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Higher Dimensional Triangular and Square Numbers

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Abstract:

Triangular numbers and square numbers are traditionally defined in two-dimensional space. In this paper, we generalize these numbers to higher dimensions and explore their properties using a coordinate system to conceptualize spaces beyond three dimensions. By generalizing triangular numbers, we establish and prove a notable combinatorial identity and recursive relationship between triangular numbers of different dimensions. Key applications discussed include network optimization and geometric partitioning. The paper concludes by finding the ratio between generalized triangular and square numbers, which geometrically corresponds to the volume ratio of a tetrahedron to a d-dimensional cube.

Keywords: Triangular numbers, Square numbers, Higher-dimensional space, Combinatorial identities.

1. Introduction

The study of polygonal numbers has a rich and ancient history, tracing back to the era of Pythagoras and his followers, based on Egyptian and Babylonian precursors. These early mathematicians sought to connect the realms of geometry and arithmetic by representing numbers through distinct geometric patterns. A figurate number is a number that can be represented by a regular and discrete geometric arrangement of equally spaced points. In two dimensions, these are known as polygonal numbers, which include the well-known triangular, square, pentagonal, hexagonal, and heptagonal numbers.

This paper is organized as follows. Section II introduces the definitions of triangular and square numbers using geometric and algebraic representations. In Section III, the generalization of square numbers using the coordinate expression is discussed. After this, the generalization of triangular numbers to arbitrary dimensions is elaborated in Section IV. Here, properties of generalized triangular numbers are also introduced. Finally, generalizations of triangular and square numbers are compared in Section V. We will discuss the applications and value of triangular numbers.

2. Original triangular and square numbers

Triangular and square numbers are defined geometrically. From stones, we can construct $n \times n$ grid. Since this structure looks like a square, we call n^2 is the n-th square number. In this paper, we denote these square numbers as

$$S(n) = n^2$$

for any natural numbers n.

Then, we know the n-th triangular number is $1 + 2 + ... + n = \frac{n(n+1)}{2}$. We also denote these triangular numbers as for any natural numbers n.

$$T(n) = \frac{n(n+1)}{2}$$

In this paper, we will generalize these numbers to arbitrary higher-dimensional spaces. Therefore, the definitions of the above numbers should be written in different ways. Since we cannot imagine a space with dimensions greater than three, we introduce coordinate systems to define triangular and square numbers.

Since the square number case is easier, we start with this case. We can describe the n-th square number as follows:

$$S(n) = |\{(a, b) : 0 \leq a, b \leq n - 1\}|.$$

Here, $|A|$ is the number of the elements of the set A. We know the set $|\{(a, b) : 0 \leq a, b \leq n - 1\}|$ forms a square and each side contains n points. Since the x coordinate can be from 0 to $n - 1$ and the y coordinate can be from 0 to $n - 1$, the total number of points is n^2.

Now, let us express the triangular numbers using the coordinate expression. In Figure 1, all triangles form the equilateral triangles. However, we can transform these points to form an isosceles right triangle as follows (Figure 2).

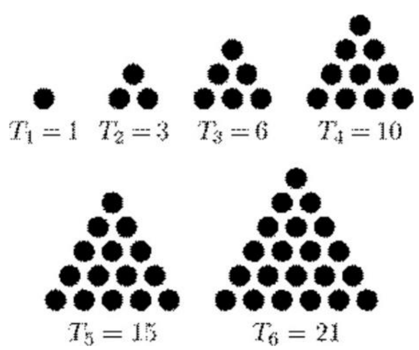

Figure 1. Triangular numbers in equilateral triangles

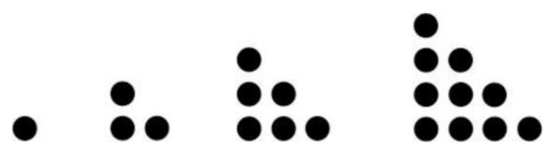

Figure 2. Transformation of the triangles in isosceles right triangles

From this structure, we can express the triangular number as follows:

$$T(n) = |\{(a, b) : 0 \leq a, b, \ 0 \leq a + b \leq n - 1\}|.$$

There was no way to generalize the concept of equilateral triangles to a general dimension; however, from the above expression, we can generalize the triangular number to arbitrary dimensions.

