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Some Product in Bipolar Valued Multi I-Fuzzy Subrings of a Ring

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Abstract:

The cited sources help to construct this paper. Here, theorems on products in bipolar valued multi-I-fuzzy subrings of rings are presented together with their attributes, which are stated and demonstrated.

Keywords: Interval-valued fuzzy subset, bipolar valued fuzzy subset, bipolar valued multi fuzzy subset, bipolar valued multi I-fuzzy subset, bipolar valued multi fuzzy subring, bipolar valued multi I-fuzzy subring, product, and strongest relation.

Introduction

The concept of a fuzzy subset of a set was first suggested by Zadeh [17] in 1965. Fuzzy sets are a helpful mathematical structure that can be used to describe a group of objects whose boundaries are not clearly defined. Since then, there have been many generalizations of this basic idea, including intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets, etc. It has also become a burgeoning field of study in other disciplines. 1, 1] are called bipolar-valued fuzzy sets. Intuitionistic fuzzy sets and bipolar-valued fuzzy sets have a similar appearance. They differ from one another, nevertheless [9, 10]. Azriel Rosenfeld introduced the fuzzy group [4]. Following that, Anthony J. M. and H. Sherwood proposed fuzzy groups redefined[2], and Chitra V. and K. Arjunan extended Q-fuzzy principles to nearring[6]. T.V. Ramakrishnan and Sabu Sebastian introduced multi fuzzy sets[12]. The concept of bipolar-valued fuzzy sets was suggested by Lee [9]. Fuzzy sets that have their membership degree range expanded from [0, 1] to [−1, 1]. Following that, Anitha M.S et al. [1] introduced bipolar-valued fuzzy subgroups of a group, while Arsham Borum and Saeid [3] introduced bipolar-valued fuzzy BCK/BCI-algebras. Balasubramanian introduced properties of Bipolar interval-valued fuzzy subgroups of a group and associates [5]. Kyoung Ja Lee introduced bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras[8]. Murugalingam.K and K. Arjunan[11] presented a study on interval-valued fuzzy subsemirings of a semiring, while Shanmugapriya.M.M & K. Arjunan[13] presented the (Q, L)-Fuzzy subnearrings of a nearing. Somasundra Moorthy's work, "A study on interval valued fuzzy, anti-fuzzy, intuitionistic fuzzy subrings of a ring, [14], writing this work benefited from the thesis. The idea of product in the bipolar valued multi I-fuzzy subring of an is explored in this article.

1. Prelirrminaries.

Definition 1.1. [17] An interval-valued fuzzy subset Ƒ of the set Ω is a function Ƒ: Ω →D[0, 1]. Here D[0, 1] denotes the family of all closed subintervals of [0, 1].

Definition 1.2. [9] *The ordered structure* $\mathfrak{T} = \{(\mathfrak{z}, \mathfrak{T}^+(\mathfrak{z}), \mathfrak{T}^-(\mathfrak{z})) : \mathfrak{z} \in \mathbb{W}\}$ *is called* a bipolar valued *fuzzy subset*(𝔹𝕍𝔽𝕊) *of* $\mathbb{w}$, *where* $\mathfrak{T}^+ : \mathbb{w} \to [0,1]$ *is a positive* membership map and $\mathfrak{T}^- : \mathbb{w} \to [-1,0]$ is a negative membership map.

Example 1.3. Let Ω = {ϖ, ω, υ} be a set. Then φ = {⟨ϖ, 0.4, –0.7⟩, ⟨ω, 0.9, –0.3⟩, ⟨υ, 0.8, –0.03⟩} is a bipolar valued fuzzy subset of Ω.

Definition 1.4. [16] *The ordered structure* $\wp = \{(\mathfrak{z}, \wp_1^+(\mathfrak{z}), \wp_2^+(\mathfrak{z}), \dots, \wp_n^+(\mathfrak{z}), \wp_1^-(\mathfrak{z}), \wp_2^-(\mathfrak{z}), \dots, \wp_n^-(\mathfrak{z})) : \mathfrak{z} \in \mathcal{M}\}$ *is called* a bipolar-valued multi-fuzzy subset (𝔹𝕍𝕄𝔽𝕊)*of* $\mathcal{M}$*with order n, where* $\wp_i^+ : \mathcal{M} \to [0,1]$ *are positive* membership maps and $\wp_i^- : \mathcal{M} \to [-1,0]$ are negative membership maps, where i = 1, 2, …, n.

