\sqrt{\mathsf{H}^\theta} \pm i p_\theta \sin 2\theta, \tag{71}$$

such that they verify:

$$\{\mathsf{H}^\theta, \mathsf{L}^\pm\}_\theta = \mp i\, 4\sqrt{\mathsf{H}^\theta}\,\mathsf{L}^\pm, \tag{72}$$

and also:

$$\{\mathsf{H}_k, (\mathsf{L}^\pm)^n\} = \{\mathsf{H}^\theta, (\mathsf{L}^\pm)^n\}_\theta = \mp n\,\alpha\,\left(\mathsf{L}^\pm\right)^n, \tag{73}$$

where:

$$\alpha = 4i\sqrt{\mathsf{H}^\theta}\frac{k^2}{\cos^2\theta} = 4i\,k\,\mathsf{M}_k\,\sec^2\phi. \tag{74}$$

The Poisson brackets $\{\cdot,\cdot\}_\theta$ and $\{\cdot,\cdot\}$ refer to the canonical variables (θ, p_θ) and $(\phi, p_\phi\,;\,\theta, p_\theta)$, respectively. It is worth noting that from Equation (72), we obtain $\{\sqrt{\mathsf{H}^\theta}, \mathsf{L}^\pm\}_\theta = \mp i\,2\,\mathsf{L}^\pm$, which corresponds to the classical analogue of Equation (41).

Similarly, we obtain shift functions $\mathsf{S}^\pm(\phi, p_\phi)$ associated with Solutions 3 and 4 from Subsection 3.1:

$$\mathsf{S}^\pm = -l_2^2\cot^2\phi - (p_\phi \mp i\,\mathsf{M}_k\tan\phi)^2. \tag{75}$$

They verify:

$$\{\mathsf{H}^\phi_{\mathsf{M}_k}, \mathsf{S}^\pm\} = \pm\,4i\,\mathsf{M}_k\,\sec^2\phi\,\mathsf{S}^\pm, \qquad \{\mathsf{H}, (\mathsf{S}^\pm)^m\} = \pm\, m\,\alpha\,\left(\mathsf{S}^\pm\right)^m. \tag{76}$$

Now, considering the functions:

$$\mathsf{X}^\pm = (\mathsf{S}^\pm)^m(\mathsf{L}^\pm)^n, \tag{77}$$

we obtain the following Poisson commutation relations:

$$\{\mathsf{H}, \mathsf{X}^\pm\} = 0 \quad \text{if } k = \frac{m}{n}, \tag{78}$$

which show that $\mathsf{X}^\pm$ are integrals of motion. This establishes the superintegrability of the classical system described by Equation (68).

4.2. Classical trajectories

From the constants of motion $\mathsf{X}^\pm$ (Equation (77)), by substituting H^θ with its expression in terms on $\mathsf{M}_k = k\sqrt{\mathsf{H}^\theta}$, we get:

$$\mathsf{X}^\pm = \left(-l_2^2\cot^2\phi - (p_\phi \mp i\,\mathsf{M}_k\tan\phi)^2\right)^m \times \left(\frac{b^2 - a^2}{\sqrt{a^2\sec^2\theta + b^2\csc^2\theta + p_\theta^2}} + \cos 2\theta \times \sqrt{a^2\sec^2\theta + b^2\csc^2\theta + p_\theta^2} \pm i\,p_\theta\sin 2\theta\right)^n. \tag{79}$$

In this system, $\mathsf{H}, \mathsf{H}^\theta, \mathsf{X}^\pm$, and M_k are constants of motion, but only three are functionally independent. By fixing the total energy $\mathsf{H} = \mathsf{E}$, both E and M_k remain constant along the classical trajectory. This allows us to express the generalised momenta in terms of the generalised coordinates:

$$p_\theta = \varepsilon_\theta\sqrt{\frac{\mathsf{M}_k^2}{k^2} - \left(\frac{a^2}{\cos^2\theta} + \frac{b^2}{\sin^2\theta}\right)}, \qquad p_\phi = \varepsilon_\phi\sqrt{\mathsf{E} - \left(\frac{l_2^2}{\sin^2\phi} + \frac{\mathsf{M}_k^2}{\cos^2\phi}\right)}, \tag{80}$$

where $\varepsilon_\theta, \varepsilon_\phi \in \{\pm 1\}$.

