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PERSISTENCE OF SOLVABILITY IN QUANTUM SYSTEMS DEFORMED BY DUNKL OPERATORS

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Abstract. We study persistence of solvability in nonrelativistic quantum systems with position-dependent mass upon introduction of a deformation by Dunkl operators. Conditions are derived for the governing Schrödinger equation of the conventional system to admit the same solutions as in the deformed case, up to a reparametrisation of coupling constants. These conditions require the position-dependent mass or the potential of the system to have a specific form. If this is the case for a particular system, then the Schrödinger equations for its conventional version and for the Dunkl-deformed partner share solutions in the same functional form.

Keywords: Schrödinger equation, Dunkl operator, bound states, position-dependent mass.

1. Introduction

In 1950, it was found that the fundamental quantum-mechanical commutation relations are not uniquely determined by the equations of motion [1]. This result, obtained for the harmonic oscillator system, allowed for a generalised definition of the momentum operator involving space reflections [2]. Further extensions of these findings concern the introduction of parafield quantisation [3, 4] and related applications, see, for example, [5–7] and references within. Independent of these studies that go back to the initial work [1], in 1989 the concept of Dunkl operators was introduced [8]. These operators are parametrised generalisations of directional derivatives that involve reflection operators. It turned out that Dunkl operators have a large amount of applications, such as in quantum-integrable Calogero-Moser-Sutherland systems [9], the higher-dimensional Schrödinger case [10], time-dependent problems [11], the path integral formalism for time-dependent systems [12], vacuum pair creation [13], and Bose-Einstein condensation [14]. In 1994 it was pointed out [15] that the two concepts studied in [1] and [8] are very closely related, that is, the generalised definition of the momentum operator provided in [2] can be written in terms of Dunkl operators. As such, these operators have been used to construct deformations of quantum mechanics, in which the standard momentum operators are replaced by generalised versions that involve Dunkl operators. There is a vast amount on literature about specific quantum systems that were studied within the Dunkl formalism. Recent examples include the massless Dirac equation in the graphene setting [16], rational extensions of the harmonic oscillator system [17], and the Klein-Gordon equation in three dimensions [18]. Further applications of the Dunkl formalism to quantum systems are given by the path integral formulation [19], the Dunkl-Fokker-Planck equation [20], the Pauli equation in the presence of a magnetic field [21], a two-parameter version of the Dunkl operator applied to two-dimensional Schrödinger equations [22], and closed-form solutions of specific systems [23, 24]. In many of the previous applications to specific quantum systems, it was observed that the Schrödinger equation remains solvable for both the conventional system as well as for its associated partner within the Dunkl formalism. This is remarkable because the latter formalism introduces additional terms in the Schrödinger equation that can lead to a loss of solvability. Therefore, the purpose of the present work is to investigate under which conditions solvability of a conventional Schrödinger equation is preserved in its Dunkl partner equation. For the sake of generality we will consider systems that include a position-dependent mass. Such masses appear in the context of transport phenomena in crystals (e.g. semiconductors), where the electrons interact with the lattice potential, rather than being completely free. The dynamics of such electrons can be described by introducing a position-dependent mass, the behaviour of which is determined by the band curvature [25–29]. For more details and further applications of position-dependent masses, the reader is referred to [30] and [31]. The remainder of this work is organized as follows: in Section 2 we will introduce the position-dependent mass Dunkl-Hamiltonian, along with the corresponding Dunkl-Schrödinger equation. Section 3 is devoted to the construction of conditions for solvability of both the conventional equation and its Dunkl partner. Several applications of our method are presented in Section 4.

2. Dunkl-Hamiltonian and Schrödinger equation

Nonrelativistic quantum systems featuring a position-dependent mass, are governed by a symmetrised Hamiltonian. In its most general hermitian form, this Hamiltonian H_0 can be written as [29, 30]:

$$H_0 = \frac{1}{2}\ \left[m(x)^{\alpha}\ p_0\ m(x)^{-1-\alpha-\gamma}\ p_0\ m(x)^{\gamma} + m(x)^{\gamma}\ p_0\ m(x)^{-1-\alpha-\gamma}\ p_0\ m(x)^{\alpha}\right] + V(x), \tag{1}$$

where p_0 stands for the momentum operator, and α, γ represent real constants. Furthermore, the position-dependent mass $m \geq 0$, and the potential V are continuous functions, except possibly at a finite number of singular points that the mass or the potential exhibit. While a point singularity is nothing unusual for a potential (a famous example is the Coulomb interaction), it is much less common in position-dependent masses due to their physical meaning. However, such singular mass functions are used in Lienard oscillator systems, which we will comment on in our example, Section 4. Note that the constants α and γ arise from an ordering ambiguity of momentum operators and position-dependent mass functions. This ambiguity is inherent to a position-dependent mass Hamiltonian. In order to obtain the latter operator in a hermitian form, there are infinitely many choices for the values of α and β. While all of these values render (1) hermitian, they affect the resulting Schrödinger equation, as will be shown in Section 4. Now, the momentum operator p_0 in (1) can be written in the usual form:

$$p_0 = -i\ \frac{d}{dx}. \tag{2}$$

We can generalise the Hamiltonian (1) by replacing the standard derivatives through the Dunkl operator. This operator can be written as:

$$D_\nu = \frac{d}{dx} + \frac{\nu}{x} - \frac{\nu}{x}\ R, \tag{3}$$

where $\nu > -\frac{1}{2}$ is a real number, and R stands for the reflection or parity operator that acts on an admissible function Ψ in the way $R\Psi(x) = \Psi(-x)$. We will now use (3) to generalise the momentum operator (2) as follows:

$$p_\nu = -i\ D_\nu. \tag{4}$$

Let us emphasize here that the definitions (3) and (4) generate a deformation of the conventional quantum theory. More precisely, it will turn out that substituting the generalised momentum operator into our Hamiltonian affects the Hilbert space that normalisable eigenfunctions are located in. The deformation of the conventional quantum theory can be controlled by the parameter ν: inspection of (3) reveals that the setting $\nu = 0$ yields the standard momentum operator (2), thus removing the deformation. In the next step, we rewrite our Hamiltonian (1) by replacing the momentum operators by their Dunkl counterparts (4). Upon substitution of (4), we obtain our Dunkl-Hamiltonian in the form:

$$H_\nu = \frac{1}{2}\ \left[m(x)^{\alpha}\ p_\nu\ m(x)^{-1-\alpha-\gamma}\ p_\nu\ m(x)^{\gamma} + m(x)^{\gamma}\ p_\nu\ m(x)^{-1-\alpha-\gamma}\ p_\nu\ m(x)^{\alpha}\right] + V(x). \tag{5}$$

