Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 6s (2024)

Prime Graceful Chromatic Number of Diverse Graphs

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Article History:

Received: *03-06-2024*

Revised: *05-07-2024*

Accepted: *25-07-2024*

Abstract:

In this paper, prime graceful coloring is introduced which aims to incorporate principles of prime and graceful coloring. The prime graceful coloring of star, path, cycle, friendship, pan and bistar graphs are proposed. This new approach is related to the prime graceful chromatic number which represent the fewest colors needed to color any graph adhering to the principles of prime graceful coloring. Also, the efficiency of new coloring are analyzed with the examples.

Keywords: Prime Coloring, Graceful Coloring, Prime Graceful Labeling, Prime Graceful Coloring.

1. Introduction

Let G be a finite simple undirected graph. Graph labeling technique was first developed by Rosa [5] who also provided numerous graph labeling techniques and Gallian J.A [2] conducted the survey on graph labeling. The proper coloring is alluded from [10]. Roger Entringer defined prime labeling and it is introduced by Tout et.all [9] and [3] gave the precise value of prime chromatic number of several graphs. Zhenming Bi et.all [1,11] introduced the concept of graceful colorings of graphs and the graceful chromatic number for wheel graph family are shown to be graceful [8]. In 2018, Selvarajan.T.M, Subramoniam.R [7] combined the characteristics of prime and graceful labeling and introduced a new labeling technique Prime Graceful Labeling and demonstrated the existence of prime graceful labeling in some graphs. Sayan Panma and Penying Rochanakul [6] defined prime-graceful number and applied prime graceful labeling to certain graphs, then Nandhini.S.P and Pooja Lakshmi.B [4] generalized the cardinality of the edges for the triangular snake graph. In this paper, we define the new technique Prime Graceful Coloring and its chromatic number are analyzed for some graphs.

2. Preliminaries

Definition: 2.1[3]

Prime Coloring is defined as G be a loop less and without multiple edges with p distinct vertices $\omega:V(G)\rightarrow\{1, 2, \dots p\}$,if each edge $e = c_i c_j$, $i\neq j$, $\gcd\{\omega(v_i), \omega(v_j)\} = 1$, $\omega(v_i)$, $\omega(v_j)$ receives distinct colors. It is denoted by $\eta(G)$.

Definition: 2.2[8]

A **graceful k-coloring** of a non-empty graph G=(V,E) is a proper vertex coloring $\omega:V(G)\rightarrow\{1, 2, \dots k\}$, $k \geq 2$ which induces a proper edge coloring $\omega^*: E(G)\rightarrow\{1,2,\dots k-1\}$ defined by $\omega^*(v_i, v_j) = |\omega(v_i) - \omega(v_j)|$ where $v_i, v_j \in V(G)$. The minimum k for which G has a graceful k-coloring is called graceful chromatic number $\chi_g(G)$.

Definition: 2.3 [4]

A graph G with p vertices and q edges is referred to have **prime graceful labeling** if the map depicts a one to one function $\omega:V(G)\to\{1, 2, \ldots k\}$. In this instance, $k = \min\{p, q\}$ such that GCD $(\omega(v_i), \omega(v_j)) = 1$ and the map reflect the induced one to one function $\omega^*: E(G)\to\{1,2,\ldots k-1\}$ defined by $\omega^*(v_iv_j) = |\omega(v_i) - \omega(v_j)|$ such that the corresponding edge labels are distinct.

Definition: 2.4

A **prime graceful k-coloring** of non-empty simple graph G is a proper vertex coloring $\omega:V(G)\to\{1, 2,3\ldots\ldots k\}$ such that GCD $(\omega(v_i), \omega(v_j)) = 1$ where $k \geq 2$ bring about a proper edge coloring $\omega^*:E(G)\to\{1,2,3\ldots k-1\}$defined by $\omega^*(v_iv_j) = |\omega(v_i)- \omega(v_j)|$, $\forall\ k\in N$ the least k for which G exhibits prime graceful k-coloring is known as the chromatic number of prime graceful coloring. It is denoted by $\chi_{pg}(G)$.

3. Main Results

Theorem: 3.1

Let $K_{1,n}$ be a star graph, then $\chi_{pg}(K_{1,n})= n+1$.

Proof:

Let $V(K_{1,n}) = \{v_i : 1 \leq i \leq n+1\}$ and $E(K_{1,n}) = \{v_iv_{i+1} : 1 \leq i \leq n\}$. Let $\omega:V(K_{1,n})\to\{c_1, c_2, \ldots, c_{n+1}\}$where the vertex of degree n is designated as c_1 (i.e),$\omega(v_1) = c_1$ and n-pendant vertices are designated as $\omega(v_i)= c_i$ which satisfy GCD $(\omega(v_1), \omega(v_i)) = 1$ for $2 \leq i \leq n+1$. Hence, adjacent vertices receive distinct colors.

