

The innovative application of Polya's problem-solving thought in the teaching of mathematical problem-solving--Take a conic section of the college entrance examination as an example

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Abstract. In view of the current situation of mathematics teaching thinking and the problems, which are easily overlooked in high school classrooms using Polya's problem-solving ideas. This paper innovates the "how to solve it" list in combination with the characteristics of the current era, which extends the "four steps to solve problems" into five steps. To take a conic section of the college entrance examination as the carrier, the five steps are applied in teaching mathematical problem-solving. It is expounded in detail to provide a method reference for optimizing the teaching thinking of high school mathematics teachers.

Keywords: Problem-solving Thinking; Teaching Thinking; Conic Section.

1. The question is raised

1.1 Mathematical Teaching Thinking and Its Present Situation

Mathematics teaching thinking is the rational understanding of mathematics teaching activities (processes) by mathematics teachers, the internal rational activities of the interaction between mathematics teachers and mathematics teaching objects (textbooks, students, etc.) ability. Mathematics teaching thinking "focuses on teaching thinking that meets the needs of students, rather than thinking only for teachers themselves or the content of the subject." [1-3]

The teaching thinking of mathematics teachers has always been an essential part of mathematics teaching, and its differences will directly affect the quality of teaching. While it is widely believed that mathematics is the gymnastics of the mind, its function as "mind gymnastics" has not been fully utilized. It is mainly due to the following three deficiencies in mathematics teaching. Firstly, it focuses on knowledge and skills while ignoring mathematical thinking. Secondly, it focuses on thinking results and neglects the thinking process [4-5].

1.2 Polya's problem-solving ideas and new mathematical problems

In the book "How To Solve It," George Polya proposed four steps for solving problems in the process of thinking in mathematics, including "understanding the problem," "devising a plan," "carrying out the plan," and "looking back" four steps [6]. "Understanding the problem" is to let students understand the meaning of the problem, and to understand the known and unknown quantities of the problem, which is the basis for solving the problem. "Devising a plan" is the core of problem-solving. "Carrying out the plan" is mainly to comprehensively express the previous step to ensure the accuracy of the results, which is the key to problem-solving. "Looking back" is a "re-understanding" of problem-solving activities and trying to draw inferences about the problem from one case to another, and then make logical presuppositions for solving this type of problem, which is the sublimation of problem-solving.

Because most of the current problem-solving process and teaching focus on how to find the ideas and methods to solve the problem, teachers often ignore the "looking back" step in the teaching process. Students only know the problem-solving tactics and feel that the problem-solving process is rushed. They do not know how the solution was discovered, nor the extensive connection between the problem and other concepts, theories, methods, and problems, let alone how the problem should be solved and how the topic will develop. Even if there is a "looking back" step in the class, it is

limited to answering the question itself and to the process of "checking whether the answer is correct" or recalling the question answered in memory. Such a "looking back" is too formal and low-level, which cannot achieve the purpose of cultivating students' divergent thinking and critical thinking, and does not meet the true meaning of "looking back" originally referred to by Polya [7]. The reason is that teachers' thinking in the teaching process is too limited, and they blindly pursue how to "instill" success without showing a complete thinking process, and they lack a deep understanding of the meaning of review after solving problems. Therefore, it is necessary to discuss further the step of "looking back" to innovate the problem-solving thought of Polya. Therefore, the teaching thinking of mathematics teachers can be effectively improved, and the effect of mathematics teaching can be improved.

2. The innovative thinking of Polya's problem-solving ideas

In view of the existing problems and defects, this paper attempts to innovate Polya's four steps to solve problems. It proposes to divide them into five steps: "observing and doubting," "multivariate analyzing doubts," "accurately solving doubts," "reviewing and doubting," and "reflecting and doubting." "Looking back" is refined into two steps "reviewing and doubting" and "reflecting and doubting."

2.1 Observing and doubting - finding the crux of the problem

From observing and doubting, based on existing knowledge, it is conducted an in-depth analysis of doubts (topics), which clarifies the mathematical problems that need to be solved in the doubts. It is known that knowledge points are related to the problems and understanding the known conditions and unknowns of the topic. Moreover, it looks for the obstacles that need to be overcome from the condition to the conclusion to find the crux of the problem.