The symmetry functions $\mathsf{X}^\pm$ are complex-valued, and therefore the constants of motion are, in general, complex numbers C. To obtain physically meaningful (real) representations, we make use of the reality condition $(\mathsf{X}^+)^* = \mathsf{X}^-$, and define:

$$\mathsf{X}^+ = \mathsf{C}, \quad \mathsf{X}^- = \mathsf{C}^*. \tag{81}$$

We then consider the real and imaginary parts of X^+:

$$Re(\mathsf{X}^+) = \frac{\mathsf{X}^+ + \mathsf{X}^-}{2} = Re(\mathsf{C}), \qquad Im(\mathsf{X}^+) = \frac{\mathsf{X}^+ - \mathsf{X}^-}{2i} = Im(\mathsf{C}). \tag{82}$$

These two real functions can be used to describe the trajectories of the system.

The classical trajectories T_0 are implicitly defined as the set of points $\mathbf{x} \in \mathbb{R}^3$ satisfying the condition:

$$\mathsf{X}(\mathbf{x}) = \mathsf{C}_0, \tag{83}$$

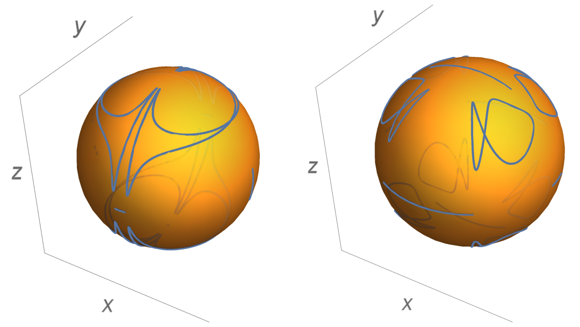
where C_0 is the fixed complex constant determined by the initial conditions.

Figures 4 and 5 show trajectories corresponding to different values of $k = \frac{m}{n}$, as presented [29]. These plots were generated using Mathematica.

5. Hyperbolic TTW $SO(2,1)$-Hamiltonian

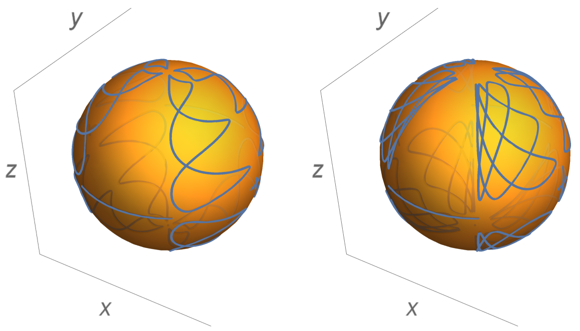
In this case, we consider the Hamiltonian introduced in [7]:

$$H := J_2^2 - J_1^2 - J_0^2 - \frac{l_2^2 - \frac{1}{4}}{s_2^2} + \frac{l_1^2 - \frac{1}{4}}{s_1^2} + \frac{l_0^2 - \frac{1}{4}}{s_0^2}, \tag{84}$$



(A). $m = 1$, $n = 1$. (B). $m = 2$, $n = 1$.

FIGURE 4. Trajectories for $m = 1$, $n = 1$ and $m = 2$, $n = 1$, respectively.



(A). $m = 1$, $n = 3$. (B). $m = 3$, $n = 3$.

FIGURE 5. Trajectories for $m = 1$, $n = 3$ and $m = 3$, $n = 3$, respectively.

where the coordinates of the ambient space $(s_i) \equiv (s_0, s_1, s_2) \in \mathbb{R}^3$ satisfy the constraint $s_0^2+s_1^2-s_2^2 = -1$. The differential operators J_i are given by:

$$\begin{aligned} J_0 &= s_1\partial_2 + s_2\partial_1, \\ J_1 &= s_2\partial_0 + s_0\partial_2, \\ J_2 &= s_0\partial_1 - s_1\partial_0, \end{aligned} \tag{85}$$

and they generate the Lie algebra $so(2,1)$, with commutation relations:

$$[J_0, J_1] = -J_2,\ [J_2, J_0] = J_1,\ [J_1, J_2] = J_0. \tag{86}$$

Next, we introduce coordinates analogous to the spherical coordinates, denoted by (ξ, θ), to proceed with the analysis:

$$\begin{aligned} s_0 &= \cos\theta \sinh\xi, \\ s_1 &= \sin\theta \sinh\xi, \\ s_2 &= \cosh\xi, \end{aligned} \tag{87}$$

where $0 \le \theta < 2\pi$ and $0 \le \xi < \infty$. The Hamiltonian (84) can be rewritten in terms of the variables (ξ, θ) as:

$$\begin{aligned} H = &-\partial_\xi^2 - \coth\xi\,\partial_\xi - \frac{l_2^2 - \frac14}{\cosh^2\xi} \\ &+ \frac{1}{\sinh^2\xi}\left[-\partial_\theta + \frac{l_1^2-\frac14}{\sin^2\theta} + \frac{l_0^2-\frac14}{\cos^2\theta}\right]. \end{aligned} \tag{88}$$