Let $\omega^*:E(K_{1,n})\to\{c_1, c_2, \ldots, c_n$. For $2 \leq i \leq n+1$, $\omega^*(v_1v_i) = c_{i-1}$. It is determined by $\omega * (v_1v_i)= |\omega(v_1) - \omega(v_i)| = c_{i-1}$. Consequently, $\omega^*(v_1v_i)$ receives distinct colors. We proved that $\chi_{pg}(K_{1,n}) \leq n + 1$. To prove $\chi_{pg}(K_{1,n}) \geq n + 1$, let us assume that $\chi_{pg}(K_{1,n}) < n + 1$, say n. We define proper vertex coloring of $K_{1,n}$ is $\omega(v_1) =c_1$, $\omega(v_i)=c_{i-1}$ $\forall\ 2 \leq i \leq n+1$. Then, it induces edge coloring $\omega * (v_1v_2)= c_1$, $\omega * (v_1v_3)= c_1$, $\omega * (v_1v_4)= c_2$, $\omega * (v_1v_5)= c_3$ and so on which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent edges are distinct. Thus, $\chi_{pg}(K_{1,n}) \geq n + 1$.

Therefore, $\chi_{pg}(K_{1,n})= n+1$.

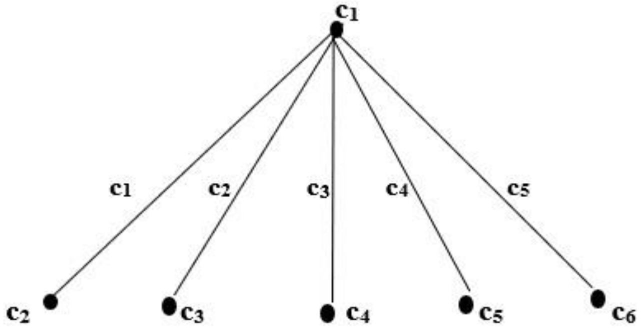

Figure 1 Analytical Evaluation of the $K_{1,5}$

Theorem: 3.2

Let $n \geq 5$ be a positive integer, then $\chi_{pg}(P_n)= 4$.

Proof:

Define a proper vertex coloring $\omega:V(P_n) \to \{c_1, c_2, c_3, c_4\}$ for $1 \leq i \leq n$

$$\omega(v_i) = \begin{cases} c_1, \text{for } i \equiv 0 \bmod 3 \\ c_2, \text{for } v_1 \\ c_3, \text{for } i \equiv 2 \bmod 3 \\ c_4, \text{for } i \equiv 1 \bmod 3, i \neq 1 \end{cases}$$

It is clear that GCD $(\omega(v_i), \omega(v_{i+1})) = 1$.

It induce a proper edge coloring $\omega^*:E(P_n) \to \{c_1, c_2, c_3\}$ as follows:

$$\omega^*(v_i v_{i+1}) = \begin{cases} c_1, \text{for } i \equiv 1 \bmod 3 \\ c_2, \text{for } i \equiv 2 \bmod 3 \\ c_3, \text{for } i \equiv 0 \bmod 3 \end{cases}$$

It satisfy $\omega * (v_i v_{i+1})= | \omega(v_i) - \omega(v_{i+1})| = c_2 - c_3 = c_3 - c_1 = c_1 - c_4 = c_4 - c_3$
$= c_1, c_2, c_3, c_1$

Hence, adjacent vertices and edges receive distinct colors. Thus, $\chi_{pg}(P_n) \leq 4$, for $n \geq 5$. To prove $\chi_{pg}(P_n) \geq 4$, let us assume that $\chi_{pg}(P_n) < 4$, say 3. We define vertex coloring of P_5 is c_2, c_1, c_3, c_2, c_1. Then, $\omega * (v_1v_2)= c_1$, $\omega * (v_2v_3)= c_2$, $\omega * (v_3v_4)= c_1$, $\omega * (v_4v_5)= c_1$ which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent edges are distinct. Thus, $\chi_{pg}(P_n) \geq 4$. Therefore, $\chi_{pg}(P_n)= 4$, if $n \geq 5$.

Corollary: 3.3 For any path, $\chi_{pg}(P_n)= 2$, if $n = 2$.