This Hamiltonian generates its associated Dunkl-Schrödinger equation by means of $(H_\nu - E)\Psi = 0$, where Ψ stands for the solution and E represents the real-valued stationary energy of the system. In explicit form, this Dunkl-Schrödinger equation reads:

$$\begin{aligned}\Psi_\nu''(x) + \left[\frac{2\ \nu}{x} - \frac{m'(x)}{m(x)}\right]\ \Psi_\nu'(x) + \Bigg\{ &\frac{\delta\ \nu - \nu}{x^2} + [-\nu + \delta\ \nu + (\alpha+\gamma)\ \delta\ \nu]\ \frac{m'(x)}{x\ m(x)} \\ &-(\alpha+\gamma+\alpha\ \gamma)\ \left[\frac{m'(x)}{m(x)}\right]^2 + (\alpha+\gamma)\ \frac{m''(x)}{2\ m(x)} + E - V(x)\Bigg\}\ \Psi_\nu(x) = 0.\end{aligned} \tag{6}$$

As mentioned above, the setting $\nu = 0$ removes the Dunkl scenario, such that the conventional Schrödinger equation for the Hamiltonian (1) is recovered. We find:

$$\Psi_0''(x) - \frac{m'(x)}{m(x)}\ \Psi_0'(x) + \left\{-(\alpha+\gamma+\alpha\ \gamma)\ \left[\frac{m'(x)}{m(x)}\right]^2 + (\alpha+\gamma)\ \frac{m''(x)}{2\ m(x)} + E - V(x)\right\} \times \Psi_0(x) = 0. \tag{7}$$

This form coincides with the result found in [30]. In order for a solution of (6) to be normalisable, it must be an element of the weighted Hilbert space $L^2_w(\mathbb{R})$ with weight function:

$$w(x) = |x|^{2\nu}. \tag{8}$$

The associated norm of a function Ψ_ν in $L^2_w(\mathbb{R})$ is then given by:

$$\|\Psi_\nu(x)\| = \left[\int_{\mathbb{R}} \Psi_\nu(x)^* \Psi_\nu(x)\ |x|^{2\nu}\ dx\right]^{\frac{1}{2}}. \tag{9}$$

Let us now simplify the form of our Dunkl-Schrödinger equation (6) by removing the first-order derivative term $\sim \Psi'_\nu$. To this end, we apply the point transformation:

$$\Psi_\nu(x) = \frac{\sqrt{m(x)}}{x^\nu}\ \xi_\nu(x), \tag{10}$$

such that the new function ξ_ν becomes a solution of the following equation:

$$\xi''_\nu(x) + U_\nu(x)\ \xi_\nu(x) = 0, \tag{11}$$

where the effective potential U_ν is given by:

$$\begin{aligned} U_\nu(x) = [E - V(x)]\ m(x) - \left(\frac{3}{4} + \alpha + \gamma + \alpha\ \gamma\right)\left[\frac{m'(x)}{m(x)}\right]^2 + (1 + \alpha + \gamma)\ \frac{m''(x)}{2\ m(x)} \\ + \nu\ \left[\frac{\delta}{x^2} + (\delta + \alpha\ \delta + \gamma\ \delta)\ \frac{m'(x)}{x\ m(x)}\right] - \frac{\nu^2}{x^2}. \end{aligned} \tag{12}$$

Note that we collected terms arising from the Dunkl formalism by means of the parameter ν. According to the effective potential (12), the latter formalism contributes an inverse square singularity at the origin, as well as possible singularities in the term $\sim \frac{m'}{xm}$. As such, the parameters ν and δ can be used to remove singularities that the potential or the position-dependent mass contain. We will discuss examples in Section 4. Next, let us point out that the particular parameter setting $\alpha = \gamma = -\frac{1}{2}$ renders the function (12) in a particularly simple form. We find:

$$U_\nu(x) = [E - V(x)]\ m(x) + \frac{\nu\ \delta - \nu^2}{x^2}. \tag{13}$$

If we set $\nu = 0$, the Dunkl formalism is removed, such that our equation (11) takes the form:

$$\xi''_0(x) + U_0(x)\ \xi_0(x) = 0. \tag{14}$$

Here, the effective potential U_0 is a special case of (12) that reads:

$$U_0(x) = [E - V(x)]\ m(x) - \left(\frac{3}{4} + \alpha + \gamma + \alpha\ \gamma\right)\left[\frac{m'(x)}{m(x)}\right]^2 + (1 + \alpha + \gamma)\ \frac{m''(x)}{2\ m(x)}. \tag{15}$$

Our next goal is to determine conditions, under which the effective potentials (12) and (15) have the same functional form, such that their associated equations have the same solutions, up to a redefinition of parameters.

3. Solvability conditions

We will now compare the forms of the reduced Dunkl-Schrödinger equation (11) and its conventional counterpart (14). More precisely, we will focus on their effective potentials U_ν and U_0, as given in (12) and (15), respectively. If these two potentials have the same functional form, but differ merely in numerical parameters, then the same is true for solutions to the corresponding Schrödinger equations. In other words, if a solution to the conventional equation is known, then it can be mapped onto a solution of the equation within the Dunkl scenario by a change of numerical parameters. The functional form of the two effective potentials (12) and (15) is, in general, different due to two terms that are contributed by the Dunkl formalism in (12): one of these terms is proportional to m'/xm, while the other term is inverse quadratic. We will now set up conditions on the position-dependent mass and the potential, such that the latter two terms appear in both effective potentials (12) and (15).

First condition. Let us focus on the term proportional to m'/xm that appears in the effective potential (12). We will now set up a condition for the position-dependent mass, such that the latter term can be written as a linear combination of existing terms in (15). This condition reads:

$$\frac{m'(x)}{x\ m(x)} = A\ \frac{m''(x)}{m(x)} + B\left[\frac{m'(x)}{m(x)}\right]^2, \tag{16}$$

where A and B are arbitrary constants. The condition can be solved in closed form for the position-dependent mass. The general solution reads:

$$m(x) = c_2 \left[c_1\ (A+1) + (A+B)\ x^{1+\frac{1}{A}}\right]^{\frac{A}{A+B}}, \tag{17}$$

introducing arbitrary parameters c_1 and c_2. Special cases of the function (17) include, but are not limited to, the following:

$$\begin{aligned} m(x) &= x^{\lambda} \\ m(x) &= \frac{1}{x^2 \pm \lambda} \\ m(x) &= x^2 + 1 \\ m(x) &= \frac{1}{(x^2 \pm \lambda)^2} \\ m(x) &= \frac{x^4}{(x^2 \pm \lambda)^2}, \end{aligned}$$

where λ represents a constant. Now, substitution of (16) into the effective potential (12) gives:

$$\begin{aligned} U_\nu(x) = [E - V(x)]\ m(x) - \left[\frac{3}{4} + \alpha + \gamma + \alpha\ \gamma - B\ \nu\ (\delta + \alpha\ \delta + \gamma\ \delta)\right] \left[\frac{m'(x)}{m(x)}\right]^2 \\ + \left[1 + \alpha + \gamma + 2\ A\ \nu\ (\delta + \alpha\ \delta + \gamma\ \delta)\right]\ \frac{m''(x)}{2\ m(x)} + \frac{\delta\ \nu - \nu^2}{x^2}, \end{aligned} \tag{18}$$

note that we omit to insert the explicit form of the mass (17) in order to avoid large expressions. Comparison of (18) with its counterpart (15) shows that now both have the same functional form except for numerical parameters, and for the inverse quadratic term. Before we conclude this paragraph, let us point out that further simplification of (18) can be reached by means of the parameter ν from the Dunkl operator (3). While, in general, we keep this parameter undetermined, it is possible to assign it specific values. For example, the coefficient terms of the mass functions in (18) vanish if the value for ν is chosen suitably, thus rendering the function U_ν in simpler form. A further example for assigning specific values to the parameter ν is to establish parity of solutions to the Dunkl-Schrödinger equation. This will be further discussed in Section 4.