By deforming this Hamiltonian à la TTW, using the real parameter $k \neq 0$ as in Section 2, we arrive at the TTW-Hamiltonian:

$$\begin{aligned} H_k = &-\partial_\xi^2 - \coth\xi\partial_\xi - \frac{l_2^2 - \frac14}{\cosh^2\xi} \\ &+ \frac{k^2}{\sinh^2\xi}\left[-\partial_\theta + \frac{l_1^2-\frac14}{\sin^2\theta} + \frac{l_0^2-\frac14}{\cos^2\theta}\right], \end{aligned} \tag{89}$$

where $0 \le \theta < \frac{\pi}{2}$ and $0 \le \xi < \infty$. In this way we have constructed a family of Hamiltonians $\{H_k\}$, depending on four real parameters (k, l_0, l_1, l_2).

The Hamiltonian may be separated in two "sub-Hamiltonians" through variable separation in the Schrödinger equation $H_k\,\Psi(\theta,\xi) = E\,\Psi(\theta,\xi)$ by assuming a factorised solution of the form $\Psi(\theta,\xi) = \psi(\theta)\,\varphi(\xi)$. This leads to two eigenvalue equations:

$$H^\theta\psi = E'\psi, \quad H^\xi_{M_k}\varphi = E\varphi, \tag{90}$$

with $E' = \beta^2$, and $M_k = k\beta$ is the separation constant, where:

$$H^\theta := -\partial_\theta + \frac{l_1^2-\frac14}{\sin^2\theta} + \frac{l_0^2-\frac14}{\cos^2\theta}, \tag{91}$$

$$H^\xi_{M_k} := -\partial_\xi^2 - \coth\xi\partial_\xi - \frac{l_2^2-\frac14}{\cosh^2\xi} + \frac{M_k^2}{\sinh^2\xi}. \tag{92}$$

It is worth noting that the Hamiltonian H^θ (Equation (91)) coincides with the Hamiltonian (Equation (11)) that appeared in the TTW $SO(3)$-Hamiltonian discussed in Section 2, whose factorisation carried out in Subsection 3.1.

5.1. FACTORISATION OF $H^\xi_{M_k}$

We further identify four distinct families of ladder operators $N^{\pm,i}, (i = 1,2,3,4)$, analogous to those arising from the TTW $SO(3)$-Hamiltonian case (Subsection 3.2):

$$H^\xi_{M_k} = N^{+,i}N^{-,i} + \mu^i. \tag{93}$$

As in the spherical case, these operators yield the same factorisation of the Hamiltonian (92), although they differ in their intertwining relations. They are:

Solution 1:

$$\begin{aligned} N^{+,1} &= \partial_\xi + \left(\frac12 - l_2\right)\tanh\xi + (1-M_k)\coth\xi, \\ N^{-,1} &= -\partial_\xi + \left(\frac12 - l_2\right)\tanh\xi - M_k\coth\xi, \\ \mu^1 &= -\frac14(2l_2+2M_k-3)(2l_2+2M_k-1). \end{aligned} \tag{94}$$

For this solution, we obtain the following expression for $N^{-,1}N^{+,1} + \mu^1$:

$$-\partial_\xi^2 - \coth\xi\,\partial_\xi - \frac{(l_2-1)^2-\frac14}{\cosh^2\xi} + \frac{(M_k-1)^2}{\sinh^2\xi}. \tag{95}$$

The Hamiltonian given in Equation (95) has the same form as $H^\xi_{M_k}$ in Equation (92), but with the parameters M_k and l_2 replaced by $M_k - 1$ and $l_2 - 1$, respectively.

Solution 2:

$$
\begin{aligned}
N^{+,2} &= \partial_\xi + \left(\frac{1}{2}+l_2\right)\tanh\xi + (1-M_k)\coth\xi,\\
N^{-,2} &= -\partial_\xi + \left(\frac{1}{2}+l_2\right)\tanh\xi - M_k\coth\xi,\\
\mu^2 &= -\frac{1}{4}(2l_2-2M_k+1)(2l_2-2M_k+3).
\end{aligned} \tag{96}
$$

In this case $N^{-,2}N^{+,2}+\mu^2$ yields:

$$
-\partial_\xi^2 - \coth\xi\,\partial_\xi - \frac{(l_2+1)^2-\frac{1}{4}}{\cosh^2\xi} + \frac{(M_k-1)^2}{\sinh^2\xi}, \tag{97}
$$

where the parameters M_k and l_2 are replaced by M_k-1 and l_2+1, respectively.