Example 1.5. Let Ω = {ϖ, ω, υ} be a set. Then φ = {⟨ϖ, 0.4, 0.5, 0.2, –0.7, –0.4, –0.1⟩, ⟨ω, 0.9, 0.5, 0.8, –0.3, –0.2, –0.8⟩, ⟨υ, 0.8, 0.1, 0.4, –0.4, –0.3, –0.6⟩} is a bipolar valued multi fuzzy subset of Ω with order 3.

Definition 1.6. [15] *The ordered structure* $\wp = \{(\mathfrak{z}, \wp_1^+(\mathfrak{z}), \wp_2^+(\mathfrak{z}), \dots, \wp_n^+(\mathfrak{z}), \wp_1^-(\mathfrak{z}), \wp_2^-(\mathfrak{z}), \dots, \wp_n^-(\mathfrak{z})) : \mathfrak{z} \in \mathcal{M}\}$ *is called* a bipolar-valued multi-I-fuzzy subset (𝔹𝕍𝕄𝕀𝔽𝕊)*of* $\mathcal{M}$*with order n, where* $\wp_i^+ : \mathcal{M} \to D[0,1]$ *are positive* membership maps and $\wp_i^- : \mathcal{M} \to D[-1,0]$ are negative membership maps, where i = 1, 2, …, n. Here D[0, 1] denotes the family of all closed subintervals of [0, 1] and $D[-1,0]$ denotes the family of all closed subintervals of [−1, 0]. Note that [0] = [0, 0], [1] = [1, 1] and [–1] = [–1, –1].

Example 1.7. Let Ω = {ϖ, ω, υ} be a set. Then φ = {⟨ϖ, [0.4, 0.6], [0.5, 0.7], [0.2, 0.6], [–0.7, –0.4], [–0.4, –0.1], [–0.3, –0.1]⟩, ⟨ω, [0.5, 0.9], [0.5, 0.7], [0.8, 0.9], [–0.3, –0.2], [–0.2, –0.1], [–0.8, –0.5]⟩, ⟨υ, [0.8, 0.9], [0.1, 0.6], [0.4, 0.7], [–0.4, –0.2], [–0.3, –0.1], [–0.6, –0.2]⟩} is a bipolar valued multi I-fuzzy subset of Ω with order 3.

Definition 1.8. [15] *A* 𝔹𝕍𝕄𝕀𝔽𝕊 $\wp = \langle \wp_1^+, \wp_2^+, \dots, \wp_n^+, \wp_1^-, \wp_2^-, \dots, \wp_n^- \rangle$ of a ring $\mathfrak{Y}$ *is said* to be a bipolar valued multi I − fuzzy subring of $\mathfrak{Y}$ (𝔹𝕍𝕄𝕀𝔽𝕊ℝ) *if* $\wp$ *has the following condition*,

(i) $\wp_i^+(\mathfrak{y} - \mathfrak{w}) \geq rmin\{\wp_i^+(\mathfrak{y}), \wp_i^+(\mathfrak{w})\}$,

(ii) $\wp_i^+(\mathfrak{y}\mathfrak{w}) \geq rmin\{\wp_i^+(\mathfrak{y}), \wp_i^+(\mathfrak{w})\}$,

(iii) $\wp_i^-(\mathfrak{y} - \mathfrak{w}) \leq rmax\{\wp_i^-(\mathfrak{y}), \wp_i^-(\mathfrak{w})\}$,

(iv) $\wp_i^-(\mathfrak{y}\mathfrak{w}) \leq rmax\{\wp_i^-(\mathfrak{y}), \wp_i^-(\mathfrak{w})\}$, *for all* $\mathfrak{y}, \mathfrak{w} \in \mathfrak{Y}$.