2.2 Multivariate analyzing doubts - exploring the path of solving the problem

Multivariate analyzing doubts, that is, multi-level and multi-angle analysis of doubts, aims to resolve the crux of the problem and find an effective solution. Recall whether there is a connection between the current problem and similar problems or similar models. If so, try to solve the problem in the same way. If not, it analyzes the relationship between the known conditions and the problem and thinks about what information can be obtained from the known and which information is related to the problem, or what conditions can be inversely deduced from the problem, and how the inversely deduced conditions are related to the known conditions. It is used to build a "bridge" between the known conditions and the problem and find the optimal solution path.

2.3 Accurately solving doubts - accurately implementing the plan

Accurately solving doubts means implementing the plan strictly and accurately to solve doubts, ensuring the accuracy of the problem-solving results, and aiming to test whether the formulated plan is feasible to achieve the purpose of problem-solving. "Accurately solving doubts" is much easier than "Multivariate analyzing doubts." It mainly implements the already formed problem-solving ideas on the test paper. The most important thing at this stage is to be patient and careful. In order to ensure that every detail is accurate, every step must be carefully checked. Every point should be clarified so that the entire problem-solving process is free from the ambiguity that may hide errors.

2.4 Reviewing and doubting - perceive the problem-solving process

Reviewing and doubting, that is, review and sort out the entire problem-solving process, reflect on whether the problem-solving process has found the right direction, whether there are loopholes in one's subject knowledge, and perceive the mathematical knowledge, mathematical methods, and mathematical ideas used in solving problems, and think about it. It can use different methods to solve

the problem and consider whether the problem can be summarized and classified to achieve the problem-solving effect of "doing one, knowing one class."

2.5 Reflecting and doubting - the problem of sublimation and expansion

Reflecting and doubting mean necessary extension and expansion of the original topic, reflecting on whether the problem-solving process can withstand scrutiny, and then discovering the rules. Reflecting and doubting are different from reviewing and doubting. It is the growth of problem-solving thinking based on reviewing and doubting. As Polya said, "Mathematics is not a pile of symbols and formulas, but a lively intellectual activity." [8] This developmental teaching link stimulates students' subjective enthusiasm for learning and improves their understanding. It has accumulated valuable experience and laid a solid foundation for solving more, deeper, and wider mathematical problems in the future.

3. Practical applications

The topic (2021 National Unified College Entrance Examination Paper Science Volume B Question 21) has a known focus F on the parabola $C: x^2 = 2py (p > 0)$, and the minimum distance between F and the point on the circle $M: x^2 + (y + 4)^2 = 1$ is 4.

(1) Ask p .

(2) If the point P is on M , PA and PB are the two tangent lines of C , A , and B are the tangent points, find the maximum value of the area of ΔPAB .

In high school teaching, most students have reached the level of solving the first question, so only the second question is analyzed to illustrate further the innovative application value of Polya's problem-solving thought.

In this paper, simulation teaching is carried out through dialogue. The code name of the teacher is "T," and the code name of the student is "S." [9-10]

Analysis of the second question:

(1) Observing and doubting

T: Read the second question carefully and talk about what information you have obtained? Moreover, what is the unknown quantity of the title? Is this a question of what?

S: PA and PB are the two tangent lines of C , A , and B are the tangent points. The unknown quantity is the maximum value of the area of ΔPAB . It is a question of the best value.

T: What is the most common method among the solutions to the conic section maximum value problem?

S: Transform the conic section maximum value problem into the maximum value problem of quadratic function or trigonometric function, and then use the basic inequality method, the matching method, element replacement method, derivative method, number-shape combination method, and other methods to find the maximum value.

T: In order to use these methods, what are the prerequisites?

S: Convert ΔPAB area expressions into quadratic or trigonometric expressions.

(2) Multivariate analyzing doubts

At this time, students may be limited to finding connections between general conditions and ΔPAB area expressions, so a teacher can guide students to use reverse thinking to start from the problem to find the relationship between known conditions and known conditions.

T: How can I find the area of ΔPAB ? Have you seen similar problems before?

S: Using the formula $S_{\Delta PAB} = \frac{1}{2}|AB| \cdot d$, d is the distance from a point P to a straight line AB .

The conic section is often asked for $|AB|$, I remember its expression is $|AB| = \sqrt{1 + k^2}|x_1 - x_2|$, and d can be expressed by directly applying the point-to-distance formula.

T: What do you need to know to get further expressions? Can these expressions be obtained based on the existing conditions?

S: We should know the equation for straight-line AB , and expressions for $x_1 + x_2$ and x_1x_2 . It is known that PA and PB are the two tangents of C , and A and B are the tangent points. Suppose the coordinates of points A and B are $(x_1, y_1), (x_2, y_2)$. According to the knowledge of derivatives, $k_{PA} = \frac{1}{2}x_1, k_{PB} = \frac{1}{2}x_2$, then the equation of the straight line PA is $y - y_1 = \frac{1}{2}x_1(x - x_1)$, and the equation of the straight line PB is $y - y_2 = \frac{1}{2}x_2(x - x_2)$.