Corollary: 3.4 For any path, $\chi_{pg}(P_n)= 3$, if $n = 3$ and 4.

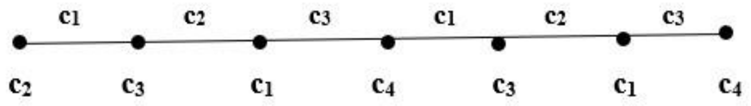

Figure 2 Analytical Evaluation of the P_7

Theorem: 3.5

Let $n \geq 5$ be a positive integer, then $\chi_{pg}(C_n)= 5$.

Proof:

The vertices and edges of C_n are $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n)=\{v_i v_{i+1}: 1 \leq i \leq n\}$.

Case 1: $n \neq 3m+3, m \in N$

Define a proper vertex coloring $\omega:V(C_n) \to \{c_1, c_2, c_3, c_4, c_5\}$ as follows:

For $1 \leq i \leq n$

$$\omega(v_i) = \begin{cases} c_1, \text{for } v_2 \\ c_2, \text{for } i \equiv 0 \bmod 3 \\ c_3, \text{for } i \equiv 2 \bmod 3 \text{ and } i \neq 2 \\ c_4, \text{for } v_1 \\ c_5, \text{for } i \equiv 1 \bmod 3 \text{ and } i \neq 1 \end{cases}$$

Such that GCD $(\omega(v_i), \omega(v_{i+1})) = 1$.

It induce a proper edge coloring $\omega^*:E(C_n) \to \{c_1, c_2, c_3\}$ as follows:

$$\omega^* (v_iv_{i+1}) = \begin{cases} c_3, \text{for } v_1v_2 \text{ and } i \equiv 0 \bmod 3 \\ c_2, \text{for } i \equiv 1 \bmod 3 \text{ and } i \neq 1 \\ c_1, \text{for } v_nv_1 \text{ and } i \equiv 2 \bmod 3 \end{cases}$$

It satisfy $\omega * (v_1v_2) = |\omega(v_1) - \omega(v_2)| = c_4 - c_1 = c_3$
$\omega * (v_2v_3) = |\omega(v_2) - \omega(v_3)| = c_1 - c_2 = c_1$
$\omega * (v_3v_4) = |\omega(v_3) - \omega(v_4)| = c_2 - c_5 = c_3$
$\omega * (v_4v_5) = |\omega(v_4) - \omega(v_5)| = c_5 - c_3 = c_2$
Hence, adjacent vertices and edges receive distinct colors.

Thus, $\chi_{pg}(C_n) \leq 5$. To prove $\chi_{pg}(C_n) \geq 5$, let us assume that $\chi_{pg}(C_n) < 5$, say 4. We define vertex coloring of C_5 is c_4, c_1, c_2, c_3, c_1. Then, $\omega * (v_1v_2) = c_3$, $\omega * (v_2v_3) = c_1$, $\omega * (v_3v_4) = c_1$, $\omega * (v_4v_5) = c_2$, $\omega * (v_5v_1) = c_3$. Since $\omega * (v_2v_3) = c_1$ and $\omega * (v_3v_4) = c_1$ receives same color which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent edges are distinct. Thus, $\chi_{pg}(C_n) \geq 5$. Therefore, $\chi_{pg}(C_n) = 5$ for $n \neq 3m+3$, $m \in N$.

Case 2: $n = 3m + 3$

Define a proper vertex coloring $\omega : V(C_n) \rightarrow \{c_1, c_2, c_3, c_4, c_5\}$ as follows:
For $1 \leq i \leq n$

$$\omega(v_i) = \begin{cases} c_1, \text{for } v_2 \\ c_2, \text{for } i \equiv 1 \bmod 3 \text{ and } i \neq 1 \\ c_3, \text{for } i \equiv 2 \bmod 3 \text{ and } i \neq 2 \\ c_4, \text{for } v_1 \\ c_5, \text{for } i \equiv 0 \bmod 3 \end{cases}$$

Such that GCD $(\omega(v_i), \omega(v_{i+1})) = 1$.
It induce a proper edge coloring $\omega^* : E(C_n) \rightarrow \{c_1, c_2, c_3\}$ as follows:

$$\omega^* (v_iv_{i+1}) = \begin{cases} c_1, \text{for } v_nv_1, i \equiv 1 \bmod 3 \text{ and } i \neq 1 \\ c_2, \text{for } i \equiv 2 \bmod 3 \text{ and } i \neq 2 \\ c_3, \text{for } v_1v_2 \text{ and } i \equiv 0 \bmod 3 \\ c_4, \text{for } v_2v_3 \end{cases}$$