Example 1.9. Let $\mathbb{z}_3 = \{0, 1, 2\}$ *be a ring with* $\oplus_3$ *and* $\otimes_3$. Then $\wp$ is defined as $\wp$ = {(0, [0.7, 0.8], [0.8, 0.9], [0.9, 1.0], [– 0.9, –0.8], [– 0.8, –0.7], [– 0.7, –0.6]), (1, [0.5, 0.6], [0.6, 0.7], [0.7, 0.8], [– 0.6, –0.5], [– 0.5, –0.4], [– 0.4, –0.3]), (2, [0.5, 0.6], [0.6, 0.7], [0.7, 0.8], [–0.6, –0.5], [– 0.5, –0.4], [– 0.4, –0.3])}, is a 𝔹𝕍𝕄𝕀𝔽𝕊ℝ of $\mathbb{z}_3$.

Definition 1.10. *Let* $\mathfrak{K} = \langle \mathfrak{K}_1^+, \mathfrak{K}_2^+, \dots, \mathfrak{K}_n^+, \mathfrak{K}_1^-, \mathfrak{K}_2^-, \dots, \mathfrak{K}_n^- \rangle$ *be* 𝔹𝕍𝕄𝕀𝔽𝕊 of the set $\mathfrak{L}_1$, the strongest 𝔹𝕍𝕄𝕀𝔽 *relation on* $\mathfrak{L}_1$, that is a 𝔹𝕍𝕄𝕀𝔽 *relation* on $\mathfrak{K}$ *is* $\wp$ = {⟨(ϱ, ζ), $\wp_1^+$(ϱ, ζ), $\wp_2^+$(ϱ, ζ),…, $\wp_n^+$(ϱ, ζ), $\wp_1^-$(ϱ, ζ), $\wp_2^-$(ϱ, ζ), …, $\wp_n^-$(ϱ, ζ)⟩ / for all ϱ, ζ ∈ $\mathfrak{L}_1$}, where $\wp_i^+$(ϱ, ζ) = rmin{$\mathfrak{K}_i^+$(ϱ), $\mathfrak{K}_i^+$(ζ)} and $\wp_i^-$(ϱ, ζ) = rmax{$\mathfrak{K}_i^-$(ϱ), $\mathfrak{K}_i^-$(ζ)}, for all ϱ, ζ ∈ $\mathfrak{L}_1$, i = 1, 2, …, n.

Definition 1.11. *Let* $\mathfrak{K} = \langle \mathfrak{K}_1^+, \mathfrak{K}_2^+, \dots, \mathfrak{K}_n^+, \mathfrak{K}_1^-, \mathfrak{K}_2^-, \dots, \mathfrak{K}_n^- \rangle$ and $\wp = \langle \wp_1^+, \wp_2^+, \dots, \wp_n^+, \wp_1^-, \wp_2^-, \dots, \wp_n^- \rangle$ *be* 𝔹𝕍𝕄𝕀𝔽𝕊s of the sets $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively. The product of $\mathfrak{K}$ and $\wp$, denoted by $\mathfrak{K} \times \wp$, is defined as $\mathfrak{K} \times \wp$ = {⟨(ϱ, ζ), $(\mathfrak{K}_1 \times \wp_1)^+$(ϱ, ζ), $(\mathfrak{K}_2 \times \wp_2)^+$(ϱ, ζ), …, $(\mathfrak{K}_n \times \wp_n)^+$(ϱ, ζ), $(\mathfrak{K}_1 \times \wp_1)^-$(ϱ, ζ),

$(\mathfrak{K}_2\times\wp_2)^-(\varrho, \zeta)$, …, $(\mathfrak{K}_n\times\wp_n)^-(\varrho, \zeta)\rangle$ / for all $(\varrho, \zeta)\in\mathfrak{L}_1 \times \mathfrak{L}_2\}$, where $(\mathfrak{K}_i\times\wp_i)^+(\varrho, \zeta) = \text{rmin}\{\mathfrak{K}_i^+(\varrho), \wp_i^+(\zeta)\}$ and $(\mathfrak{K}_i\times\wp_i)^-(\varrho, \zeta) = \text{rmax}\{\mathfrak{K}_i^-(\varrho), \wp_i^-(\zeta)\}$, i = 1, 2, …, n.