Solution 3:

$$
\begin{aligned}
N^{+,3} &= \partial_\xi + \left(\frac{1}{2}+l_2\right)\tanh\xi + (M_k+1)\coth\xi,\\
N^{-,3} &= -\partial_\xi + \left(\frac{1}{2}+l_2\right)\tanh\xi + M_k\coth\xi,\\
\mu^3 &= -\frac{1}{4}(1+2l_2+2M_k)(3+2l_2+2M_k).
\end{aligned} \tag{98}
$$

Here, the expression $N^{-,3}N^{+,3}+\mu^3$ results in:

$$
-\partial_\xi^2 - \coth\xi\,\partial_\xi - \frac{(l_2+1)^2-\frac{1}{4}}{\cosh^2\xi} + \frac{(M_k+1)^2}{\sinh^2\xi}, \tag{99}
$$

where the parameters M_k and l_2 are replaced by M_k+1 and l_2+1, respectively.

Solution 4:

$$
\begin{aligned}
N^{+,4} &= \partial_\xi + \left(\frac{1}{2}-l_2\right)\tanh\xi + (M_k+1)\coth\xi,\\
N^{-,4} &= -\partial_\xi + \left(\frac{1}{2}-l_2\right)\tanh\xi + M_k\coth\xi,\\
\mu^4 &= -\frac{1}{4}(2l_2-2M_k-3)(2l_2-2M_k-1).
\end{aligned} \tag{100}
$$

Evaluating $N^{-,4}N^{+,4}+\mu^4$ results in:

$$
-\partial_\xi^2 - \coth\xi\,\partial_\xi - \frac{(l_2-1)^2-\frac{1}{4}}{\cosh^2\xi} + \frac{(M_k+1)^2}{\sinh^2\xi}, \tag{101}
$$

where the parameters M_k and l_2 are replaced by M_k+1 and l_2-1, respectively.

We have identified eight operators that shift the parameters $\beta\, k$ and l_2 by ±1, allowing movement in both directions along the $k\beta$ and l_2 axes.

We also construct a hierarchy of Hamiltonians, $\{H^\xi_{M_k;n,m}\}_{n,m\in\mathbb{Z}}$, associated with the initial Hamiltonian $H^\xi_{M_k}$ (Equation (92)), by the repeatedly applying the intertwining operators $N^{\pm,3}$ as described in Equation (13). The elements of this hierarchy are explicitly given by:

$$
\begin{aligned}
H^\xi_{M_k;n,m} = &-\partial_\xi^2 - \coth\xi\,\partial_\xi\\
&- \frac{(l_2+m)^2-\frac{1}{4}}{\cosh^2\phi} + \frac{(k\beta+n)^2}{\sinh^2\phi},
\end{aligned} \tag{102}
$$

where we have used the relation $M_k = k\,\beta$. For $m=n=0$, we obtain the initial Hamiltonian $H^\xi_{M_k}$ given by Equation (92).

The fundamental states of the Hamiltonians $H^\xi_{M_k;0,m}$ are obtained via Equation (16) and are given by [7]:

$$
\varphi^0_{(m)}(\xi) = \cosh^{l_2+m+\frac{1}{2}}\xi\,\sinh^{k\beta}\xi, \tag{103}
$$

with $\beta = l_0+l_1+n'+\frac{1}{2}$ (Equation (32)), where we have taken a fixed, but arbitrary value of $n=n'\in\mathbb{N}$ (see Subsection 3.1), The excited states of the original Hamiltonian $H^{k\beta}_{\phi,0,0}$, obtained using Equation (32), are [32]:

$$
\begin{aligned}
\varphi^m_{(0)}(\xi) = &\,N\cosh^{l_2+\frac{1}{2}}\xi\,\sinh^{\beta\,k}\xi\\
&\times P_m^{(l_2+\frac{1}{2},\beta\,k)}(\cosh 2\xi),
\end{aligned} \tag{104}
$$

where N is the normalisation constant and $P_m^{(l_0,l_1)}$ are Jacobi polynomials. The energy of these states is:

$$
\begin{aligned}
E^{\beta\,k,m}_{(0)} = &-\left(\beta k + l_2 + 2m + \frac{1}{2}\right)\\
&\times\left(\beta k + l_2 + 2m + \frac{3}{2}\right).
\end{aligned} \tag{105}
$$

We can define generalised operators $N^{\pm,i}_{n,m}$ in terms of the operators $N^\pm_i$ given in Equations (94), (96), (98), and (100), by replacing $k\beta$ with $k\beta+n$ and l_2 by l_2+m with $n,m\in\mathbb{Z}$. That is:

$$
N^{\pm,i}_{n,m} := N^\pm_i(k\beta\to k\beta+n, l_2\to l_2+m), \tag{106}
$$

for $i=1,\ldots,4$.