Second condition. Since the effective potential (15) does not contain an inverse quadratic term, we must generate it by means of choosing the potential V in a suitable form, or by constraining the position-dependent mass (17). This way we obtain three different conditions that will be distinguished now.

- The first option to include an inverse quadratic term is to constrain the form of the potential V. The corresponding condition takes the form:

$$V(x) = \frac{C}{x^2\ m(x)} + W(x), \tag{19}$$

  where C is an arbitrary constant, and W contains the remaining terms of the potential, if any. Note that W is not allowed to contain any term of the form $\sim [x^2 m]^{-1}$. In order to compare the two effective potentials for the Dunkl context and the conventional scenario, let us now substitute (19) into (18) and (15). The first potential takes the form:

$$\begin{aligned} U_\nu(x) = [E - W(x)]\ m(x) - \left[\frac{3}{4} + \alpha + \gamma + \alpha\ \gamma - B\ \nu\ (\delta + \alpha\ \delta + \gamma\ \delta)\right] \left[\frac{m'(x)}{m(x)}\right]^2 \\ + \left[1 + \alpha + \gamma + 2\ A\ \nu\ (\delta + \alpha\ \delta + \gamma\ \delta)\right]\ \frac{m''(x)}{2\ m(x)} + \frac{\delta\ \nu - \nu^2 - C}{x^2}, \end{aligned} \tag{20}$$

  while the second potential reads:

$$\begin{aligned} U_0(x) = [E - W(x)]\ m(x) - \left(\frac{3}{4} + \alpha + \gamma + \alpha\ \gamma\right) \left[\frac{m'(x)}{m(x)}\right]^2 \\ + (1 + \alpha + \gamma)\ \frac{m''(x)}{2\ m(x)} - \frac{C}{x^2}. \end{aligned} \tag{21}$$

  We observe that the two effective potentials have the same functional form. Recall that the two conditions for this are given by (17) and (19).

- Instead of constraining the potential V, we can also use the position-dependent mass to generate an inverse quadratic term in (15). Note that this will result in a compatibility condition for the expression (17). Starting out with the term $\sim \frac{m''}{m}$, we must satisfy:

$$\begin{aligned} &\frac{m''(x)}{m(x)} = \frac{C}{x^2} \\ &\rightsquigarrow \; m(x) = c_3 \; x^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4C}}, \end{aligned} \tag{22}$$

  introducing arbitrary constants c_3 and C. Since this condition as well as (17) are imposed on the position-dependent mass function simultaneously, both conditions must be compatible. This can be achieved by setting the parameters in (17) as follows:

$$c_1 = 0, \qquad c_2 = \frac{c_3 \; (-1 \pm \sqrt{1+4\;C})}{2}, \qquad A = \frac{2}{-1 \pm \sqrt{1+4\;C}}, \qquad B = 0. \tag{23}$$

  Substitution of these settings into the function U_ν, as given in (18), gives the result:

$$\begin{aligned} U_\nu(x) = c_3 \; x^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4C}} \; [E - V(x)] - \frac{1}{8\;x^2} \; \Big[& \; 3 + 4\;\alpha + 4\;\gamma + 4\;\alpha\;\gamma \\ & + 2\;C\;(1+2\;\alpha)\;(1+2\;\gamma) - 4\;\delta\;\nu\;(3+\alpha+\gamma) \\ & + 8\;\nu^2 \pm \sqrt{1+4\;C}\;(3+4\;\alpha+4\;\gamma+4\;\alpha\;\gamma-4\;\delta\;\nu-4\;\alpha\;\delta\;\nu-4\;\gamma\;\delta\;\nu)\;\Big]. \end{aligned} \tag{24}$$

  The settings (23) render the remaining effective potential U_0 in the form:

$$\begin{aligned} U_0(x) = c_3 \; x^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4C}} \; [E - V(x)] - \frac{1}{8\;x^2} \; \Big[& \; 3 + 4\;\alpha + 4\;\gamma + 4\;\alpha\;\gamma \\ & + 2\;C\;(1+2\;\alpha)\;(1+2\;\gamma) \pm \sqrt{1+4\;C}\;(3+4\;\alpha+4\;\gamma+4\;\alpha\;\gamma)\;\Big]. \end{aligned} \tag{25}$$

  In summary, if the conditions (17) and (22) with (23) are met, then the two functions (24) and (25) have the same functional form.

- The last option to produce an inverse quadratic term in the effective potential (15) is to use the position-dependent mass by means of the term $(m'/m)^2$ that appears in (15). The corresponding condition is given by:

$$\begin{aligned} &\left[\frac{m'(x)}{m(x)}\right]^2 = \frac{C}{x^2} \\ &\rightsquigarrow \; m(x) = c_3 \; x^{\pm\sqrt{C}}. \end{aligned} \tag{26}$$

  This is a particular case of our previous condition (22), such that we do not need to discuss it further. There are more possibilities for setting up conditions leading to an inverse quadratic term in (18), such as a linear combination of (19) and (22). However, in any case, the compatibility with the position-dependent mass given in (17) must be established.

Summary. In conclusion, the effective potentials for the Dunkl case and for the conventional scenario have the same functional form, if the condition (17), and one of the conditions, (19), (22), are fulfilled. In this case, the Dunkl-Schrödinger equation (6) and its conventional partner (7) share the same solutions, up to a change of numerical parameters. Let us now summarise the two main routes for fulfilling the solvability conditions.