2. Some Theorems.

Theorem 2.1. $If\ \mathfrak{K} = \langle \mathfrak{K}_1^+, \mathfrak{K}_2^+, \dots, \mathfrak{K}_n^+, \mathfrak{K}_1^-, \mathfrak{K}_2^-, \dots, \mathfrak{K}_n^-\rangle$ and $\wp = \langle \wp_1^+, \wp_2^+, \dots, \wp_n^+, \wp_1^-, \wp_2^-, \dots, \wp_n^-\rangle\ are$ $\mathbb{BVMIFSR}$s of the rings $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively, then $\mathfrak{K} \times \wp$ is a $\mathbb{BVMIFSR}$ of the ring $\mathfrak{L}_1 \times \mathfrak{L}_2$.

Proof. Let ϱ, υ be in $\mathfrak{L}_1$ and ζ, ξ be in $\mathfrak{L}_2$. Then (ϱ, ζ) and (υ, ξ) are in $\mathfrak{L}_1\times\mathfrak{L}_2$.

For all i, i = 1, 2, …, n,

$(\mathfrak{K}_i\times\wp_i)^+[(\varrho, \zeta)-(\upsilon, \xi)] = (\mathfrak{K}_i\times\wp_i)^+(\varrho-\upsilon, \zeta-\xi) = \text{rmin}\{\mathfrak{K}_i^+(\varrho-\upsilon), \wp_i^+(\zeta-\xi)\} \geq \text{rmin}\ \{\text{rmin}\{\mathfrak{K}_i^+(\varrho), \mathfrak{K}_i^+(\upsilon)\}, \text{rmin}\{\wp_i^+(\zeta), \wp_i^+(\xi)\}\} = \text{rmin}\{\text{rmin}\{\mathfrak{K}_i^+(\varrho), \wp_i^+(\zeta)\}, \text{rmin}\{\mathfrak{K}_i^+(\upsilon), \wp_i^+(\xi)\}\} = \text{rmin}\{(\mathfrak{K}_i\times\wp_i)^+(\varrho, \zeta), (\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi)\}$, for all (ϱ, ζ), (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

And $(\mathfrak{K}_i\times\wp_i)^+[(\varrho, \zeta)(\upsilon, \xi)] = (\mathfrak{K}_i\times\wp_i)^+(\varrho\upsilon, \zeta\xi) = \text{rmin}\{\mathfrak{K}_i^+(\varrho\upsilon), \wp_i^+(\zeta\xi)\} \geq \text{rmin}\ \{\text{rmin}\{\mathfrak{K}_i^+(\varrho), \mathfrak{K}_i^+(\upsilon)\}, \text{rmin}\{\wp_i^+(\zeta), \wp_i^+(\xi)\}\} = \text{rmin}\{\text{rmin}\{\mathfrak{K}_i^+(\varrho), \wp_i^+(\zeta)\}, \text{rmin}\{\mathfrak{K}_i^+(\upsilon), \wp_i^+(\xi)\}\} = \text{rmin}\ \{(\mathfrak{K}_i\times\wp_i)^+(\varrho, \zeta), (\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi)\}$, for all (ϱ, ζ), (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

Also $(\mathfrak{K}_i\times\wp_i)^-[(\varrho, \zeta)-(\upsilon, \xi)] = (\mathfrak{K}_i\times\wp_i)^-(\varrho-\upsilon, \zeta-\xi) = \text{rmax}\{\mathfrak{K}_i^-(\varrho-\upsilon), \wp_i^-(\zeta-\xi)\} \leq \text{rmax}\ \{\text{rmax}\{\mathfrak{K}_i^-(\varrho), \mathfrak{K}_i^-(\upsilon)\}, \text{rmax}\{\wp_i^-(\zeta), \wp_i^-(\xi)\}\} = \text{rmax}\{\text{rmax}\{\mathfrak{K}_i^-(\varrho), \wp_i^-(\zeta)\}, \text{rmax}\{\mathfrak{K}_i^-(\upsilon), \wp_i^-(\xi)\}\} = \text{rmax}\{(\mathfrak{K}_i\times\wp_i)^-(\varrho, \zeta), (\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi)\}$, for all (ϱ, ζ), (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

