Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 6s (2024)

Evaluating Star Vertex Cochromatic Number in Prism, Sunlet and Derived Graphs

R. Sabitha[1*], V. Kowsalya[2]

[1]Research Scholar, [2]Associate Professor, PG & Research Department of Mathematics, Sri Ramakrishna College of Arts & Science (Autonomous), Formerly SNR Sons College, Coimbatore.

E-Mail ID: [1*]sabitha.r@srcas.ac.in, [2]kowsalya@srcas.ac.in

Article History:

Received: 12-01-2025

Revised: 15-02-2025

Accepted: 01-03-2025

Abstract: In this discussion, the star cochromatic number q is found for the following graphs: Prism Graph $q[Y_m]$, Line Graph of Prism Graph $q[L(Y_m)]$, Middle Graph of Prism Graph $q[M(Y_m)]$, Sunlet Graph $q[S_m]$, Line Graph of Sunlet Graph $q[L(S_m)]$, Middle Graph of Sunlet Graph $q[M(S_m)]$.

Keywords: Prism Graph, Sunlet Graph, Line Graph, Middle Graph, Star Cocoloring, Star Cochromatic Index.

AMS Classification: 05C07, 05C15, 05C69, 05C76.

1. Introduction

This study examines finite, simple and undirected graphs. Whitney initially presented the concept of a line graph in 1932 [1]. In 1983, Maria Chudnovsky [5] proposed using line graphs as a very fundamental lesson in graph theory. T. Hamada and I. Yoshimura were the ones who first suggested the center graph of the graph [6].

Vertex partitioning is an important concept in graph theory. Partitioning and imposing on vertex sets yielded new concepts and outcomes. In 1977, Lesinak and H. Straight introduced the concept Cocoloring and found some basic results [3]. Kowsalya. V, Vernold Vivin. J and Venkatachalam. M [2] found the star chromatic number for sunlet graph families and their line, middle, central and total graphs. Vernold Vivin. J, Kowsalya. V, and Vimal Kumar. S [7] found the star chromatic number for prism graph families and their derived graphs. Star Cocoloring concept was introduced by M. Poobalaranjani[4].

A clique is defined as a subset W of a simple graph G = (V, E) that produces a complete subgraph of G. If W has cardinality K, it is referred to as k-clique. A subset U of V is considered independent if it creates an empty subgraph of G. If U has cardinality K, it is referred to k-independent set.

2. Preliminaries

2.1: Prism Graph: The Prism graph is a planar, polyhedral graph that resembles the skeleton of an m-prism. An m-prism graph is the same as the generalised Petersen graph $P_{\{n,1\}}$ with 2m vertices and 3m edges. It is denoted by Y_m

2.2: Sunlet Graph: The sunlet graph on 2m vertices is obtained by attaching n pendant edges to the cycle C_n and it is denoted by S_m.

2.3: Line Graph: Line graph L(G) is

- L(G) is a graph in which each vertex corresponds to an edge of G.
- Two L(G) vertices are said to be adjacent if their respective lines share an endpoint in G.

2.4: Middle Graph: The newly added middle vertices of G's surrounding edges are joined to form the Middle Graph M(G). It is made by accurately splitting each edge of G once.

2.5: Star Cocoloring: Let G be a graph k,l,r be non-negative integers then a (k,l,r) - s - cocoloring of G is a partition of the vertex set of G into sets $I_1, I_2, I_3, \dots, I_k, C_1, C_2, C_3, \dots, C_l, S_1, S_2, S_3, \dots, S_r$ such that I_k is an independent set each C_l is a clique and each S_r is a star $K_{\{1,t\}}$ where $t \geq 3$.

2.6: Star Cochromatic Number: The Star Cochromatic Number q is defined as

$$z^*(G) = \min\{q: there\ exists\ k, l, r\ such\ that\ k, l \geq 0\ , r \geq 1; G\ is\ (k, l, r) - s - colorable\ and\ k + l + r = q\ \}$$

3. Main Results:

3.1: Star Cochromatic Number of Prism Graphs

Theorem 3.1.1: The Star Cochromatic Number of Prism Graph $q[Y_m]$ for $m \geq 3$ is $q[Y_m] = \left\lceil \frac{m}{2} \right\rceil$

Proof:

Let Y_m be prism graph with 2m vertices and 3m edges and $V[Y_m] = \{v_n: 1 \leq n \leq m\} \cup \{u_n: 1 \leq n \leq m\}$. Consider the color class $C = \{c_1, c_2, c_3, \dots, c_{\left\lceil \frac{m}{2} \right\rceil}\}$.