It satisfy $\omega * (v_1v_2) = |\omega(v_1) - \omega(v_2)| = c_4 - c_1 = c_3$
$\omega * (v_2v_3) = |\omega(v_2) - \omega(v_3)| = c_1 - c_5 = c_4$
$\omega * (v_3v_4) = |\omega(v_3) - \omega(v_4)| = c_5 - c_2 = c_3$
$\omega * (v_4v_5) = |\omega(v_4) - \omega(v_5)| = c_2 - c_3 = c_1$
Hence, adjacent vertices and edges receive distinct colors. Thus, $\chi_{pg}(C_n) \leq 5$. To prove $\chi_{pg}(C_n) \geq 5$, let us assume that $\chi_{pg}(C_n) < 5$, say 4. We define vertex coloring of C_6 is c_4, c_1, c_2, c_4, c_3, c_2. Then, $\omega * (v_1v_2) = c_3$, $\omega * (v_2v_3) = c_1$, $\omega * (v_3v_4) = c_2$, $\omega * (v_4v_5) = c_1$, $\omega * (v_5v_6) = c_1$ and $\omega * (v_6v_1) = c_2$. Since $\omega * (v_4v_5) = c_1$ and $\omega * (v_5v_6) = c_1$ receives same color which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent edges are distinct. Thus, $\chi_{pg}(C_n) \geq 5$. Therefore, $\chi_{pg}(C_n) = 5$ for $n = 3m+3$, $m \in N$.

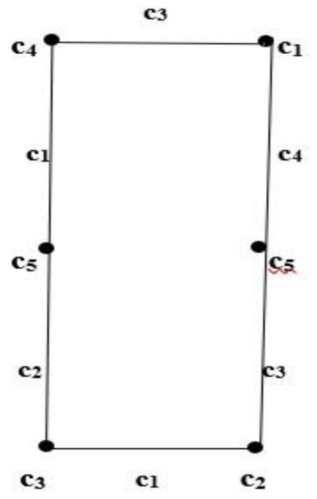

Figure 3 Analytical Evaluation of the C_6

Theorem: 3.6

Let F_n be a friendship graph, then for $n \geq 2$ $\chi_{pg}(F_n)= 2n+1$.

Proof:

The vertices and edges of F_n are $V(F_n) = \{ v_i : 1 \leq i \leq 2n+1\}$ and $E(F_n) = \{v_i v_{i+1} : 1 \leq i \leq 3n\}$.

Let $\omega: V(F_n) \to \{c_1, c_2, \ldots, c_{2n+1}\}$ be a proper vertex coloring of F_n.

It is defined as $\omega(v_1) = c_1$ (i.e), vertex of degree 2n, $\omega(v_{2i}) = c_{2i}$ $\forall$ $1 \leq i \leq n$.

For i=1, $\omega(v_{2i+1}) = c_{2n+1}$ and $\omega(v_{2i+1}) = c_{2i-1}$ $\forall$ $2 \leq i \leq n$.

Since these colors are consecutive colors, $GCD(\omega(v_1), \omega(v_{2i})) = 1$, $GCD(\omega(v_1), \omega(v_{2i+1})) = 1$ and $GCD(\omega(v_{2i}), \omega(v_{2i+1})) = 1$

Let $\omega^*: E(F_n) \to \{c_1, c_2, \ldots, c_{2n}\}$ be a proper edge coloring of F_n. It is defined as,

$\omega * (v_i v_{i+1}) = | \omega(v_i) - \omega(v_{i+1})|$ $\forall$ $1 \leq i \leq n$.

$\omega^*(v_1 v_{2i}) = c_{2i-1}$,

$\omega^*(v_1 v_{2i+1}) = c_{2i}$

For i=1, $\omega^*(v_{2i} v_{2i+1}) = c_{2n-1}$ and for $i \neq 1$, $\omega^*(v_{2i} v_{2i+1}) = c_1$.

Consequently, $\omega^*(v_i v_{i+1})$ receives distinct colors.

Thus, $\chi_{pg}(F_n) \leq 2n + 1$. To prove $\chi_{pg}(F_n) \geq 2n + 1$, let us assume that $\chi_{pg}(F_n) < 2n + 1$, say 2n. We must assign 2n colors for $\{v_{2i}, v_{2i+1}: 1 \leq i \leq n\}$ for proper vertex coloring. Since one vertex is adjacent to remaining vertices, we cannot assign the same color which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent vertices are distinct. Thus, $\chi_{pg}(F_n) \geq 2n + 1$. Therefore, $\chi_{pg}(F_n) = 2n+1$.