And $(\mathfrak{K}_i\times\wp_i)^-[(\varrho, \zeta)(\upsilon, \xi)] = (\mathfrak{K}_i\times\wp_i)^-(\varrho\upsilon, \zeta\xi) = \text{rmax}\{\mathfrak{K}_i^-(\varrho\upsilon), \wp_i^-(\zeta\xi)\} \leq \text{rmax}\ \{\text{rmax}\{\mathfrak{K}_i^-(\varrho), \mathfrak{K}_i^-(\upsilon)\}, \text{rmax}\{\wp_i^-(\zeta), \wp_i^-(\xi)\}\} = \text{rmax}\{\max\{\mathfrak{K}_i^-(\varrho), \wp_i^-(\zeta)\}, \text{rmax}\{\mathfrak{K}_i^-(\upsilon), \wp_i^-(\xi)\}\} = \text{rmax}\{(\mathfrak{K}_i\times\wp_i)^-(\varrho, \zeta), (\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi)\}$, for all (ϱ, ζ), (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$. Hence $\mathfrak{K}\times\wp$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{L}_1\times\mathfrak{L}_2$**.**

Theorem 2.2. $If\ \wp_1, \wp_2, \dots, \wp_m\ are$ $\mathbb{BVMIFSR}$s of the rings $\mathfrak{L}_1, \mathfrak{L}_2, \dots, \mathfrak{L}_m$ respectively, then $\wp_1 \times \wp_2 \times \dots \times \wp_m$ is a $\mathbb{BVMIFSR}$ of the ring $\mathfrak{L}_1 \times \mathfrak{L}_2 \times \dots \times \mathfrak{L}_m$.

Proof. $From\ the\ theorem$ 2.1, $the\ proof\ is\ trivial.$

Theorem 2.3. $If\ \mathfrak{K}\times\wp$ is a $\mathbb{BVMIFSR}$ with degree n of a ring $\mathfrak{L}_1\times\mathfrak{L}_2$, then for all i, i = 1, 2, …, n, $(\mathfrak{K}_i\times\wp_i)^+(-\upsilon, \xi^{-1}) = (\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi)$, $(\mathfrak{K}_i\times\wp_i)^-(-\upsilon, \xi^{-1}) = (\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi)$, $(\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi) \leq (\mathfrak{K}_i\times\wp_i)^+(0, 1)$ and $(\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi) \geq (\mathfrak{K}_i\times\wp_i)^-(0, 1)$, for all (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$, where (0, 1) is the identity element of $\mathfrak{L}_1\times\mathfrak{L}_2$.

Proof. Let (υ, ξ) be in $\mathfrak{L}_1\times\mathfrak{L}_2$ and (0, 1) be the identity element of $\mathfrak{L}_1\times\mathfrak{L}_2$.

For all i, i = 1, 2, …, n,

$(\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi) = (\mathfrak{K}_i\times\wp_i)^+(-(-\upsilon), (\xi^{-1})^{-1}) \geq (\mathfrak{K}_i\times\wp_i)^+(-\upsilon, \xi^{-1}) \geq (\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi)$. Thus $(\mathfrak{K}_i\times\wp_i)^+(-\upsilon, \xi^{-1}) = (\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi)$, for all (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

And $(\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi) = (\mathfrak{K}_i\times\wp_i)^-(-(-\upsilon), (\xi^{-1})^{-1}) \leq (\mathfrak{K}_i\times\wp_i)^-(-\upsilon, \xi^{-1}) \leq (\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi)$. Thus $(\mathfrak{K}_i\times\wp_i)^-(-\upsilon, \xi^{-1}) = (\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi)$, for all (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

Also $(\mathfrak{K}_i\times\wp_i)^+(0, 1) = (\mathfrak{K}_i\times\wp_i)^+(\upsilon - \upsilon), \xi\xi^{-1}) = \text{rmin}\ \{\mathfrak{K}_i^+(\upsilon - \upsilon), \wp_i^+(\xi\xi^{-1})\} \geq \text{rmin}\ \{\text{rmin}\{\mathfrak{K}_i^+(\upsilon), \mathfrak{K}_i^+(\upsilon)\}, \text{rmin}\ \{\wp_i^+(\xi), \wp_i^+(\xi)\}\} = \text{rmin}\ \{\mathfrak{K}_i^+(\upsilon), \wp_i^+(\xi)\} = (\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi)$. Thus $(\mathfrak{K}_i\times\wp_i)^+(\upsilon, \xi) \leq (\mathfrak{K}_i\times\wp_i)^+(0, 1)$, for all (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