3. Generalization of square numbers

Since generalizing square numbers is easier than triangular numbers, we introduce this part first. For the two-dimensional space, we used the coordinate plane, and the square number was expressed in the following form:

$$S(n) = |\{(a, b) : 0 \leq a, b \leq n - 1\}|.$$

Since this square number is defined in the two-dimensional space, to emphasize the dimension from now on we denote

$$S_2(n) = |\{(a, b) : 0 \leq a, b \leq n - 1\}|.$$

We will denote the square number defined in the d-dimensional space $d \geq 2$ as $S_d(n)$. The definition of this number can be generalized from $S_2(n)$. If we only increase the dimension of the coordinate system, then we can define $S_d(n)$ for $d \geq 2$ as follows:

$$S_d(n) = |\{(x_1, x_2, \dots, x_d) : 0 \leq x_1, x_2, \dots, x_d \leq n - 1\}|.$$

We know each x_i can be from 0 to $n - 1$ and they can be independently determined. For this reason, we have $S_d(n) = n^d$. We can explain this number both algebraically and geometrically. For instance, when $d = 3$, then n^3 points form a cube in three-dimensional space. Also, when $d = 1$, then we just get $S_d(n) = 1$ and it implies one-dimensional square numbers are just natural numbers.

4. Generalization of triangular numbers

In this section, we study the generalization of triangular numbers to higher dimensional space. Recall that the triangular number was expressed as follows in Section 1:

$$T(n) = |\{(a, b) : 0 \leq a, b,\ 0 \leq a + b \leq n - 1\}|.$$

To emphasize the original triangular number is defined on the two dimensional space, we again denote $T_2(n) = T(n)$. From the above expression of the triangular number defined in the two-dimensional space, we can define the d-dimensional triangular number as

$$T_d(n) = |\{(x_1, x_2, \dots, x_d) : 0 \leq x_1, x_2, \dots, x_d,\ 0 \leq x_1 + \dots + x_d \leq n - 1\}|.$$

When $d = 3$, then we can construct a tetrahedron using $T_d(n)$ points. We have one interesting property.

If we consider $T_d(n + 1)$, then we have

$$T_d(n + 1) = |\{(x_1, x_2, \dots, x_d) : 0 \leq x_1, x_2, \dots, x_d,\ 0 \leq x_1 + \dots + x_d \leq n\}|.$$

The set on the right-hand side can be decomposed into two parts:

$$\{(x_1, x_2, \dots, x_d) : 0 \leq x_1, x_2, \dots, x_d,\ 0 \leq x_1 + \dots + x_d \leq n - 1\}$$

and

$$\{(x_1, x_2, \dots, x_d) : 0 \leq x_1, x_2, \dots, x_d,\ x_1 + \dots + x_d = n\}$$

The number of elements of the above set is $T_d(n)$ from the definition of the d-dimensional triangular number. So, we have the following identity:

$$\{(x_1, x_2, \dots, x_{d-1}, 0) : 0 \le x_1, x_2, \dots, x_{d-1},\ 0 \le x_1 + \dots + x_{d-1} \le n\},$$
$$\{(x_1, x_2, \dots, x_{d-1}, 1) : 0 \le x_1, x_2, \dots, x_{d-1},\ 0 \le x_1 + \dots + x_{d-1} \le n - 1\},$$
$$\dots$$
$$\{(x_1, x_2, \dots, x_{d-1}, n-1) : 0 \le x_1, x_2, \dots, x_{d-1},\ 0 \le x_1 + \dots + x_{d-1} = 1\},$$
$$\{(x_1, x_2, \dots, x_{d-1}, n) : 0 \le x_1, x_2, \dots, x_{d-1},\ 0 \le x_1 + \dots + x_{d-1} = 0\},$$