T: Can you simplify the equation? Which hidden condition is not used?

S: The two points A, B are on the parabola $C: y = \frac{1}{4}x^2$, and the two straight-line equations can be simplified as $l_{PA}: y = \frac{1}{2}x_1x - y_1, l_{PB}: y = \frac{1}{2}x_2x - y_2$.

T: Can you find the equation of straight line AB if you know the equation of straight line PA and straight-line PB ?

S1: It is known that P is the intersection of the straight line PA and the straight line PB . Let the coordinates of the point P be (x_0, y_0) , and substitute it into the equation of the straight line PA and

the straight line PB to get $\begin{cases} y_0 = \frac{1}{2}x_1x_0 - y_1 \\ y_0 = \frac{1}{2}x_2x_0 - y_2 \end{cases}$, so the equation of straight line AB is $y_0 = \frac{1}{2}x_0x -$

y , that is, $y = \frac{1}{2}x_0x - y_0$.

T: Some students do not understand why they can directly write the equation of a straight line AB . Please explain.

S1: According to the knowledge of plane geometry, it is known that two points have only one straight line. That is, two points determine a straight line. Because $A(x_1, y_1), B(x_2, y_2)$ are two different points, and their coordinates satisfy the equation $y = \frac{1}{2}x_0x - y_0$, that is, the coordinates of A and B points are the solution of the equation $y = \frac{1}{2}x_0x - y_0$, the equation of line AB is $y = \frac{1}{2}x_0x - y_0$.

T: How to find the expression of $x_1 + x_2$ and x_1x_2 ?

S: The equation of the simultaneous line AB and the parabola C can be obtained according to Veda's theorem.

(3) Accurately solving doubts

Let $A(x_1, y_1), B(x_2, y_2), P(x_0, y_0)$, there is $x_0^2 + (y_0 + 4)^2 = 1$.

Given $C: y = \frac{1}{4}x^2$, derivation of the function gives $y' = \frac{1}{2}x$.

\therefore The equation of the line PA is $y - y_1 = \frac{1}{2}x_1(x - x_1)$, that is, $y = \frac{1}{2}x_1x - y_1$.

Similarly, the equation of the straight line PB can be obtained as $y = \frac{1}{2}x_2x - y_2$.

\therefore Point P is the common point of the two lines. There is $\begin{cases} y_0 = \frac{1}{2}x_1x_0 - y_1 \\ y_0 = \frac{1}{2}x_2x_0 - y_2 \end{cases}$.

\therefore The coordinates of the points A and B satisfy the equation $y_0 = \frac{1}{2}x_0x - y$.

\therefore The equation of the straight line AB is $y_0 = \frac{1}{2}x_0x - y$, that is, $y = \frac{1}{2}x_0x - y_0$.

Combine $\begin{cases} y = \frac{1}{2}x_0x - y_0 \\ y = \frac{1}{4}x^2 \end{cases}$, get $x^2 - 2x_0x + 4y_0 = 0$,

According to Veda's theorem, we can get $x_1 + x_2 = 2x_0, x_1x_2 = 4y_0$.

Therefore,

$$|AB| = \sqrt{1 + \left(\frac{1}{2}x_0\right)^2} \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = \sqrt{1 + \left(\frac{1}{2}x_0\right)^2} \sqrt{4x_0^2 - 16y_0} = \sqrt{4 + x_0^2} \sqrt{x_0^2 - 4y_0}.$$

The distance from the $d = \frac{|x_0^2 - 4y_0|}{\sqrt{4 + x_0^2}}$ From the point P to the straight line AB .

$$\therefore S_{\Delta PAB} = \frac{1}{2} |AB| \cdot d = \frac{1}{2} |x_0^2 - 4y_0| \sqrt{x_0^2 - 4y_0} = \frac{1}{2} (x_0^2 - 4y_0)^{\frac{3}{2}}.$$

And $x_0^2 - 4y_0 = 1 - (y_0 + 4)^2 - 4y_0 = -y_0^2 - 12y_0 - 15 = -(y_0 + 6)^2 + 21$.

It is known that $-5 \leq y_0 \leq -3$, so when $y_0 = -5$, $S_{\Delta PAB}$ obtains the maximum value of $\frac{1}{2} \times 20^{\frac{3}{2}} = 20\sqrt{5}$.