Define mapping $\sigma: \{v_n: 1 \leq n \leq m\} \cup \{u_n: 1 \leq n \leq m\} \rightarrow c_k\ \forall\ k = 1,2,3, \dots$

Case-(i): $m \equiv 0\ mod\ 4$

- $\sigma(u_{4k}, u_{4k-1}, u_{4k+1}, v_{4k}) = c_{2k}$
- $\sigma(v_{4k-2}, v_{4k-3}, v_{4k-1}, u_{4k-2}) = c_{2k-1}$

Case-(ii): $m \equiv 1\ mod\ 4$

- $\sigma(u_{4k-2}, u_{4k-1}, u_{4k-3}, v_{4k-2}) = c_{2k-1}$
- $\sigma(v_{4k}, v_{4k+1}, v_{4k-1}, u_{4k}) = c_{2k}$
- $\sigma(v_1, u_m) = c_{\left\lceil \frac{m}{2} \right\rceil}$

Case-(iii): $m \equiv 2\ mod\ 4$

Assign the coloring as follows:

- $\sigma(u_{4k+1}, u_{4k}, u_{4k+2}, v_{4k+1}) = c_{2k}$
- $\sigma(v_{4k+3}, v_{4k+2}, v_{4k+4}, u_{4k+3}) = c_{2k-1}$

There exists an uncolored set of vertices $\{v_1, v_2, v_3, v_4, v_m, u_1, u_2, u_3\}$. In which the vertices $\{v_2, v_1, v_3, u_2\}$ forms $k_{1,3}$ which is to be colored with $c_{\lceil \frac{m}{2} \rceil -1}$ and the vertices $\{u_1, u_3, v_4, u_m\}$ forms an independent set is colored with new color $c_{\lceil \frac{m}{2} \rceil}$

Case-(iv): $\boldsymbol{m \equiv 3 \bmod 4}$

Assign the coloring as follows:

- $\sigma(u_{4k}, u_{4k-1}, u_{4k+1}, v_{4k}) = c_{2k}$
- $\sigma(v_{4k-2}, v_{4k-3}, v_{4k-1}, u_{4k-2}) = c_{2k-1}$
- $\sigma(u_1, u_m) = c_{\lceil \frac{m}{2} \rceil}$

To Prove: $z^*[Y_m] < \lceil \frac{m}{2} \rceil$; $z^*[Y_m]$ exists $(m-12) - K_{1,3}$ stars, each color class is colored with one color. Hence, there exists an independent set remains uncolored we need one more color to complete the graph. Therefore, assumption is contradictory. Hence $q[Y_m] = \lceil \frac{m}{2} \rceil$.

3.2: Star Cochromatic Number of Line Graph of Prism Graphs

Theorem 3.2.1: The Star Chromatic Number of Line Graph of Prism Graph $q[L(Y_m)]$ for $m \geq 3$ is $q[L(Y_m)] = \lceil \frac{m+2}{2} \rceil$

Proof:

Let Y_m be prism graph with 2m vertices and 3m edges and $V[Y_m] = \{v_n : 1 \leq n \leq m\} \cup \{u_n : 1 \leq n \leq m\}$ and $[L(Y_m)]$ be line graph of prism graph with 3m vertices and 6m edges where $V[L(Y_m)] = \{v_n : 1 \leq n \leq m\} \cup \{u_n : 1 \leq n \leq m\} \cup \{w_n : 1 \leq n \leq m\}$. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_{\lceil \frac{m+2}{2} \rceil}\}$. Define mapping $\sigma : \{v_n : 1 \leq n \leq m\} \cup \{u_n : 1 \leq n \leq m\} \cup \{w_n : 1 \leq n \leq m\} \rightarrow c_k \; \forall \; k = 1,2,3, \ldots$

Case-(i): $\boldsymbol{m = even}$

Assign the coloring as follows:

- $\sigma(u_{4k-3}, v_{4k-3}, v_{4k-2}, w_{4k-3}, w_{4k-2}) = c_{2k-1}$
- $\sigma(u_{4k-1}, v_{4k-1}, v_{4k}, w_{4k-1}, w_{4k}) = c_{2k}$
- $\sigma(u_{2k}) = c_{\lceil \frac{m+2}{2} \rceil}$

Case-(ii): $\boldsymbol{m = odd}$

- $\sigma(u_{4k-2}, v_{4k-3}, v_{4k-2}, w_{4k-3}, w_{4k-2}) = c_{2k-1}$
- $\sigma(u_{4k}, v_{4k-1}, v_{4k}, w_{4k-1}, w_{4k}) = c_{2k}$
- $\sigma(u_{2k-1}) = c_{\lceil \frac{m+2}{2} \rceil -1}$
- $\sigma(v_m, w_m) = c_{\lceil \frac{m+2}{2} \rceil}$