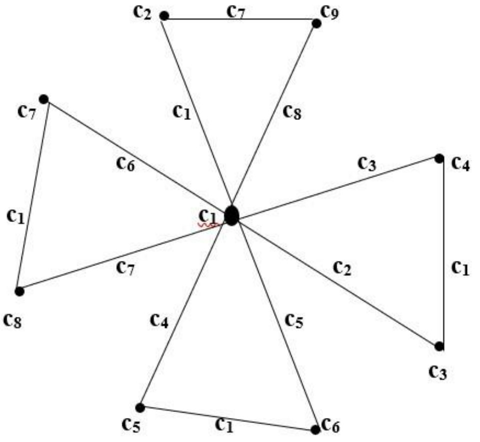

Figure 4 Analytical Evaluation of the F_4

Theorem: 3.7

Let n ≥ 3 be a positive integer, then

$$\chi_{pg}(\text{n-pan graph}) = \begin{cases} 4, \text{if } n = 3 \\ 5, \text{if } n \neq 3 \end{cases}$$

Proof:

The vertices and edges of pan graph are $V(G) = \{v_i : 1 \leq i \leq n+1\}$ and $E(G)=\{v_iv_{i+1}: 1 \leq i \leq n+1\}$.

Case 1: n =3

Define a proper vertex coloring ω:V(G) →{c_1, c_2, c_3, c_4}as follows:

For $1 \leq i \leq n+1$, Such that GCD ($\omega(v_i)$, $\omega(v_{i+1})$) = 1. It induce a proper edge coloring ω*:E(G)→{c_1, c_2, c_3} as follows:

$$\omega(v_i) = \begin{cases} c_1, \text{for } v_2 \\ c_2, \text{for } v_4 \\ c_3, \text{for } v_1 \\ c_4, \text{for } v_3 \end{cases}$$

$$\omega^*(v_iv_{i+1}) = \begin{cases} c_2, \text{for } v_1v_2 \\ c_3, \text{for } v_2v_3 \\ c_1, \text{for } v_3v_1 \text{ and } v_1v_4 \end{cases}$$

Hence, adjacent vertices and edges receive distinct colors. Therefore, χ_{pg}(n-pan graph)= 4 for n=3

Case 2: n ≠ 3m+3, m∈N

Define a proper vertex coloring ω:V(G) →{c_1, c_2, c_3, c_4, c_5}as follows: For $1 \leq i \leq n+1$

$$\omega(v_i) = \begin{cases} c_1, \text{for } v_2 \\ c_2, \text{for } i \equiv 0 \text{ mod } 3 \\ c_3, \text{for } i \equiv 2 \text{ mod } 3, v_{n+1} \text{ and } i \neq 2 \\ c_4, \text{for } v_1 \\ c_5, \text{for } i \equiv 1 \text{ mod } 3 \text{ and } i \neq 1 \end{cases}$$

Such that GCD ($\omega(v_i)$, $\omega(v_{i+1})$) = 1. It induce a proper edge coloring ω*:E(G)→{c_1, c_2, c_3} as follows:

$$\omega^*(v_iv_{i+1}) = \begin{cases} c_1, \text{for } i \equiv 2 \text{ mod } 3 \text{ and } v_nv_1 \\ c_2, \text{for } i \equiv 1 \text{ mod } 3, v_2v_{n+1} \text{ and } i \neq 1 \\ c_3, \text{for } v_1v_2 \; and \; i \equiv 0 \text{ mod } 3 \end{cases}$$

It satisfy $\omega * (v_i v_{i+1}) = |\ \omega(v_i) - \omega(v_{i+1})|$

Hence, adjacent vertices and edges receive distinct colors.

Thus, χ_{pg}(n-pan graph)≤5. To prove χ_{pg}(n-pan graph)≥ 5, let us assume that $\chi_{pg}(C_n) < 5$, say 4. We define vertex coloring as c_4, c_1, c_2, c_3, c_1 for 5-pan graph and the vertex with degree 1 is assigned with the color c_3. Then, $\omega * (v_1v_2) = c_3$, $\omega * (v_2v_3) = c_1$, $\omega * (v_2v_6) = c_2$, $\omega * (v_3v_4) = c_1$, $\omega * (v_4v_5) = c_2$, $\omega * (v_5v_1) = c_3$. Since $\omega * (v_2v_3) = c_1$ and $\omega * (v_3v_4) = c_1$ receives same color which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent edges are distinct. Thus, χ_{pg}(n-pan graph)≥ 5. Therefore, χ_{pg}(n-pan graph)= 5 for n ≠3m+3, m∈N.