- We impose the two constraints (17) and (19):

$$m(x) = c_2 \; \Big[c_1 \; (A+1) + (A+B) \; x^{1+\frac{1}{A}}\Big]^{\frac{A}{A+B}}, \qquad V(x) = \frac{C}{x^2 \; m(x)} + W(x). \tag{27}$$

  The resulting effective potential of the reduced equation (11) takes the form (20):

$$\begin{aligned} U_\nu(x) = [E - W(x)] \; m(x) - \left[\frac{3}{4} + \alpha + \gamma + \alpha\;\gamma - B\;\nu\;(\delta + \alpha\;\delta + \gamma\;\delta)\right] \left[\frac{m'(x)}{m(x)}\right]^2 \\ + \left[1 + \alpha + \gamma + 2\;A\;\nu\;(\delta + \alpha\;\delta + \gamma\;\delta)\right] \frac{m''(x)}{2\;m(x)} + \frac{\delta\;\nu - \nu^2 - C}{x^2}. \end{aligned} \tag{28}$$

- We impose the constraint (22) along with (23):

$$m(x) = c_3\ x^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4C}}. \tag{29}$$

The resulting effective potential of the reduced equation (11) takes the form (24):

$$\begin{aligned} U_\nu(x) = c_3\ x^{\frac{1}{2}\pm\frac{1}{2}\sqrt{1+4C}}\ [E-V(x)] - \frac{1}{8\ x^2}\ \Big[\ & 3+4\ \alpha+4\ \gamma+4\ \alpha\ \gamma \\ & +2\ C\ (1+2\ \alpha)\ (1+2\ \gamma)-4\ \delta\ \nu\ (3+\alpha+\gamma) \\ & +8\ \nu^2 \pm \sqrt{1+4\ C}\ (3+4\ \alpha+4\ \gamma+4\ \alpha\ \gamma-4\ \delta\ \nu-4\ \alpha\ \delta\ \nu-4\ \gamma\ \delta\ \nu)\ \Big]. \end{aligned} \tag{30}$$

4. Applications

We will now present several examples for quantum systems, where solutions of the governing Dunkl-Schrödinger equation can be derived directly from the conventional context. In each example, we verify that our solvability conditions are satisfied, and afterwards construct bound state solutions for the Dunkl-Schrödinger equation from the conventional system. In each of the following examples, we will point out existing applications for the respective potential and the position-dependent mass in conventional quantum systems. The study of the systems within the Dunkl formalism is motivated by the interest in deformed quantum theories rather than in experimental realisability. For this reason, we cannot give a physical or experimental realisation of the latter systems.

4.1. Inverse-quadratic potential with confinement

In the first example we will consider a system constrained onto the finite interval given by $|x| < 1$. After verifying our solvability conditions, we find the general solution to the conventioanl Schrödinger equation. Afterwards, we construct the general solution of the associated Dunkl-Schrödinger equation by means of reparametrisation.

The Dunkl quantum model. Our system is governed by the Dunkl-Schrödinger equation (6) for the following position-dependent mass m and potential V:

$$m(x) = \frac{1}{1-x^2}, \qquad V(x) = \frac{V_1}{x^2}, \tag{31}$$

where V_1 is a real constant. We observe that the mass is singular at the endpoints of the domain, while the potential is singular at the origin. The specific combination of potential and mass given in (31), appears, for example, in isotonic oscillator models [32]. More generally, position-dependent masses, as in (31), have applications in quantum versions of the Mathews-Lakshmanan oscillator system [33, 34], see also [35].

Solvability conditions and effective potentials. In order to determine if solutions of the Dunkl-Schrödinger equation for the settings (31) can be found from the conventional system, we will verify the solvability conditions (17) and (19). As far as the first condition is concerned, we compare the position-dependent mass in (31) with the general form (17). Both of these functions coincide if the following settings are in effect:

$$c_1 = \frac{1}{2}, \qquad c_2 = -1, \qquad A = 1, \qquad B = -2. \tag{32}$$

We proceed by verifying our second solvability condition (19). Substituting (31) leads to:

$$\frac{V_1}{x^2} = \frac{C\ (1-x^2)}{x^2} + W(x) = \frac{C}{x^2} - C + W(x). \tag{33}$$

It is straightforward to see that the left and the right side become equal, provided we set $W = C = V_1$. Since our two conditions (17) and (19) are satisfied, the potentials U_ν in (20) and U_0 in (15) with our settings (31) must have the same functional form except for constant parameters. We verify this by substituting the latter settings into (20), which gives the result:

$$\begin{aligned} U_\nu(x) = \frac{1}{x^2\ (x^2-1)^2}\ \Big\{ & V_1+\delta\ \nu-\nu^2+x^2\ [1-E-V_1+\alpha+\gamma-\delta\ \nu \\ & +\delta\ \nu\ (1+2\ \alpha+2\ \gamma)+2\ \nu^2]+x^4\ [E-\alpha-\gamma-4\ \alpha\ \gamma \\ & -\delta\ \nu\ (1+2\ \alpha+2\ \gamma)-\nu^2]\Big\}. \end{aligned} \tag{34}$$

In the next step, we obtain the counterpart U_0 either by inserting the settings (31) into (15), or by setting $\nu = 0$ in (34). We obtain:

$$U_0(x) = \frac{1}{x^2\ (x^2-1)^2}\ \Big\{V_1 + x^2\ [1 - E - V_1 + \alpha + \gamma] + x^4\ [E - \alpha - \gamma - 4\ \alpha\ \gamma]\Big\}. \tag{35}$$

Comparison with (34) shows that the functions have the same functional form, such that the reduced Schrödinger equations (11) and (14) have the same solutions, up to a redefinition of numerical parameters.

Reduced equations and general solutions. Before we state these solutions, let us simplify subsequent calculations by setting $\alpha = \gamma = -\frac{1}{2}$. Upon substituting these values into (34) and (35), the associated reduced Schrödinger equations take the form:

$$\xi_\nu''(x) + \left(\frac{E - V_1}{x^2 - 1} + \frac{\delta\ \nu - \nu^2 + V_1}{x^2}\right)\xi_\nu(x) = 0, \tag{36}$$

$$\xi_0''(x) + \left(\frac{E - V_1}{x^2 - 1} + \frac{V_1}{x^2}\right)\xi_0(x) = 0. \tag{37}$$

We observe that in (36) the parameters ν and δ from the Dunkl operator affect the singularity at the origin that is contributed by the potential. In particular, the latter parameters can be chosen such that the singularity is removed. Now, once we know a solution of the conventional equation (37), we can generate a solution to the reduced Dunkl-Schrödinger equation (36) by a reparametrisation of the coefficients to the terms $\sim (x^2-1)^{-1}$ and $\sim x^{-2}$. We observe that the parameters ν and δ from the Dunkl operator (3) affect the behaviour of the effective potential in equation (36). In particular, for obtaining bound states of our system, the singularity at the origin should be attractive, meaning that:

$$\delta\ \nu - \nu^2 + V_1 < 0. \tag{38}$$

This condition can be satisfied by making V_1 negative with sufficiently large value in modulus. We will comment on this further below when determining the stationary energies. Now, the general solution of (37) is known and can be written in the form:

$$\begin{aligned}\xi_{0,\mathrm{gen}}(x) = {} & C_1\ x^{\frac{1}{2}-\frac{g_2}{2}}\,{}_2F_1\left(-\frac{g_1}{4} - \frac{g_2}{4}, \frac{g_1}{4} - \frac{g_2}{4}, 1 - \frac{g_2}{2}, x^2\right) \\ & + C_2\ x^{\frac{1}{2}+\frac{g_2}{2}}\,{}_2F_1\left(-\frac{g_1}{4} + \frac{g_2}{4}, \frac{g_1}{4} + \frac{g_2}{4}, 1 + \frac{g_2}{2}, x^2\right),\end{aligned} \tag{39}$$

where the following abbreviations are in use:

$$g_1 = \sqrt{1 - 4\ E}, \qquad g_2 = \sqrt{1 - 4\ V_1}. \tag{40}$$

Since the effective potentials (34) and (35) have the same functional form, we can now construct the general solution of the reduced Dunkl-Schrödinger equation (36). After a reparametrisation, we find:

$$\begin{aligned}\xi_{\nu,\mathrm{gen}}(x) = {} & C_1\ x^{\frac{1}{2}-\frac{f_2}{2}}\,{}_2F_1\left(-\frac{f_1}{4} - \frac{f_2}{4}, \frac{f_1}{4} - \frac{f_2}{4}, 1 - \frac{f_2}{2}, x^2\right) \\ & + C_2\ x^{\frac{1}{2}+\frac{f_2}{2}}\,{}_2F_1\left(-\frac{f_1}{4} + \frac{f_2}{4}, \frac{f_1}{4} + \frac{f_2}{4}, 1 + \frac{f_2}{2}, x^2\right),\end{aligned} \tag{41}$$

where ${}_2F_1$ denotes the hypergeometric function [36], and C_1, C_2 stand for arbitrary constants. Furthermore, the parameters f_1, f_2 are defined as:

$$f_1 = \sqrt{1 - 4\ E + 4\ \nu\ (\nu - \delta)}, \qquad f_2 = \sqrt{1 - 4\ V_1 + 4\ \nu\ (\nu - \delta)}. \tag{42}$$

Recall that we are interested in the solution of the actual Dunkl-Schrödinger equation (6), the general solution $\Psi_{\nu,\mathrm{gen}}$ of which is obtained from (41) by means of point transformation (10). Insertion of the position-dependent mass from (31) gives the relationship:

$$\Psi_{\nu,\mathrm{gen}}(x) = \frac{1}{\sqrt{x^2+1}\ x^\nu}\ \xi_{\nu,\mathrm{gen}}(x). \tag{43}$$

Rather than considering the general solution of our problem, let us now restrict ourselves to the case of bound states.

Bound states of the Dunkl-Schrödinger equation. Solutions of bound state type can be extracted from (41) by setting $C_1 = 0$ in order to remove singular terms, and by requiring the first argument of the remaining hypergeometric function to equal a nonpositive integer, recall that this will make the latter function degenerate to a polynomial. The condition reads:

$$-\frac{f_1}{4} + \frac{f_2}{4} = -n. \tag{44}$$

Upon substituting the abbreviations (42), we can solve this condition with respect to the stationary energy $E = E_{\nu,n}$. This yields:

$$\begin{aligned} E_{\nu,n} &= -4\ n^2 + V_1 - 2\ n\ f_2 \\ &= -4\ n^2 + V_1 - 2\ n\ \sqrt{1 - 4\ V_1 + 4\ \nu\ (\nu - \delta)}. \end{aligned} \tag{45}$$

We observe that the root on the right side contains a term that appears in (38). If the latter condition is fulfilled, then the stationary energies (45) take real values only, which is in accordance with bound state energies. However, if V_1 takes positive values, then (45) can become complex-valued, depending on the parameters ν and δ from the Dunkl operator (3). Now we insert this energy, and the setting $C_1 = 0,\ C_2 = 1$ into (41). Combination with (43) gives the solutions of our Dunkl-Schrödinger equation (6) in the form:

$$\Psi_{\nu,n}(x) = \frac{1}{\sqrt{x^2+1}}\ x^{-\nu+\frac{1}{2}+\frac{f_2}{2}}\ {}_2F_1\left(-n, \frac{f_1}{4} + \frac{f_2}{4}, 1 + \frac{f_2}{2}, x^2\right). \tag{46}$$

These solutions include the standard case (without Dunkl formalism), that we obtain by setting $\nu = 0$ in (46). Note that the negative power of ν in the exponent on the right side of (46) does not create a singularity at the origin, as long as V_1 is negative. This is due to the occurrence of ν in the constant f_2, see (42). If the Dunkl formalism is present, meaning $\nu \neq 0$, then one more detail must be considered before we have the solution in its final form. This detail is the solution's parity. As we can see by inspecting the explicit form (46), our solution does not have parity due to the exponent of the monomial term. In order for the solution to have the correct parity, this exponent must be an odd integer if $\delta = -1$ (odd parity), and it must equal an even integer for $\delta = 1$ (even parity). In the first of the two cases the aforementioned condition reads:

$$\begin{aligned} -\nu + \frac{1}{2} + \frac{f_2}{2} &= 2\ k + 1 \\ \rightsquigarrow\ \nu &= -\frac{1}{2} - k - \frac{V_1}{4\ k}, \end{aligned} \tag{47}$$

where k is an arbitrary integer. In the remaining case $\delta = 1$ we must have:

$$\begin{aligned} \nu + \frac{1}{2} + \frac{f_2}{2} &= 2\ k \\ \rightsquigarrow\ \nu &= \frac{1}{2} - k - \frac{V_1}{4\ k}. \end{aligned} \tag{48}$$

It is important to point out that the value of k does not have to be the same for odd and even solutions. Hence, by choosing different values of k in (47) and (48), one can obtain the same value of ν in both cases. In summary, the function (46) is a solution of our Dunkl-Schrödinger equation (6) for $\nu \neq 0$ and the present settings (31), provided the parameter ν complies with (47) for $\delta = -1$, and it complies with (48) for $\delta = 1$. Particular cases of our solutions are shown in Figures 1 and 2.

We omit to show graphs of solutions to the standard Schrödinger equation, as these are similar to the ones displayed in the above figures.