To Prove: $z^*[L(Y_m)] < \left\lceil \frac{m+2}{2} \right\rceil$, say $\left\lceil \frac{m+2}{2} \right\rceil - 2$; $z^*[L(Y_m)]$ exists $\left(\frac{m}{2}\right) - K_{1,3}$ stars, each color class is colored with one color. Hence, there exists an independent set and clique remains uncolored we need two more colors to complete the graph. Therefore, assumption is contradictory. Hence $z^*[L(Y_m)] = \left\lceil \frac{m+2}{2} \right\rceil$.

3.3: Star Cochromatic Number of Middle Graph of Prism Graphs

Theorem 3.3.1: The Star Chromatic Number of Middle Graph of Prism Graph $q[M(Y_m)]$ for $m \equiv 0 \; mod \; 2$ is $q[Y_m] = \left\lceil \frac{m+4}{2} \right\rceil$

Proof:

Let Y_m be prism graph with 2m vertices and 3m edges and $V[Y_m] = \{v_n: 1 \le n \le m\} \cup \{u_n: 1 \le n \le m\}$ and $[M(Y_m)]$ be middle graph of prism graph where subdividing each edge exactly once and join the adjacent vertices, the vertex set of $V[M(Y_m)] = \{v_n: 1 \le n \le m\} \cup \{v'_n: 1 \le n \le m\} \cup \{u_n: 1 \le n \le m\} \cup \{u'_n: 1 \le n \le m\} \cup \{w'_n: 1 \le n \le m\}$. Consider the color class $C = \{c_1, c_2, c_3, \dots, c_{\left\lceil \frac{m+4}{2} \right\rceil}\}$. Define mapping $\sigma: \{v_n: 1 \le n \le m\} \cup \{v'_n: 1 \le n \le m\} \cup \{u_n: 1 \le n \le m\} \cup \{u'_n: 1 \le n \le m\} \cup \{w'_n: 1 \le n \le m\} \rightarrow c_k \; \forall \; k = 1,2,3, \dots$

Assign the coloring as follows:

- $\sigma(w'_{4k-3}, v'_{4k-3}, v'_{4k-2}, u'_{4k-3}, v'_{4k-2}, v_{4k-2}, u_{4k-2}) = c_{2k-1}$
- $\sigma(w'_{4k-1}, v'_{4k-1}, v'_{4k}, u'_{4k-1}, u'_{4k}, u_{4k}, v_{4k}) = c_{2k}$
- $\sigma(u_{2k-1}, v_{2k-1}) = c_{\left\lceil \frac{m+4}{2} \right\rceil - 1}$
- $\sigma(w'_{2k}) = c_{\left\lceil \frac{m+4}{2} \right\rceil}$

To Prove: $z^*[M(Y_m)] < \left\lceil \frac{m+4}{2} \right\rceil$, say $\left\lceil \frac{m+4}{2} \right\rceil - 1$; $z^*[M(Y_m)]$ exists $\left(\frac{m}{2}\right) - K_{1,3}$ stars, is colored with $\frac{m}{2}$ colors. Hence, there exists an independent set remains uncolored we need two more colors to complete the graph. Therefore, assumption is contradictory. Hence $z^*[M(Y_m)] = \left\lceil \frac{m+4}{2} \right\rceil$.

4.1: Star Cochromatic Number of Sunlet Graphs

Theorem 4.1.1: The Star Chromatic Number of Sunlet Graph $q[S_m]$ for $m \ge 3$ is $q[S_m] = \left\lceil \frac{m+3}{3} \right\rceil$

Proof:

Let S_m be the sunlet graph with 2m vertices and 3m edges and $V[S_m] = \{v_n: 1 \le n \le m\} \cup \{u_n: 1 \le n \le m\}$. Consider the color class $C = \{c_1, c_2, c_3, \dots, c_{\left\lceil \frac{m+3}{3} \right\rceil}\}$. Define mapping $\sigma: \{v_n: 1 \le n \le m\} \cup \{u_n: 1 \le n \le m\} \rightarrow c_k \; \forall \; k = 1,2,3, \dots$

Case-(i): $\boldsymbol{m \equiv 0 \ mod\ 3}$

- $\sigma(v_{6k-4}, v_{6k-5}, v_{6k-3}, u_{6k-4}) = c_{2k-1}$
- $\sigma(v_{6k-1}, v_{6k-2}, v_{6k}, u_{6k-1}) = c_{2k}$
- $\sigma(u_{3k-2}, u_{3k}) = c_{\left\lceil\frac{m+3}{3}\right\rceil}$

To Prove: $z^*[S_m] < \left\lceil\frac{m+3}{3}\right\rceil$; $z^*[S_m]$ exists $3m - K_{1,3}$ stars, each color class is colored with one color. There exists an independent set (u_{3k-2}, u_{3k}) remains uncolored we need one more color to complete the graph. Therefore, assumption is contradictory. Hence $q[S_m] = \left\lceil\frac{m+3}{3}\right\rceil$.