And $(\mathfrak{K}_i\times\wp_i)^-(0, 1) = (\mathfrak{K}_i\times\wp_i)^-(\upsilon - \upsilon), \xi\xi^{-1}) = \text{rmax}\{\mathfrak{K}_i^-(\upsilon - \upsilon), \wp_i^-(\xi\xi^{-1})\} \leq \text{rmax}\ \{\text{rmax}\{\mathfrak{K}_i^-(\upsilon), \mathfrak{K}_i^-(\upsilon)\}, \text{rmax}\{\wp_i^-(\xi), \wp_i^-(\xi)\}\} = \text{rmax}\{\mathfrak{K}_i^-(\upsilon), \wp_i^-(\xi)\} = (\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi)$. Thus $(\mathfrak{K}_i\times\wp_i)^-(\upsilon, \xi) \geq (\mathfrak{K}_i\times\wp_i)^-(0, 1)$, for all (υ, ξ) in $\mathfrak{L}_1\times\mathfrak{L}_2$.

Theorem 2.4. Let $\mathfrak{Y}$ and $\mathfrak{W}$ $be\ any\ two$ $\mathbb{BVMIFS}$s of the rings $\mathfrak{H}_1$ and $\mathfrak{H}_2$

respectively. If $\mathfrak{Y} \times \mathfrak{W}$ is a $\mathbb{BVMIFSR}$ of the ring $\mathfrak{H}_1 \times \mathfrak{H}_2$, then at least one of the following two statements must hold; (i) For all i = 1, 2, …, n, $\mathfrak{W}_i^+(\mathfrak{o}) \geq \mathfrak{Y}_i^+(\varrho), \mathfrak{W}_i^-(\mathfrak{o}) \leq \mathfrak{Y}_i^-(\varrho)$, for all $\varrho \in \mathfrak{H}_1$, (ii) $\mathfrak{W}_i^+(\zeta) \leq \mathfrak{Y}_i^+(\mathfrak{e}), \mathfrak{W}_i^-(\zeta) \geq \mathfrak{Y}_i^-(\mathfrak{e})$, for all $\zeta \in \mathfrak{H}_2$, where $\mathfrak{e}, \mathfrak{o}$ *are identity elements of* $\mathfrak{H}_1$ and $\mathfrak{H}_2$.

Proof. By contraposition, suppose that none of the statements (i) and (ii) holds. For $\varrho \in \mathfrak{H}_1$ and $\zeta \in \mathfrak{H}_2$ such that $\mathfrak{W}_i^+(\mathfrak{o}) < \mathfrak{Y}_i^+(\varrho), \mathfrak{W}_i^-(\mathfrak{o}) > \mathfrak{Y}_i^-(\varrho)$ and $\mathfrak{W}_i^+(\zeta) > \mathfrak{Y}_i^+(\mathfrak{e}), \mathfrak{W}_i^-(\zeta) < \mathfrak{Y}_i^-(\mathfrak{e})$. For all i = 1, 2, …, n,

$(\mathfrak{Y}_i\times\mathfrak{W}_i)^+(\varrho, \zeta)$ = rmin$\{\mathfrak{Y}_i^+(\varrho), \mathfrak{W}_i^+(\zeta)\}$> rmin$\{\mathfrak{Y}_i^+(\mathfrak{e}), \mathfrak{W}_i^+(\mathfrak{o})\}$ = $(\mathfrak{Y}_i\times\mathfrak{W}_i)^+(\mathfrak{e}, \mathfrak{o})$.

Also $(\mathfrak{Y}_i\times\mathfrak{W}_i)^+(\varrho, \zeta)$ = rmax$\{\mathfrak{Y}_i^+(\varrho), \mathfrak{W}_i^+(\zeta)\}$< rmax$\{\mathfrak{Y}_i^+(\mathfrak{e}), \mathfrak{W}_i^+(\mathfrak{o})\}$= $(\mathfrak{Y}_i\times\mathfrak{W}_i)^+(\mathfrak{e}, \mathfrak{o})$.