Similarly to the sphere case (Subsection 3.4), we can consider index-free operators $N^{a,b}$ defined, for example, in terms of the operators $N^{-,i}_{m,n}$ for all $n,m\in\mathbb{Z}$ as follows:

$$
N^{a,b}\,\varphi_{m,n} := N^{-,i}_{m,n}\,\varphi_{m,n}, \quad a,b=\pm1, \tag{107}
$$

such that $N^{1,1} = N^{-,3}_{m,n}$, $N^{1,-1} = N^{-,2}_{m,n}$, $N^{-1,1} = N^{-,4}_{m,n}$, and $N^{-1,-1} = N^{-,1}_{m,n}$. By composing two of these operators, we can construct the shift operators, defined in Equation (22), which move only in a single direction. In our case, we choose the direction along $k\beta$ (i.e. the n direction).

Considering the composition of these operators acting as $N^{a,b}\varphi_{m,n} = \varphi_{m+a,n+b}$, where $\varphi_{m,n}\in\mathcal{H}^\xi_{M_k;n,m}$, we obtain that:

$$
N^{a,b}N^{-a,b}\varphi_{m,n} = N^{a,b}\varphi_{m-a,n+b} = \varphi_{m,n+2b}, \tag{108}
$$

with $a,b=\pm1$. Equation (107) allows us to define the shift operators (22) as:

$$
S^\pm := N^{a,\pm1}\,N^{-a,\pm1}, \tag{109}
$$

such that:

$$
S^\pm : \mathcal{H}^\xi_{M_k;m,n} \to \mathcal{H}^\xi_{M_k;m,n\pm2}. \tag{110}
$$

Taking into account Equation (40), we can obtain the shift operators $S_{\pm 2m}$ defined in Equation (23) as:

$$S_{\pm 2m} = \left(S^{\pm}\right)^{m}. \tag{111}$$

Thus, together with the operators $L^{\pm}_{\pm 2n}$ (Equation (23)) and $S_{\pm 2m}$ (Equation (111)), we can construct the symmetries $X^{\pm}$ (Equation (15)), that commute with the Hamiltonian. This allows us to prove that the Hamiltonian system H_k (Equation (89)) is superintegrable whenever $k = \frac{m}{n}$ is a rational number.

As mentioned in the TTW $SO(3)$-Hamiltonian case, we also find that $[X^+, X^-] \neq 0$. By evaluating the double commutators $[X^{\pm}, [X^+, X^-]]$, one paves the way for constructing the algebra of integrals of motion. Incidentally, [33] shows that, for both initial Hamiltonians ($SO(3)$ and $SO(2,1)$, when $k = 1$, the algebra of integrals of motion coincides with the Racah algebra $\mathcal{R}(3)$. Furthermore, in [34] a new algebraic method to describe the symmetry of a quadratically superintegrable system on the two-spherecommonly associated with $\mathcal{R}(3)$. Instead of relying on explicit operator realisations, the symmetry algebra is built directly from the enveloping algebra of $su(3)$, using polynomials of degrees 2–4 in a maximal Abelian subalgebra. This leads to a new six-dimensional cubic algebra with integer structure constants, which in specific realisations reduces to $\mathcal{R}(3)$. Moreover, a contraction of this cubic algebra to the symmetry algebra of a Smorodinsky-Winternitz model on the sphere is shown.

In a recent work [35], it was demonstrated that for integer values of the TTW parameter k, the TTW-Hamiltonian, together with two independent integrals of motion and their commutator generate a finite-dimensional polynomial algebra of $k+1$ order. This algebra exhibits polynomial, rather than linear, closure and is referred to as the hidden algebra $\mathfrak{g}(k)$. For $k = 1, 2, 3, 4$ the polynomial structure has been explicitly established, and it is conjectured that the same holds for all positive integer k. The polynomial degree increases with k, reflecting the higher-order nature of the additional integral of motion.

The specific cases we have analysed in this paper fall within this framework and will be the subject of a forthcoming publication elsewhere.

5.2. Associated classical Hamiltonian

The classical counterpart of the Hamiltonian (89), including the "coupling constant" k, is:

$$\mathsf{H}_k = p_\xi^2 - \frac{l_2^2}{\cosh^2\xi} + \frac{k^2\,\mathsf{H}^\theta}{\sinh^2\xi}, \tag{112}$$

where H^θ is given by Equation (69), and $\mathsf{H}^\xi_{\mathsf{M}_k}$ is defined by:

$$\mathsf{H}^\xi_{\mathsf{M}_k} = p_\xi^2 - \frac{l_2^2}{\cosh^2\xi} + \frac{\mathsf{M}_k^2}{\sinh^2\xi}, \tag{113}$$

with $\mathsf{M}_k^2 = k^2\,\mathsf{H}^\theta$. It is worth noting that, unlike in the quantum case, it now depends on θ.