Case-(ii): $\boldsymbol{m \equiv 1 \ mod\ 3}$

- $\sigma(v_{6k-4}, v_{6k-5}, v_{6k-3}, u_{6k-4}) = c_{2k-1}$
- $\sigma(v_{6k-1}, v_{6k-2}, v_{6k}, u_{6k-1}) = c_{2k}$
- $\sigma(v_m, u_m) = c_{\left\lceil\frac{m+3}{3}\right\rceil-1}$
- $\sigma(u_{3k-2}, u_{3k}) = c_{\left\lceil\frac{m+3}{3}\right\rceil}$

Case-(iii): $\boldsymbol{m \equiv 2 \ mod\ 3}$

- $\sigma(v_{6k-4}, v_{6k-5}, v_{6k-3}, u_{6k-4}) = c_{2k-1}$
- $\sigma(v_{6k-1}, v_{6k-2}, v_{6k}, u_{6k-1}) = c_{2k}$
- $\sigma(v_m, v_{m-1}) = c_{\left\lceil\frac{m+3}{3}\right\rceil-1}$
- $\sigma(u_{3k-2}, u_{3k}, u_m) = c_{\left\lceil\frac{m+3}{3}\right\rceil}$

To Prove: $z^*[S_m] < \left\lceil\frac{m+3}{3}\right\rceil$; $z^*[S_m]$ exists $3m - K_{1,3}$ stars, each color class is colored with one color. Hence, there exists an independent set and clique remains uncolored we need two more colors to complete the graph. Assumption is contradictory. Hence $q[Y_m] = \left\lceil\frac{m+3}{3}\right\rceil$.

4.2: Star Cochromatic Number of Line Graph of Sunlet Graphs

Theorem 4.2.1: The Star Chromatic Number of Line Graph of Sunlet Graph $q[L(S_m)]$ for $m \geq 4$ is $q[Y_m] = \left\lceil\frac{m}{2}\right\rceil$

Proof:

Let S_m be sunlet graph with 2m vertices and 3m edges and $V[S_m] = \{v_n: 1 \leq n \leq m\} \cup \{u_n: 1 \leq n \leq m\}$ and $[L(S_m)]$ be line graph of prism graph with 3m vertices where $V[L(S_m)] = \{v_n: 1 \leq n \leq m\} \cup \{u_n: 1 \leq n \leq m\}$. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_{\left\lceil\frac{m}{2}\right\rceil}\}$. Define mapping $\sigma: \{v_n: 1 \leq n \leq m\} \cup \{u_n: 1 \leq n \leq m\} \rightarrow c_k \ \forall \ k = 1,2,3, \ldots$

Case-(i): $\boldsymbol{m \equiv 0 \; mod \; 3}$

- $\sigma(v_{6k-4}, v_{6k-5}, v_{6k-3}, v_{6k-5}, u_{6k-4}) = c_{2k-1}$
- $\sigma(v_{6k-1}, v_{6k-2}, v_{6k}, v_{6k-2}, u_{6k-1}) = c_{2k}$
- $\sigma(u_{3k}) = c_{\lceil \frac{m}{2} \rceil}$

Case-(ii): $\boldsymbol{m \equiv 1 \; mod \; 3}$

- $\sigma(v_{6k-4}, v_{6k-5}, v_{6k-3}, v_{6k-5}, u_{6k-4}) = c_{2k-1}$
- $\sigma(v_{6k-1}, v_{6k-2}, v_{6k}, v_{6k-2}, u_{6k-1}) = c_{2k}$
- $\sigma(v_m, u_m) = c_{\lceil \frac{m}{2} \rceil - 1}$
- $\sigma(u_{3k}) = c_{\lceil \frac{m}{2} \rceil}$

Case-(iii): $\boldsymbol{m \equiv 2 \; mod \; 3}$

- $\sigma(v_{6k-4}, v_{6k-5}, v_{6k-3}, v_{6k-5}, u_{6k-4}) = c_{2k-1}$
- $\sigma(v_{6k-1}, v_{6k-2}, v_{6k}, v_{6k-2}, u_{6k-1}) = c_{2k}$
- $\sigma(v_m, v_{m-1}) = c_{\lceil \frac{m}{2} \rceil - 1}$
- $\sigma(u_{3k}, u_m, u_{m-1}) = c_{\lceil \frac{m}{2} \rceil}$