Case 3: $n = 3m + 3$

Define a proper vertex coloring $\omega: V(G) \to \{c_1, c_2, c_3, c_4, c_5\}$ as follows:

For $1 \le i \le n+1$, Such that GCD $(\omega(v_i), \omega(v_{i+1})) = 1$. It induce a proper edge coloring $\omega^*: E(G) \to \{c_1, c_2, c_3\}$ as follows:

$$\omega(v_i) = \begin{cases} c_1, \text{for } v_2 \\ c_2, \text{for } i \equiv 1 \bmod 3 \text{ and } i \neq 1 \\ c_3, \text{for } i \equiv 2 \bmod 3, v_{n+1} \text{ and } i \neq 2 \\ c_4, \text{for } v_1 \\ c_5, \text{for } i \equiv 0 \bmod 3 \end{cases}$$

$$\omega^* (v_i v_{i+1}) = \begin{cases} c_4, \text{for } v_2v_3 \\ c_3, \text{for } v_1v_2 \textit{ and } i \equiv 0 \bmod 3 \\ c_2, \text{for } i \equiv 2 \bmod 3, v_2v_{n+1} \text{ and } i \neq 2 \\ c_1, \text{for } v_nv_1, i \equiv 1 \bmod 3 \text{ and } i \neq 1 \end{cases}$$

It satisfy $\omega * (v_i v_{i+1}) = |\ \omega(v_i) - \omega(v_{i+1})|$

Hence, adjacent vertices and edges receive distinct colors. Thus, χ_{pg}(n-pan graph)≤5. To prove χ_{pg}(n-pan graph)≥ 5, let us assume that χ_{pg}(n-pan graph)< 5, say 4. We define vertex coloring as c_4, c_1, c_2, c_3, c_1, c_3 for 6-pan graph and the vertex with degree 1 is assigned with the color c_3. Then, $\omega * (v_1v_2) = c_3$, $\omega * (v_2v_3) = c_1$, $\omega * (v_2v_6) = c_2$, $\omega * (v_3v_4) = c_1$, $\omega * (v_4v_5) = c_2$, $\omega * (v_5v_6) = c_2$, $\omega * (v_6v_1) = c_1$. Since $\omega * (v_2v_3) = c_1$ and $\omega * (v_3v_4) = c_1$ receive same color which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent edges are distinct. Thus, χ_{pg}(n-pan graph)≥ 5. Therefore, χ_{pg}(n-pan graph)= 5 for n =3m+3, m∈N.

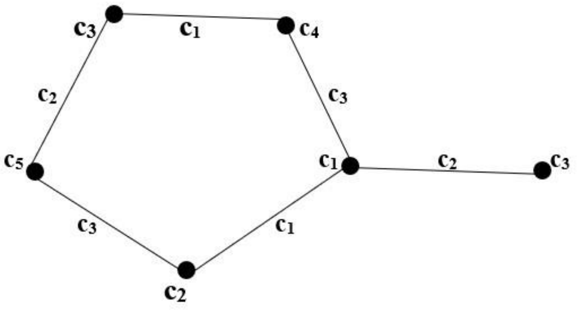

Figure 5 Analytical Evaluation of the 5 - Pan graph

Theorem: 3.8

Let $B_{n,n}$ be a bistar graph, then

$$\chi_{pg}(B_{n,n}) = \begin{cases} n+2, \text{ when n is odd} \\ n+3, \text{ when n is even} \\ n+4, \text{ when } n \equiv 1 \bmod 6 \text{ and } n \neq 1 \\ n+5, \text{ when } n \equiv 0 \bmod 6 \end{cases}$$

Proof:

The Bistar graph has 2n+2 vertices and 2n+1 edges.

Case 1: n is odd

Let $V(B_{n,n}) = \{ u_i : 1 \leq i \leq n+1\} \cup \{ v_i : 1 \leq i \leq n+1\}$. Define a proper vertex coloring $\omega : V(B_{n,n}) \rightarrow \{c_1, c_2, c_3 \ldots\ldots c_{n+2}\}$. $\omega(u_1) = c_1$ and $\omega(v_1) = c_{n+2}$ and $\omega(u_i) = c_i$, $\omega(v_i) = c_i$ $\forall\ 2 \leq i \leq n+1$. Such that GCD $\omega(u_1, u_i) = 1$, GCD $\omega(v_1, v_i) = 1$ and GCD $\omega(u_1, v_1) = 1$.