For H^θ (Equation (69)), we obtained the ladder functions $\mathsf{L}^{\pm}$ (Equation (71)) in Subsection 4.1, and for $\mathsf{H}^{\mathsf{M}_k}_\xi$ (Equation (113)) we have the shift functions:

$$\begin{aligned}\mathsf{S}^{\pm} &= (-l_2\tanh\xi \mp \mathsf{M}_k\coth\xi + ip_\xi)\\ &\quad\times(l_2\tanh\xi \mp \mathsf{M}_k\coth\xi + ip_\xi).\end{aligned} \tag{114}$$

Both operators satisfy:

$$\{\mathsf{H}_k, \mathsf{S}^{\pm}\} = \pm\alpha\,\mathsf{S}^{\pm}, \quad \{\mathsf{H}, \mathsf{L}^{\pm}\} = \mp k\alpha\,\mathsf{L}^{\pm}, \tag{115}$$

with:

$$\alpha = \frac{4\mathsf{M}_k\, i}{\sinh^2\xi}. \tag{116}$$

Then, the functions $\mathsf{X}^{\pm} := (\mathsf{S}^{\pm})^m(\mathsf{L}^{\pm})^n$ satisfy:

$$\{\mathsf{H}_k, \mathsf{X}^{\pm}\} = 0 \quad \text{if } k = \frac{m}{n} \in \mathbb{Q}. \tag{117}$$

Thus, for each rational value of k, there exist two independent integrals of motion, explicitly given by:

$$\begin{aligned}\mathsf{X}^{\pm} &= \big(\mathsf{M}_k^2\coth^2\xi - l_2^2\tanh^2\xi\\ &\quad\mp 2i\mathsf{M}_k p_\xi\coth\xi - p_\xi^2\big)^m\\ &\quad\times\left(\frac{b^2-a^2}{\sqrt{\dfrac{a^2}{\cos^2\theta}+\dfrac{b^2}{\sin^2\theta}+p_\theta^2}} + \cos 2\theta\right.\\ &\quad\left.\times\sqrt{\frac{a^2}{\cos^2\theta}+\frac{b^2}{\sin^2\theta}+p_\theta^2} \pm ip_\theta\sin 2\theta\right)^n,\end{aligned} \tag{118}$$

where $a, b \in \mathbb{R}$ (see Equation (68)), which satisfy the reality condition $(\mathsf{X}^+)^* = \mathsf{X}^-$. This condition will be used to compute the classical trajectories, similarly to the spherical case.

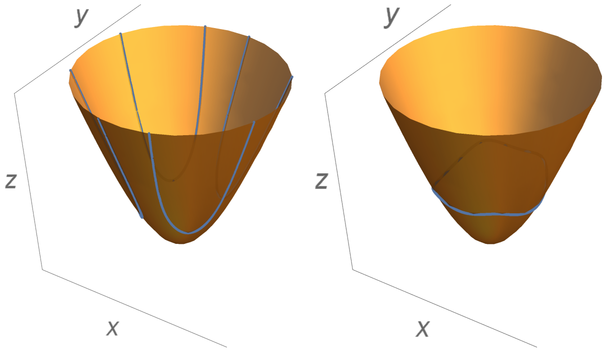
For our system, $\mathsf{H}_k, \mathsf{H}^\theta, \mathsf{X}^{\pm}$, and M_k are all constants of motion, but only three are functionally independent. By fixing the value of $\mathsf{H}_k = \mathsf{E}$, both E and M_k remain constant along the trajectory. Thus, the generalised momenta can be expressed in terms of the generalised coordinates as follows:

$$\begin{aligned}p_\theta &= \varepsilon_\theta\sqrt{\frac{\mathsf{M}_k^2}{k^2} - \left(\frac{\alpha^2}{\cos^2\theta}+\frac{\beta^2}{\sin^2\theta}\right)},\\ p_\xi &= \varepsilon_\xi\sqrt{\mathsf{E} + \left(\frac{l_2^2}{\cosh^2\xi} - \frac{\mathsf{M}_k^2}{\sinh^2\xi}\right)},\end{aligned} \tag{119}$$

with $\varepsilon_\theta, \varepsilon_\xi \in \{\pm 1\}$.