4.2. Inverse-quadratic potential without confinement

In our next example, we will focus on a system that is closely related to the previous one. In particular, we will use the same potential, but modify the position-dependent mass, such that our system is defined on the whole real line. Since the approach to the present example is the same as for the last case, we will omit details. Our starting point is the position-dependent mass and the potential that define our system. These functions are given by:

$$m(x) = \frac{1}{1+x^2}, \qquad V(x) = \frac{V_1}{x^2}, \tag{49}$$

where V_1 is a real-valued coupling constant. The mass profile given in (49) can be defined on the real line, and it has been used in the literature, see, for example, [30] and references therein. In the next step, we verify that our solvability conditions are fulfilled. The first condition (17) can be satisfied by setting:

$$c_1 = -\frac{1}{2}, \qquad c_2 = -1, \qquad A = 1, \qquad B = -2, \tag{50}$$

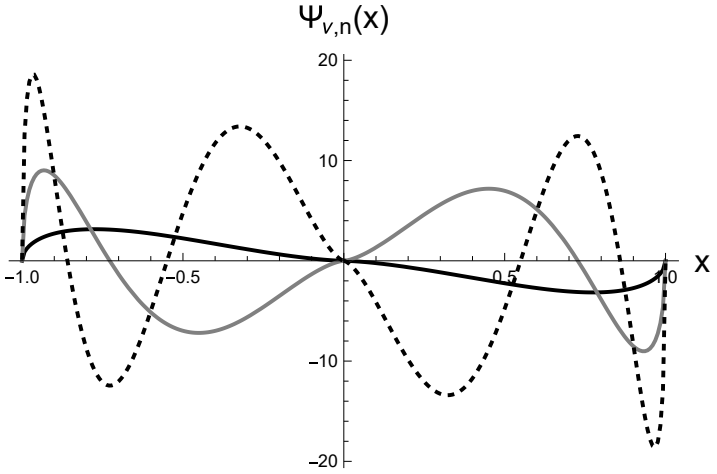

FIGURE 1. Graphs of odd-parity ($\delta = -1$) solutions (46) for the settings $V_1 = -1$ and $k = -1$, corresponding to $\nu = \frac{1}{4}$. The solutions are given for $n = 1$ (black solid curve), $n = 2$ (gray curve), $n = 3$ (dashed curve).

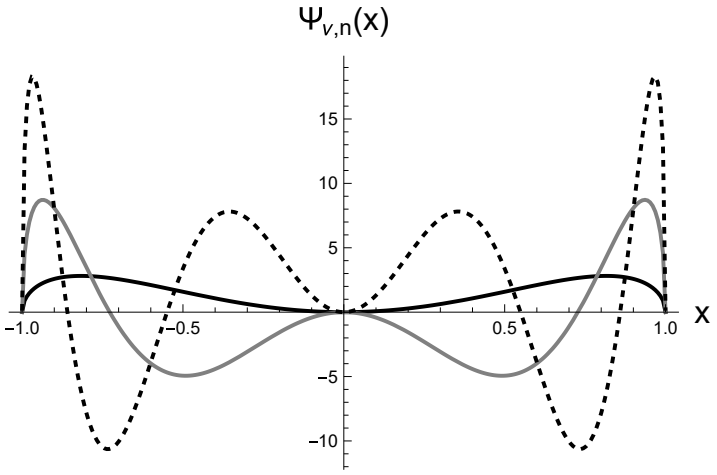

FIGURE 2. Graphs of even-parity ($\delta = 1$) solutions (46) for the settings $V_1 = -1$ and $k = 1$, corresponding to $\nu = -\frac{1}{4}$. The solutions are given for $n = 1$ (black solid curve), $n = 2$ (gray curve), $n = 3$ (dashed curve).

while the second condition (19) in combination with (49) takes the form:

$$\frac{V_1}{x^2} = \frac{C\ (1+x^2)}{x^2} + W(x) = \frac{C}{x^2} + C + W(x). \tag{51}$$

We choose $W = -C$ and $C = V_1$ to comply with this condition. Consequently, the effective potentials for the Dunkl case (20), and for the conventional scenario (15) must have the same functional form. Substituting the present settings (49) for (20):

$$\begin{aligned} U_\nu(x) = \frac{1}{x^2\ (1+x^2)} \Big\{ & -V_1 + \delta\ \nu - \nu^2 + x^2\ [-1 + E - V_1 - \alpha - \gamma \\ & - 2\ \delta\ \nu\ (\alpha+\gamma) - 2\ \nu^2] + x^4\ [E - \delta\ \nu - \nu^2 - \gamma\ (1 + 2\ \delta\ \nu) \\ & - \alpha\ (1 + 4\ \gamma + 2\ \delta\ \nu)]\Big\}. \end{aligned} \tag{52}$$

Similarly, the effective potential for the conventional case reads:

$$U_0(x) = \frac{1}{x^2\ (1+x^2)} \Big\{ -V_1 + x^2\ (-1 + E - V_1 - \alpha - \gamma) + x^4\ (E - \alpha - \gamma - 4\ \alpha\ \gamma)\Big\}. \tag{53}$$

Comparison of the two functions (52) and (53) shows that they have the same functional form. For the sake of simplicity let us now set $\alpha = \gamma = -\frac{1}{2}$. Substitution of the resulting effective potential (52) renders the reduced Dunkl-Schrödinger equation (36) in the form:

$$\xi_\nu''(x) + \left(\frac{E + V_1}{x^2+1} + \frac{\delta\ \nu - \nu^2 - V_1}{x^2}\right) \xi_\nu(x) = 0, \tag{54}$$

while its conventional partner reads:

$$\xi_0''(x) + \left(\frac{E+V_1}{x^2+1} - \frac{V_1}{x^2}\right)\xi_0(x) = 0. \tag{55}$$

Upon comparing (54) with its counterpart from the previous example (36), we see that the sign of V_1 in the term $\sim x^{-2}$ has changed. As a consequence, for the existence of bound states the condition:

$$\delta\ \nu - \nu^2 - V_1 < 0, \tag{56}$$

must be satisfied. While in the previous example this required negative values of V_1, in the present case V_1 must be positive. The general solution of (55) is very similar to (39). We find:

$$\begin{aligned}\xi_{0,\mathrm{gen}}(x) = C_1\ x^{\frac{1}{2}-\frac{g_2}{2}}\,{}_2F_1\left(-\frac{g_1}{4}-\frac{g_2}{4},\frac{g_1}{4}-\frac{g_2}{4},1-\frac{g_2}{2},-x^2\right)\\ + C_2\ x^{\frac{1}{2}+\frac{g_2}{2}}\,{}_2F_1\left(-\frac{g_1}{4}+\frac{g_2}{4},\frac{g_1}{4}+\frac{g_2}{4},1+\frac{g_2}{2},-x^2\right),\end{aligned} \tag{57}$$

where the constants g_1 and g_2 stand for:

$$g_1 = \sqrt{1-4\ E}, \qquad\qquad g_2 = \sqrt{1+4\ V_1}. \tag{58}$$

We can now reparametrise the effective potential in equation (55) in order to construct the general solution of (54). This gives:

$$\begin{aligned}\xi_{\nu,\mathrm{gen}}(x) = C_1\ x^{\frac{1}{2}-\frac{f_2}{2}}\,{}_2F_1\left(-\frac{f_1}{4}-\frac{f_2}{4},\frac{f_1}{4}-\frac{f_2}{4},1-\frac{f_2}{2},-x^2\right)\\ + C_2\ x^{\frac{1}{2}+\frac{f_2}{2}}\,{}_2F_1\left(-\frac{f_1}{4}+\frac{f_2}{4},\frac{f_1}{4}+\frac{f_2}{4},1+\frac{f_2}{2},-x^2\right),\end{aligned} \tag{59}$$

with the abbreviations:

$$f_1 = \sqrt{1-4\ E+4\ \nu\ (\nu-\delta)}, \qquad\qquad f_2 = \sqrt{1+4\ V_1+4\ \nu\ (\nu-\delta)}. \tag{60}$$