So, we have

$$|\{(x_1, x_2, \dots, x_d) : 0 \le x_1, x_2, \dots, x_d,\ x_1 + \dots + x_d = n\},$$
$$= |\{(x_1, x_2, \dots, x_{d-1}) : 0 \le x_1, x_2, \dots, x_{d-1},\ x_1 + \dots + x_{d-1} \le n\},$$

Since the right-hand side of the above equality is $T_{d-1}(n)$ from the definition of the higher dimensional triangular number, we eventually get the relationship:

$$T_d(n+1) = T_d(n) + T_{d-1}(n+1)$$
$$T_d(n+1) - T_d(n) = |\{(x_1, x_2, \dots, x_d) : 0 \le x_1, x_2, \dots, x_d, x_1 + \dots + x_d = n\}|.$$

To simplify the above identity, we again decompose the set on the right-hand side. Since can be from 0 to n, we can decompose the set as follows:

We can interpret the above relation geometrically. When $d = 3$, we know $T_d(n+1)$ and $T_d(n)$ are tetrahedrons with side length $n + 1$ and n, respectively. If we compare these two structures, then the only difference is the triangular surface. If we stack one more layer of points on the triangular surface of $T_d(n)$, then we get $T_d(n+1)$. Also, we know the dimension of the triangular surface of $T_d(n)$ is two-dimensional, so $T_{d-1}(n+1)$ will be added.

Now, we investigate the explicit form of $T_d(n)$. Rather than using the geometric property, we will again use the coordinate system expression:

$$T_d(n) = |\{(x_1, x_2, \dots, x_d) : 0 \le x_1, x_2, \dots, x_d, 0 \le x_1 + \dots + x_d \le n - 1\}|,$$

since using the algebraic properties is better for higher dimensional space.

The above expression can be rewritten as

$$T_d(n) = |\{(x_1, x_2, \dots, x_{d+1}) : 0 \le x_1, x_2, \dots, x_{d+1}, x_1 + \dots + x_{d+1} = n - 1\}|$$

by introducing one more variable x_{d+1}. If $x_1, x_2, \dots, x_d$ are determined then x_{d+1} will be determined by $x_{d+1} = n - 1 - x_1 - \dots - x_d$ automatically. So, the number of elements of two sets in two expressions is the same.

Let assume that there are $n - 1$ candies are on the line and we would like to distribute these candies to $d + 1$ people. Then, instead of considering how to distribute these candies, we consider d partitions to divide these $n - 1$ candies. If we insert these d partitions between

$n - 1$ candies then the number of candies for each person will be determined. Instead of inserting d partitions between candies, we add d candies so considering $n + d - 1$ candies first. After that, we choose d candies and replace them with partitions. So, the number of ways to insert partitions is ${}_{n+d-2}C_{d} = \frac{(n+d-1)!}{(n-1)!d!}$. Finally, we get the explicit form of $T_{d}(n)$ as follows:

$$T_{d}(n) = \frac{(n+d-1)!}{(n-1)!d!}$$

When $d = 1$, we get

$$T_{1}(n) = \frac{n!}{(n-1)!1!} = n$$

and it implies one-dimensional triangular numbers are just natural numbers. So, we can conclude that

$$S_{1}(n) = T_{1}(n) = n$$

where $S_{1}(n)$ is a one-dimensional square number.

When $d = 2$, we get

$$T_{2}(n) = \frac{(n+1)!}{(n-1)!2!} = \frac{n(n+1)}{2}$$

and we can recover the result of the original triangular numbers.

When $d = 3$, we can obtain

$$T_{3}(n) = \frac{(n+2)!}{(n-1)!3!} = \frac{n(n+1)(n+2)}{6}.$$

Recall that we proved $T_{d}(n + 1) = T_{d}(n) + T_{d-1}(n + 1)$. If we use

$$T_{d}(n) = \frac{(n+d-1)!}{(n-1)!d!} = {}_{n+d-2}C_{d},$$

then we can prove the following identity using the generalization of triangular numbers:

$${}_{n+d-1}C_{d} = {}_{n+d-2}C_{d} + {}_{n+d-2}C_{d-1}.$$

5. Comparison between generalization of square and triangular numbers

In this section, we compare two generalized numbers. After we expand all factorial terms in $T_{d}(n)$, we can rewrite $T_{d}(n)$ as follows:

$$T_{d}(n) = \frac{n(n+1)\ldots(n+d-1)}{d!}$$

The ratio between $T_d(n)$ and $S_d(n)$ can be considered as follows:

$$\lim_{n \to \infty} \frac{T_d(n)}{S_d(n)} = \frac{1}{d!}$$

When n approaches infinity $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{d-1}{n}$ approaches zero. The meaning of the above limit can be interpreted geometrically. We consider a square. If we divide the square into two parts, then we get the triangle. The ratio between two areas of a triangle and the square is $\frac{1}{2}$.

Similarly, we consider the cube. We can divide this cube into six congruent tetrahedrons. Then, the ratio between two volumes of a tetrahedron and the cube is $\frac{1}{3!} = \frac{1}{6}$. In general, if we consider d-dimensional cube, then we can divide this cube into $d!$ congruent generalized tetrahedrons such that the volume ratio between the tetrahedron and the cube is $\frac{1}{d!}$.

6. Conclusion and Discussion

This paper explored the generalization of triangular and square numbers to higher-dimensional spaces through coordinate systems, building upon their initial geometric definitions in two dimensions. Square numbers and triangular numbers in two-dimensional case are respectively defined as $|\{(a, b) : 0 \leq a, b \leq n - 1\}|$ and $|\{(a, b) : 0 \leq a, b, 0 \leq a + b \leq n - 1\}|$.

For square numbers, this concept is then generalized to d-dimension, resulting in

$$S_d(n) = |\{(x_1, x_2, \ldots, x_d) : 0 \leq x_1, x_2, \ldots, x_d \leq n - 1\}| = n^d.$$

Triangular numbers follow a similar pattern in d-dimension, resulting in

$$T_d(n) = |\{(x_1, x_2, \ldots, x_d) : 0 \leq x_1, x_2, \ldots, x_d, 0 \leq x_1 + \ldots + x_d \leq n - 1\}| = \frac{(n+d-1)!}{(n-1)!d!}.$$

The paper establishes the identity $T_d(n + 1) = T_d(n) + T_{d-1}(n + 1)$, proving the recursive relationship between triangular numbers of different dimensions.

Another significant finding is the comparison between generalized triangular and square numbers. The ratio between these numbers approaches $\frac{1}{d!}$ as n approaches infinity. This ratio has a geometric interpretation: dividing a d-dimensional cube into congruent generalized tetrahedrons reveals that the volume ratio between a tetrahedron and the cube is $\frac{1}{d!}$.

Triangular numbers also have practical applications in addition to combinatorial identities and can be particularly insightful when generalized to higher dimensions. In a fully connected network of n computing devices, the number of necessary connections corresponds to the triangular number T_{n-1}, akin to the handshake problem. In a three-dimensional lattice of computing nodes, each node's connections can be shown as a higher-dimensional analog of triangular numbers, such as tetrahedral numbers, $T_{n-1,3}$ representing the sum of combinations $C_{n-1\ 3}$. This approach minimizes the total number of connections and ensures efficient communication pathways.

Another application can be that the maximum number of pieces, p, obtainable with n straight cuts in higher-dimensional spaces aligns with the generalized triangular number $T_{n,d} + 1$.

This principle, known in two dimensions as the "lazy caterer's sequence," becomes a powerful tool for solving problems in higher-dimensional cutting and partitioning tasks. For instance, consider making cuts in a three-dimensional space (a cube).

With $n = 3$ straight cuts, the number of pieces, p, can be calculated using the tetrahedral number plus one.

$$p = T_{3,3} + 1 = 3 * (3 + 1) * (3 + 2)/6 + 1 = 11.$$

This can be extended to higher dimensions to determine the maximum number of regions formed by n cuts in d-dimensional spaces as well.

The importance and meaning of triangular numbers lie in their fundamental role in understanding geometric and arithmetic relationships, their applications in various fields, and their elegant recursive properties that extend into higher dimensions.

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