Define a proper edge coloring $\omega^* : E(B_{n,n}) \rightarrow \{ c_1, c_2, c_3 \ldots\ldots c_{n+1}\}$

$\omega*(u_1 v_1) = c_1 - c_{n+2} = c_{n+1}$, $\omega*(u_1 u_i) = c_1 - c_i = c_{i-1}$, $\omega*(v_1 v_i) = c_{n+2} - c_i = c_{n+2-i}$

Hence, adjacent vertices and edges receive distinct colors. We proved that $\chi_{pg}(B_{n,n}) \leq n+2$. To prove $\chi_{pg}(B_{n,n}) \geq n+2$, let us assume that $\chi_{pg}(B_{n,n}) < n+2$ say $n+1$. We define proper vertex coloring of $B_{n,n}$ is $\omega(u_i) = c_i$, $\omega(v_i) = c_i$ $\forall\ 2 \leq i \leq n+1$ since $\omega(u_1)$ and $\omega(v_1)$ are adjacent vertices we cannot assign the same color c_1 which is a contradiction with the definition of prime graceful coloring since the color of any two adjacent vertices are distinct. Thus, $\chi_{pg}(B_{n,n}) \geq n+2$. Therefore, $\chi_{pg}(B_{n,n}) = n+2$ when n is odd.

Case 2: n is even

Let $V(B_{n,n}) = \{ u_i : 1 \leq i \leq n+1\} \cup \{ v_i : 1 \leq i \leq n+1\}$.

Define a proper vertex coloring $\omega : V(B_{n,n}) \rightarrow \{c_1, c_2, c_3 \ldots\ldots c_{n+3}\}$. $\omega(u_1) = c_1$ and $\omega(v_1) = c_{n+3}$

And $\omega(u_i) = c_i$, $\omega(v_i) = c_{i+1}$ $\forall\ 2 \leq i \leq n+1$. Such that GCD $\omega(u_1, u_i) = 1$, GCD $\omega(v_1, v_i) = 1$ and GCD $\omega(u_1, v_1) = 1$.

Define a proper edge coloring $\omega^* : E(B_{n,n}) \rightarrow \{ c_1, c_2, c_3 \ldots\ldots c_{n+2}\}$

$\omega*(u_1 v_1) = c_1 - c_{n+3} = c_{n+2}$, $\omega*(u_1 u_i) = c_1 - c_i = c_{i-1}$, $\omega*(v_1 v_i) = c_{n+3} - c_{i+1} = c_{n+2-i}$

Hence, adjacent vertices and edges receive distinct colors. We proved that $\chi_{pg}(B_{n,n}) \leq n+3$. To prove $\chi_{pg}(B_{n,n}) \geq n+3$, let us assume that $\chi_{pg}(B_{n,n}) < n+3$ say $n+2$. We define proper vertex coloring of $B_{n,n}$ is $\omega(u_1) = c_1$, $\omega(u_i) = c_i$, $\omega(v_i) = c_i$ $\forall\ 2 \leq i \leq n+1$ and $\omega(v_1) = c_{n+2}$. Since v_1 is adjacent to v_i $\forall\ 2 \leq i \leq n+1$ for each edge $v_i v_j$ GCD $\omega(v_1, v_i) \neq 1$. Thus, $\chi_{pg}(B_{n,n}) \geq n+3$. Therefore, $\chi_{pg}(B_{n,n}) = n+3$ when n is even.

Case 3: $n \equiv 1 \bmod 6$ and $n \neq 1$

Let $V(B_{n,n}) = \{ u_i : 1 \leq i \leq n+1\} \cup \{ v_i : 1 \leq i \leq n+1\}$. Define a proper vertex coloring $\omega : V(B_{n,n}) \rightarrow \{c_1, c_2, \ldots\ldots c_{n+4}\}$. $\omega(u_1) = c_1$ and $\omega(v_1) = c_{n+4}$ $\omega(u_i) = c_i$, $\omega(v_i) = c_{i+2}$ $\forall\ 2 \leq i \leq n+1$. Such that GCD $\omega(u_1, u_i) = 1$, GCD $\omega(v_1, v_i) = 1$ and GCD $\omega(u_1, v_1) = 1$.