Since the symmetry functions $\mathsf{X}^{\pm}$ are complex, the constants of motion will be complex numbers C. To obtain real representations, and considering the reality condition for $\mathsf{X}^{\pm}$, we set $\mathsf{X}^+ = \mathsf{C}$, and $\mathsf{X}^- = \mathsf{C}^*$, we can consider:

$$\begin{aligned}Re(\mathsf{X}^+) &= \frac{\mathsf{X}^+ + \mathsf{X}^-}{2} = Re(\mathsf{C}),\\ Im(\mathsf{X}^+) &= \frac{\mathsf{X}^+ - \mathsf{X}^-}{2i} = Im(\mathsf{C}).\end{aligned} \tag{120}$$



(A). $m = 1$, $n = 1$. (B). $m = 2$, $n = 1$.

FIGURE 6. Trajectories for $m = 1$, $n = 1$ and $m = 2$, $n = 1$, respectively.

Thus, the trajectories T_0 are implicitly defined by the set of points $\mathbf{x} \in \mathbb{R}^3$ satisfying:

$$\mathsf{X}(\mathbf{x}) = \mathsf{C}_0, \tag{121}$$

where C_0 is a fixed complex constant.

In Figures 6–8 we present trajectories plotted using Mathematica for various values of $k = \frac{m}{n}$.

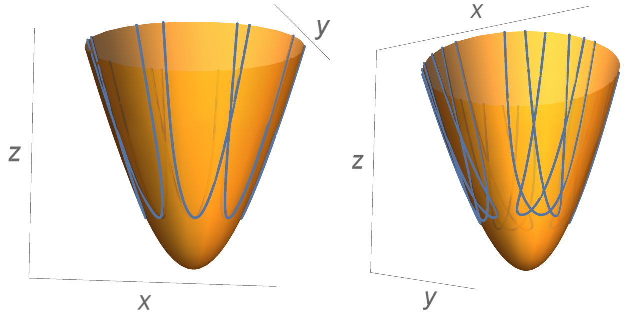
6. CONCLUSIONS

We have examined in detail two new families of Hamiltonians defined on curved spaces, derived from well-known superintegrable Hamiltonians on the sphere [6] and hyperbolic 2-space [7] through a TTW procedure analogous to that used in [9] to generate the TTW Hamiltonians from the Smorodinsky-Winternitz system [12]. This procedure involves deforming the initial Hamiltonian by a real parameter $k \neq 0$, which recovers the initial system in the limit $k \to 1$.

To prove the superintegrability of these new TTW Hamiltonian families, we construct generalised ladder and shift operators via the factorisation method. These operators yield two symmetries of the TTW Hamiltonian, demonstrating that superintegrability is preserved when k is rational.

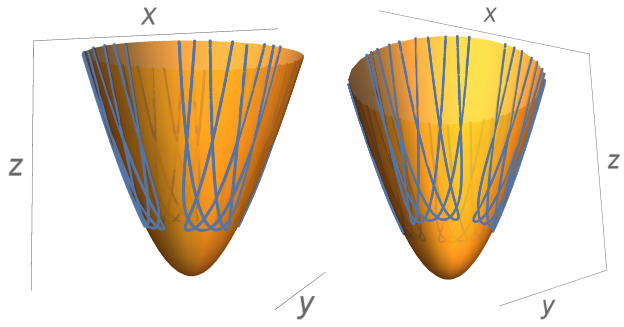
Furthermore, we study the classical counterparts of these Hamiltonians by following a procedure parallel to the quantum case and guided by the correspondence principle [31]. We derive classical analogues of the ladder and shift quantum operators as classical functions, and identify two functions in involution with the Hamiltonian, proving classical superintegrability for rational k. Additionally, the classical trajectories are obtained through an algebraic approach.

An open and mathematically significant problem, particularly in light of [33–35], is the explicit determination of the polynomial algebra underlying the integrals of motion in both cases. A deeper understanding of this algebraic structure could provide new insights into the symmetry properties and superintegrability of these two Hamiltonian systems.



(A). $m = 1$, $n = 2$. (B). $m = 3$, $n = 3$.

FIGURE 7. Trajectories for $m = 1$, $n = 2$ and $m = 3$, $n = 3$, respectively.



(A). $m = 4$, $n = 4$. (B). $m = 4$, $n = 5$.

FIGURE 8. Trajectories for $m = 4$, $n = 4$ and $m = 4$, $n = 5$, respectively.