(4) Reviewing and doubting

The teacher organizes students to review the problem-solving method of this sub-question and asks S2 to review how this problem-solving idea was born.

S2: I used reverse thinking to think about the problem you mentioned, guessing and assuming that the expression of $S_{\Delta PAB}$ can be written out, and then the maximum value can be obtained according to the properties of the function. And it is known that $S_{\Delta PAB} = \frac{1}{2} |AB| \cdot d$, then the problem direction is transformed into an expression for finding $|AB|$ and d .

T: A "good thought" produces good results. During the inspection, I saw that S3 used different methods and obtained results. We asked him to share his ideas.

S3: In daily problem solving, I am accustomed to directly setting the equation of the straight line AB . According to the meaning of the question, I know that the straight line AB and the parabola C must have two intersections, so set $l_{AB}: y = kx + b$, according to the slope of the straight line PA and the straight line PB can be solved. The coordinates of point P is $(\frac{x_1 + x_2}{2}, \frac{x_1 x_2}{4})$, and then the coordinates of point P can be simplified to $(2k, -b)$. Simultaneous equations, Veda's theorem, and other methods are shown as follows. It is similar to the above. The expression of $S_{\Delta PAB}$ is finally converted into an expression containing b .

T: Very good! The above two students have made a brief exposition of their problem-solving ideas, I believe some should inspire everyone. What do you gain from solving this problem?

S4: I feel that reverse thinking is a common way of thinking in problem-solving. This question is not limited to looking for a "bridge" between conditions and problems but inversely looking for a "bridge" between problems and conditions, which is the breakthrough point of this problem.

S5: I feel that it is essential to transform your mind. This problem transforms the problem of the maximum value of the ΔPAB area into a function to find the maximum value and converts it into the maximum value of the expression to find the ΔPAB area. Continuous transformation is the key to the success of the problem.

(5) Reflecting and doubting

T: Please review the problem-solving process again after completing the problem-solving process. Maybe you will generate some new ideas and gain some unexpected discoveries.

S6: (raises his hand) In this question, the focus of the parabola and the center of the circle are both on the y -axis. If they are replaced by both on the x -axis, the method must be the same.

T: If you change it according to the method you said, can a new topic be adapted so that the final result remains the same?

S6:

It is known that the focus F of the parabola $C: y^2 = 2px (p > 0)$, and the minimum distance between F and the point on the circle $M: (x + 4)^2 + y^2 = 1$ is 4. If the point P is on M , PA and PB are the two tangents of C , and A and B are the tangent points, find the maximum value of the area of ΔPAB .

T: Very good! Did any of the other students have any surprises? Try to adapt a question.

S7: Replacing the circle M with a straight line can also be solved by using the method of this problem.

It is known that the focus F of the parabola $C: x^2 = 2py (p > 0)$, and the minimum distance between F and the point on the line $l: 3x - 4y - 16 = 0$ is 4. If the point P is on M , PA and PB are the two tangents of C , and A and B are the tangent points, find the maximum value of the area of ΔPAB .

T: Have you ever thought about extending and promoting the method of solving this problem? How did this idea come about?

S8: I am thinking, is the solution of the area maximum value problem in the conic section transformed into the function maximum value problem? In the past problems of conic sections, I found that the area formula was often written $S = \frac{1}{2} |AB| \cdot d$, and then the most value was calculated based on the knowledge of the function.

T: That is right! In other words, it is aimed at solving the problem of the maximum value of the area in the conic curve, which is the problem of "the maximum value of the function." This will provide a solution path for subsequent model-like problems.

4. Summary and Outlook

This paper explores the problems existing in mathematics teaching thinking and the current problem-solving teaching using Polya's problem-solving thought and puts forward the innovative thinking of "whether it is possible to 're-create' the 'four steps of problem-solving.'" On the original basis, the step of "Looking back" is innovatively decomposed into two steps of "reviewing and doubting" and "reflecting and doubting," and the names of the remaining three steps are changed. Finally, taking the topic of the conic section of the college entrance examination as an example, this paper discusses the innovative application of Polya's problem-solving thought in mathematical problem-solving, which strengthens the process of students inferring others from one case. It provides a method reference for optimizing high school mathematics teachers' teaching thinking.

Polya's problem-solving thought provides a method for students to do problems and for teachers to give lectures. Whether it is past, present, or future, research on this problem-solving thought has a particular value. With the changing times and the emergence of various problems in teaching, Polya's problem-solving thought will continue to evolve, radiate and generate new values in the future.

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