Define a proper edge coloring $\omega^* : E(B_{n,n}) \rightarrow \{ c_1, c_2, c_3 \ldots\ldots c_{n+3}\}$

$\omega*(u_1 v_1) = c_1 - c_{n+4} = c_{n+3}$, $\omega*(u_1 u_i) = c_1 - c_i = c_{i-1}$, $\omega*(v_1 v_i) = c_{n+4} - c_{i+2} = c_{n+2-i}$

Hence, adjacent vertices and edges receive distinct colors. We proved that $\chi_{pg}(B_{n,n}) \leq n+4$. To prove $\chi_{pg}(B_{n,n}) \geq n+4$, let us assume that $\chi_{pg}(B_{n,n}) < n+4$ say $n+3$. We define proper vertex coloring of $B_{n,n}$ is $\omega(u_1) = c_1$, $\omega(u_i) = c_i$, $\omega(v_i) = c_i$ $\forall\ 2 \leq i \leq n+1$ and $\omega(v_1) = c_{n+3}$. Since v_1 is adjacent to v_i $\forall\ 2 \leq i \leq n+1$ for each edge $v_i v_j$ GCD $\omega(v_1, v_i) \neq 1$. Thus, $\chi_{pg}(B_{n,n}) \geq n+4$. Therefore, $\chi_{pg}(B_{n,n}) = n+4$ when $n \equiv 1 \bmod 6$ and $n \neq 1$.

Case 4: $n \equiv 0 \bmod 6$

Let $V(B_{n,n}) = \{ u_i : 1 \leq i \leq n+1\} \cup \{ v_i : 1 \leq i \leq n+1\}$. Define a proper vertex coloring $\omega : V(B_{n,n}) \rightarrow \{c_1, c_2, c_3 \ldots\ldots c_{n+5}\}$. $\omega(u_1) = c_1$ and $\omega(v_1) = c_{n+5}$ and $\omega(u_i) = c_i$, $\omega(v_i) = c_{i+3}$ $\forall\ 2 \leq i \leq n+1$. Such that GCD $\omega(u_1, u_i) = 1$, GCD $\omega(v_1, v_i) = 1$ and GCD $\omega(u_1, v_1) = 1$.

Define a proper edge coloring $\omega^* : E(B_{n,n}) \rightarrow \{ c_1, c_2, c_3 \ldots\ldots c_{n+4}\}$

$\omega*(u_1 v_1) = c_1 - c_{n+5} = c_{n+4}$, $\omega*(u_1 u_i) = c_1 - c_i = c_{i-1}$, $\omega*(v_1 v_i) = c_{n+5} - c_{i+3} = c_{n+2-i}$

Hence, adjacent vertices and edges receive distinct colors. We proved that $\chi_{pg}(B_{n,n}) \leq n+5$. To prove $\chi_{pg}(B_{n,n}) \geq n+5$, let us assume that $\chi_{pg}(B_{n,n}) < n+5$ say $n+4$. We define proper vertex coloring of $B_{n,n}$ is $\omega(u_1) = c_1$, $\omega(u_i) = c_i$, $\omega(v_i) = c_i$ $\forall\ 2 \le i \le n+1$ and $\omega(v_1) = c_{n+4}$. Since v_1 is adjacent to v_i $\forall\ 2 \le i \le n+1$ for each edge v_iv_j GCD $\omega(v_1, v_i) \neq 1$. Thus, $\chi_{pg}(B_{n,n}) \geq n + 5$. Therefore, $\chi_{pg}(B_{n,n}) = n + 5$ when $n \equiv 0 \bmod 6$.

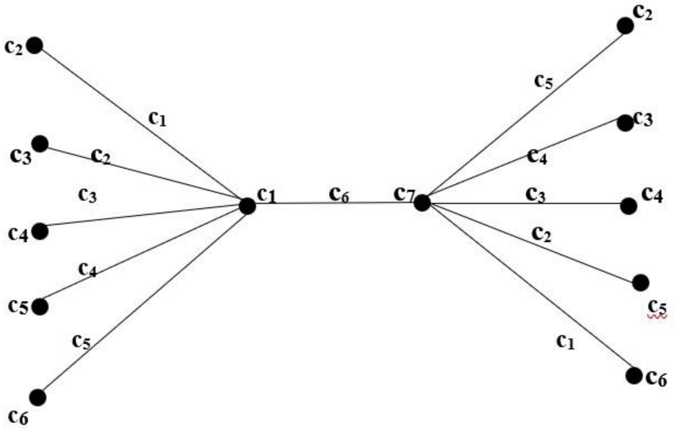

Figure 6 Analytical Evaluation of the $B_{5,5}$

4. Conclusion

This paper demonstrates that several graph classes admits prime graceful coloring. Prime graceful coloring offer a unique structure where vertex colors are relatively prime and induce a proper edge coloring which potentially leads to application in areas yet to be explored. Further research can investigate prime graceful coloring in more complex graph families and explore their potential uses in areas where efficient and unique coloring schemes are crucial.

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