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Appendix A. Generalisation of Theorem 1

Let us consider a Hamiltonian of the form:

$$H_k = H^x_{M_k} + \frac{k^2(H^y - E')}{f(x)}, \qquad H^x_{M_k} = H^x + \frac{M_k^2}{f(x)}, \tag{122}$$

where (x, y) are the configuration space variables, H^x and H^y are Hamiltonians, depending only on x and y, respectively, and $f(x)$ is a smooth function. Here, $M_k = k\beta$, $k \neq 0$ is a real number, and $E' = \beta^2$ is an eigenvalue of H^y. Suppose that there exist ladder and shift operators L^y_n and S^x_m, respectively, acting as Equation (23):

$$\begin{aligned} L^y_{\pm n} : \mathcal{H}^y &\to \mathcal{H}^y, & S^x_{\pm m} : \mathcal{H}^x_{M_k} &\to \mathcal{H}^x_{M_k \pm m}, \\ \psi_\beta(y) &\mapsto \psi_{\beta\pm n} = L_{\pm n}\psi_\beta(y), & \varphi_{M_k}(x) &\mapsto \varphi_{M_k\pm m}(x) = S^x_{\pm m}\varphi^{M_k}(x), \end{aligned} \tag{123}$$

where $\psi_\beta \in \mathcal{H}^\theta$ is an eigenvector of H^y with eigenvalue β^2 and $\varphi_{M_k} \in \mathcal{H}^x_{M_k}$ is an eigenvector of $H^x_{M_k}$.

Then, if $k = \frac{m}{n}$ is a rational number, there are two symmetry operators ($X^\pm$) of the Hamiltonian H_k, (i.e. operators that commute with H_k), defined as:

$$X^\pm := L^y_{\pm n}\, S^x_{\pm m}, \qquad m, n \in \mathbb{N}^* \equiv \mathbb{N} - \{0\}. \tag{124}$$

Since there are $2 \times 2 - 1$ independent symmetries, (namely $X^\pm$ and H_k), the system is maximally superintegrable.

Proof. By computing the action of the commutator $[H_k, L^y_n S^x_m]$ on an eigenfunction $\Psi(x, y) = \varphi_{M_k}(x)\psi_\beta(y)$ of H_k with eigenvalue E, we obtain:

$$[H_k, L^y_n S^x_m]\varphi_{M_k}\psi_\beta = (H_k - E)L^y_n S^x_m \varphi_{M_k}\psi_\beta, \tag{125}$$

since $\varphi_{M_k}\psi_\beta$ is an eigenfunction of H_k with eigenvalue E. This follows directly from the nested structure of the Hamiltonian (122) and the fact that $H^x_{M_k}\varphi_{M_k} = E\varphi_{M_k}$.

By computing $L^y_n S^x_m \varphi_{M_k}\psi_\beta$, we obtain:

$$L^y_n S^x_m \varphi_{M_k}\psi_\beta = \psi_{\beta+n}\varphi_{M_k+m}, \tag{126}$$

from the definitions of the ladder and shift operators L^y_n (Equation (21)) and S^x_m (Equation (13)), where m and n are integers. Taking this fact into account, the Hamiltonian H_k (Equation (122)) takes the form:

$$H_k = H^x + \frac{(M_k + m)^2}{f(x)} + \frac{-m^2 - 2M_k m}{f(x)} + \frac{k^2(H^y - E')}{f(x)} = H^x_{M_k+m} + \frac{-m^2 - 2M_k m}{f(x)} + \frac{k^2(H^y - E')}{f(x)}, \tag{127}$$

since $H^x + (M_k + m)^2/f(x) = H^x_{M_k+m}$. Applying the Hamiltonian H_k (Equation (127)) to the function $\psi_{\beta+n}\varphi_{M_k+m}$ (Equation (126)) we find that:

$$H_k\varphi_{M_k+m}\psi_{\beta+n} = \left(E + \frac{-m^2 - 2M_k m}{f(x)} + \frac{k^2(E' + 2\beta n + n^2 - E')}{f(x)}\right)\varphi_{M_k+m}\psi_{\beta+n}. \tag{128}$$

For the commutator in Equation (125) to vanish, it is sufficient that the additional term on the right-hand side of Equation (128) vanishes, namely:

$$-m^2 - 2M_k m + k^2(2\beta n + n^2), \tag{129}$$

which can be rearranged as:

$$-(M_k + m)^2 + (M_k + kn)^2 = 0, \tag{130}$$

recalling that $k\beta = M_k$. Equation (130) admits two possible solutions:

$$M_k + m = M_k + kn, \tag{131}$$

$$M_k + m = -M_k - kn. \tag{132}$$

The second solution (Equation (132)) is valid only for specific values of ψ_β and M_k, and not in general. Therefore, the appropriate and general solution is the first one, Equation (131), which leads to the condition:

$$k = \frac{m}{n}. \tag{133}$$

This result shows that we must construct ladder operators L^y_n and shift operators S^x_m such that they preserve the quantity M_k. In our two systems, as previously observed, both m and n must be even integers. Moreover, since both operators depends of different variables, they commute. ∎