Hence, the general solution of the Dunkl-Schrödinger equation (6) can be determined by means of (59), in combination with the point transformation (10), and our settings (49). We have:

$$\Psi_{\nu,\mathrm{gen}}(x) = \frac{1}{\sqrt{x^2+1}\ x^\nu}\ \xi_{\nu,\mathrm{gen}}(x). \tag{61}$$

Bound states can be extracted from this general solution by requiring $C_1 = 0$, $C_2 = 1$, and:

$$\frac{f_1}{4} + \frac{f_2}{4} = -n - 1, \tag{62}$$

introducing a nonnegative integer n. After solving this constraint for the stationary energy $E = E_{\nu,n}$, we obtain:

$$\begin{aligned}E_{\nu,n} &= -4\ n^2 - 8\ n - 2\ n\ f_2 - V_1 - 4 - 2\ f_2\\ &= -4\ n^2 - 8\ n - (2\ n+2)\ \sqrt{1+4\ V_1+4\ \nu\ (\nu-\delta)} - V_1 - 4.\end{aligned} \tag{63}$$

These energies are real-valued, as long as V_1 takes positive values, see (56). After inserting this expression into the solution (61), and applying the identity [37], our bound states can be written in the form:

$$\begin{aligned}\Psi_{\nu,n}(x) = \frac{1}{\sqrt{x^2+1}}\ x^{-\nu+\frac{f_2+1}{2}}\ (x^2+1)^{-1-\frac{3f_1}{4}-\frac{f_2}{4}}\\ \times\ {}_2F_1\left(-n,\frac{3f_1}{4}+\frac{f_2}{4}+1,\frac{f_2}{2}+1,\frac{x^2}{x^2+1}\right).\end{aligned} \tag{64}$$

Note that the hypergeometric function degenerates to a polynomial, since n is a nonnegative integer. Also, recall that the negative power of ν in (64) does not create a singularity at the origin, as long as V_1 is positive. The remaining task is to establish parity of (46), according to the value of the parameter δ. This can be done by imposing suitable restrictions on the exponent of the monomial term in our solution. In the case of odd parity, corresponding to $\delta = -1$, the restriction reads:

$$\begin{aligned}-\nu + \frac{1}{2} + \frac{f_2}{2} &= 2\ k+1\\ \rightsquigarrow\ \nu &= -\frac{1}{2} - k + \frac{V_1}{4\ k},\end{aligned} \tag{65}$$

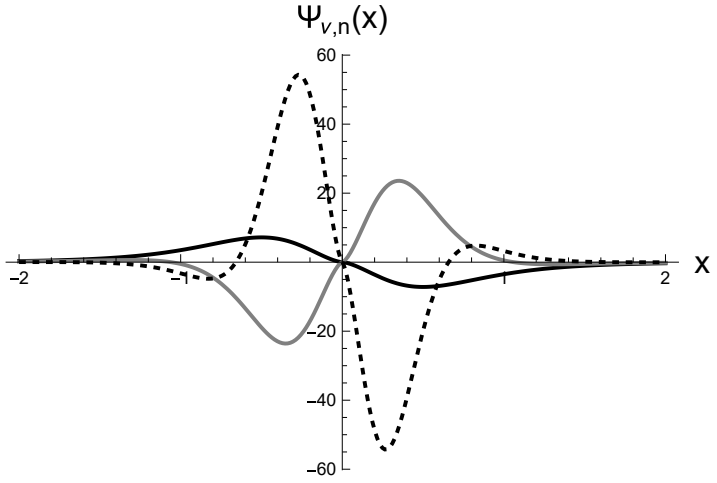

Figure 3. Graphs of odd-parity ($\delta = -1$) solutions (64) for the settings $V_1 = 1$ and $k = -1$, corresponding to $\nu = \frac{1}{4}$. The solutions are given for $n = 0$ (black solid curve), $n = 1$ (gray curve), $n = 2$ (dashed curve).

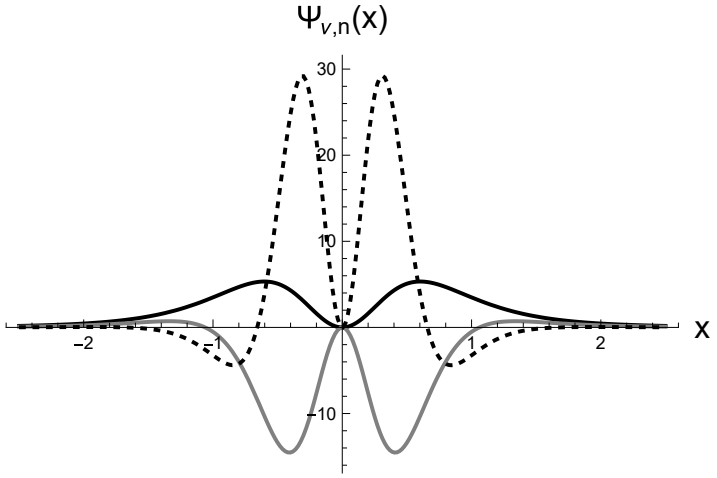

Figure 4. Graphs of even-parity ($\delta = 1$) solutions (64) for the settings $V_1 = 1$ and $k = 1$, corresponding to $\nu = -\frac{1}{4}$. The solutions are given for $n = 0$ (black solid curve), $n = 1$ (gray curve), $n = 2$ (dashed curve).

where k is an integer that must be chosen such that $\nu > -\frac{1}{2}$. For even parity of our solution ($\delta = 1$), we must have:

$$-\nu + \frac{1}{2} + \frac{f_2}{2} = 2\ k$$
$$\rightsquigarrow\ \nu = \frac{1}{2} - k + \frac{V_1}{4\ k}, \tag{66}$$

recall that we require $\nu > -\frac{1}{2}$. Figures 3 and 4 show examples of our bound state solutions (64) for particular values of the parameter ν.