To Prove: $z^*[L(S_m)] < \left\lceil \frac{m}{2} \right\rceil$, say $\left\lceil \frac{m}{2} \right\rceil - 1$; $z^*[L(S_m)]$ exists $\left(\frac{m}{3}\right) - K_{1,3}$ stars, each color class is colored with $\frac{m}{3}$ colors. Hence, there exists an independent set remains uncolored we need one more colors to complete the graph. Therefore, assumption is contradictory. Hence $z^*[L(S_m)] = \left\lceil \frac{m}{2} \right\rceil$.

4.3: Star Cochromatic Number of Middle Graph of Sunlet Graphs

Theorem 4.3.1: The Star Chromatic Number of Middle Graph of Sunlet Graph $q[M(S_m)]$ for $m \equiv 0 \; mod \; 3$ is $q[M(S_m)] = \left\lceil \frac{m+4}{3} \right\rceil$

Proof:

Let S_m be sunlet graph with 2m vertices and 3m edges and $V[S_m] = \{v_n : 1 \le n \le m\} \cup \{u_n : 1 \le n \le m\}$ and $[M(S_m)]$ be middle graph of sunlet graph where subdividing each edge exactly once and join the adjacent vertices, the vertex set of $V[M(S_m)] = \{v_n : 1 \le n \le m\} \cup \{v'_n : 1 \le n \le m\} \cup \{u_n : 1 \le n \le m\} \cup \{u'_n : 1 \le n \le m\}$. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_{\lceil \frac{m+4}{3} \rceil}\}$. Define mapping $\sigma : \{v_n : 1 \le n \le m\} \cup \{v'_n : 1 \le n \le m\} \cup \{u_n : 1 \le n \le m\} \cup \{u'_n : 1 \le n \le m\} \to c_k$ $\forall \; k = 1,2,3, \ldots$

Assign the coloring as follows:

- $\sigma(v'_{6k-4}, u'_{6n-4}, u'_{6n-3}, v'_{6k-5}, v'_{6k-3}, v_{6k-4}, v_{6k-3}) = c_{2k-1}$
- $\sigma(v'_{6k-1}, u'_{6n-1}, u'_{6n}, v'_{6k-2}, v'_{6k}, v_{6k-1}, v_{6k}) = c_{2k}$

- $\sigma(u_k, v_{3k-2}) = c_{\left\lceil\frac{m+4}{3}\right\rceil}$

To Prove: $z^*[M(S_m)] < \left\lceil\frac{m+4}{3}\right\rceil$, say $\left\lceil\frac{m+4}{3}\right\rceil - 1$; $z^*[M(S_m)]$ exists $\left(\frac{m}{3}\right) - K_{1,3}$ stars, is colored with $\frac{m}{3}$ colors. Hence, there exists an independent set remains uncolored we need one more color to complete the graph. Assumption is contradiction. Hence $z^*[M(S_m)] = \left\lceil\frac{m+4}{3}\right\rceil$.

5. Conclusion:

In this article, we examined the star cochromatic number of prism graphs, the line graph of prism graphs, the middle graph of prism graphs, the star cochromatic index for sunlet graphs, the line graph of sunlet graphs and the middle graph of sunlet graphs. Further improvements will encompass more central, line, total and middle graphs in various graph families.

References

[1] Hassler Whitney, *Congruent graphs and the connectivity of graphs*, Amer. J. Math. 1932.

[2] Kowsalya. V, Vernold Vivin. J and Venkatachalam. M, *On Star Chromatic Number of Sunlet Graph Families*, Ars Combinatoria, Vol-123 (431-437), 2015.

[3] Lesinak and Straight, *A Cochromatic Number of a Graph*, 1977.

[4] M. Poobalaranjani, *A Study on Cocoloring and Variants of Cocoloring of Graphs*, Ph. D Thesis, 2020

[5] Maria Chudnovsky, Neil Robertson, Paul Seymour, Robin Thomas, *The strong perfect graph theorem*, Ann. of Math, 2006.

[6] T. Hamada and I. Yoshimura, *Traversability and connectivity of the Middle Graphs of a Graph*, Discrete Mathematics, 14, 1976.

[7] Vernold Vivin. J, Kowsalya. V, and Vimal Kumar. S, *On Star Chromatic Number of Prism Graph Families*, TWMS Journal of Applied and Engineering Mathematics, Vol.9, No.3 687-692, 2019.