4.3. Lienard-type oscillator potential with spiked mass

Our next example focuses on a quantum system that was first studied in [38]. This system, the classical version of which is a nonlinear oscillator of Lienard type, is characterised by a spiked mass function and a harmonic oscillator potential. These functions are given by:

$$m(x) = \frac{2}{x^4}, \qquad V(x) = \lambda\ x^2, \tag{67}$$

introducing a real-valued constant $\lambda > 0$. Note that here the mass function does not have the same physical meaning as in semiconductor applications that were mentioned in the introduction, but rather appears as a byproduct when converting the classical Hamiltonian function of the Lienard oscillator into a quantum Hamiltonian operator. More precisely, in the course of the latter conversion, a first-derivative term is generated in the associated Schrödinger equation, the coefficient of which can be interpreted as a position-dependent mass. For this reason, it is not problematic if the position-dependent mass exhibits a singularity. In fact, as

mentioned at the beginning of Section 2, a singular mass function is common in Lienard oscillator or related systems. We will now determine if we can construct solutions of the Dunkl-Schrödinger equation (6) with (67), from the known solutions of the conventional equation (7). To this end, let us verify that the first solvability condition (17) is true. We find that the setting:

$$c_1 = 0, \qquad c_2 = \frac{1}{2}, \qquad A = 0, \qquad B = -\frac{3}{2}, \tag{68}$$

makes the position-dependent masses (17) and (67) coincide, as desired. In the next step, we verify our second solvability condition (19) by substituting, mass and potential from (67). We obtain:

$$\lambda\, x^2 = \frac{C\, x^2}{2} + W(x). \tag{69}$$

We can fulfill this condition by setting $C = 2\lambda$ and $W = 0$. As such, the effective potential U_ν in (20) and its counterpart U_0 in (15) with the present settings (67) must have the same functional form. Inserting the latter settings into the explicit form of the effective potentials shows that this is true. We have:

$$\begin{aligned} U_\nu(x) &= \frac{2\, E}{x^4} + \frac{2\,[1 + 3\,\gamma + \alpha\,(3 + 9\,\gamma) + \lambda] + \delta\,\nu\,(3 + 4\,\alpha + 4\,\gamma) + \nu^2}{x^2}, \\ U_0(x) &= \frac{2\, E}{x^4} + \frac{2\,[1 + 3\,\gamma + \alpha\,(3 + 9\,\gamma) + \lambda]}{x^2}. \end{aligned} \tag{70}$$

Consequently, the Schrödinger equations for the Dunkl context (6) and without it have the same solutions, up to a redefinition of numerical parameters. Since these solutions were found and analyzed in [24, 38], we do not discuss them here any further.

4.4. Constant mass system

In our final example we consider the simplest scenario of a constant mass m, that without restriction we can set equal to one, meaning:

$$m(x) = 1. \tag{71}$$

Furthermore, we leave the potential V of our system undetermined except for requiring the existence of an inverse quadratic term. We write:

$$V(x) = \frac{C}{x^2} + W(x), \tag{72}$$

where C is a free constant, and W stands for an arbitrary function. The combination of mass (71) and potential (72) occurs in radial equations that result from a variable separation in spherical coordinates of the three-dimensional Schrödinger equation. Let us now verify our solvability conditions (17) and (19). It is straightforward to see that the settings:

$$c_1 = 0, \qquad c_2 = -m, \qquad A = -1, \qquad B = 0, \tag{73}$$

give a constant function, such that (17) is satisfied. The remaining condition (19) is automatically fulfilled after inserting (72). Hence, the effective potentials (12) and (15) have the same functional form. In fact, their explicit form reads:

$$\begin{aligned} U_\nu(x) &= m\,[E - W(x)] + \frac{\delta\,\nu - \nu^2 - C}{x^2}, \\ U_0(x) &= m\,[E - W(x)] - \frac{C}{x^2}. \end{aligned} \tag{74}$$

Consequently, solvability of the conventional equation (14) is preserved when transitioning to the Dunkl scenario (11).

5. Concluding remarks

We will now make several comments on the results found in this note, discussing its properties, and possible generalisations.

- Upon introduction of a deformed quantum theory, the associated Schrödinger equation for a specific system can change its solvability properties. If the deformation is created by replacing the conventional derivative (e.g. by a Dunkl operator), then the associated Schrödinger equation can have a more complicated structure, as compared to its conventional (non-deformed) counterpart. If the deformation is due to another mechanism, for example the introduction of energy dependence in the potential, the norm takes a non-conventional form [27]. In both cases, the solvability properties (regarding physically meaningful solutions) of the governing equation typically change.

- There is no general method for transitioning from a conventional, solvable Schrödinger equation to its deformed counterpart. In order to construct such a transition for the solutions, the deformed Schrödinger equation must be compared with its conventional partner, such that, conditions for the persistence of solvability can be derived. This leads to constraints on the parameters in the equation, such as the potential or, as in the present case, the position-dependent mass.
- Our solvability conditions derived in Section 3 open the possibility of constructing solutions to the governing equations in Dunkl-Schrödinger systems from their conventional counterparts. In addition, our method provides an explanation for the solvability of systems within the Dunkl formalism that were recently studied in the literature.
- A restriction to the approach we are taking in this work lies in the particular shape of the position-dependent mass that must comply with the four-parameter class (17). Even though the latter class contains many commonly used mass profiles as special cases (see Section 3), there is a vast amount of profiles that cannot be written in the form (17). An example is given by hyperbolic functions that are often used as smooth steps (hyperbolic tangent) or finite wells (hyperbolic secant).
- A further restriction of our method concerns the potential in the Schrödinger equation that must contain an inverse quadratic term, see (19). While this is compatible with certain classes of systems, such as those governed by radial equations after a separation in spherical coordinates, in general condition (19) is not satisfied. As examples we can mention exponential potentials that include the hyperbolic and trigonometric cases.
- Besides the two restrictions, there are also ways to extend the method proposed in this work, such as its generalisation to equations different from the Schrödinger case, for example, the Klein-Gordon and the Dirac equation. Here, the simplest case is the one-dimensional Klein-Gordon equation. Since it is not built from a Hamiltonian, the equation does not contain derivatives of the position-dependent mass. Therefore, the solvability conditions will simply require that an inverse quadratic term is present in the effective potential. This term can be contributed either by the position-dependent mass or the potential. A similar situation occurs in the higher-dimensional case, if spherical symmetry can be used. For other systems, involving for example angular-dependent potentials, the solvability conditions can become much more complicated. The same is true for the Dirac equation, where the latter conditions depend on the symmetries of the potential and the vector potential coupled to the system. Specific systems involving the latter two equations have already been considered in recent literature, as mentioned in the introduction. These topics will be investigated in future research.

List of symbols

H_ν Hamiltonian for parameter $\nu > -\frac{1}{2}$

α, γ Constants in the Hamiltonian

V Potential of the system

E Stationary energy

D_ν Dunkl operator for parameter $\nu > -\frac{1}{2}$

δ Constant determining parity of the solution

m Position-dependent mass

U_ν Potential of reduced Schrödinger equation for parameter $\nu > -\frac{1}{2}$

$c_1, c_2, c_3, A, B, \lambda$ Constants in the constrained position-dependent mass

C Constant in the constrained potential V

W Term in the constrained potential V

f_1, f_2, g_1, g_2, V_1 Abbreviations used in the examples

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