Thus $\mathfrak{Y} \times \mathfrak{W}$ is not a $\mathbb{BVMIFSR}$ of the ring $\mathfrak{H}_1 \times \mathfrak{H}_2$.

Hence either $\mathfrak{W}_i^+(\mathfrak{o}) \geq \mathfrak{Y}_i^+(\varrho), \mathfrak{W}_i^-(\mathfrak{o}) \leq \mathfrak{Y}_i^-(\varrho)$, for all $\varrho \in \mathfrak{H}_1$ or

$\mathfrak{W}_i^+(\zeta) \leq \mathfrak{Y}_i^+(\mathfrak{e}), \mathfrak{W}_i^-(\zeta) \geq \mathfrak{Y}_i^-(\mathfrak{e})$, for all $\zeta \in \mathfrak{H}_2$.

Theorem 2.5. *Let* $\mathfrak{P}$ and $\mathfrak{W}$ *be any two* $\mathbb{BVMIFS}$s of the rings $\mathfrak{O}_1$ and $\mathfrak{O}_2$ respectively and $\mathfrak{P} \times \mathfrak{W}$ be a $\mathbb{BVMIFSR}$ of the ring $\mathfrak{O}_1 \times \mathfrak{O}_2$. Then the following are true;

(i) For all i = 1, 2, …, n, if $\mathfrak{W}_i^+(\mathfrak{o}) \geq \mathfrak{P}_i^+(\varrho), \mathfrak{W}_i^-(\mathfrak{o}) \leq \mathfrak{P}_i^-(\varrho)$, for all $\varrho \in \mathfrak{O}_1$, then $\mathfrak{P}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{O}_1$; (ii) if $\mathfrak{W}_i^+(\zeta) \leq \mathfrak{P}_i^+(\mathfrak{e}), \mathfrak{W}_i^-(\zeta) \geq \mathfrak{P}_i^-(\mathfrak{e})$, for all $\zeta \in \mathfrak{O}_2$, then $\mathfrak{W}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{O}_2$; where $\mathfrak{e}, \mathfrak{o}$ *are identity elements of* $\mathfrak{O}_1$ and $\mathfrak{O}_2$.

Proof. Let ϱ, υ be in $\mathfrak{O}_1$. Then $(\varrho, \mathfrak{o})$ and $(\upsilon, \mathfrak{o})$ are in $\mathfrak{O}_1\times\mathfrak{O}_2$. For all i, i = 1, 2, …, n,

(i) $\mathfrak{P}_i^+(\varrho-\upsilon)$ = rmin$\{\mathfrak{P}_i^+(\varrho-\upsilon), \mathfrak{W}_i^+(\mathfrak{o}-\mathfrak{o})\}$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\varrho-\upsilon, \mathfrak{o}-\mathfrak{o})$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^+[(\varrho, \mathfrak{o})-(\upsilon, \mathfrak{o})]$ ≥ rmin$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\varrho, \mathfrak{o}), (\mathfrak{P}_i\times\mathfrak{W}_i)^+(\upsilon, \mathfrak{o})\}$ = rmin{rmin$\{\mathfrak{P}_i^+(\varrho), \mathfrak{W}_i^+(\mathfrak{o})\}$, rmin$\{\mathfrak{P}_i^+(\upsilon), \mathfrak{W}_i^+(\mathfrak{o})\}$} = rmin$\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_1$.

And $\mathfrak{P}_i^+(\varrho\upsilon)$ = rmin$\{\mathfrak{P}_i^+(\varrho\upsilon), \mathfrak{W}_i^+(\mathfrak{o}\mathfrak{o})\}$= $(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\varrho\upsilon, \mathfrak{o}\mathfrak{o})$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^+[(\varrho, \mathfrak{o})(\upsilon, \mathfrak{o})]$ ≥ rmin$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\varrho, \mathfrak{o}), (\mathfrak{P}_i\times\mathfrak{W}_i)^+(\upsilon, \mathfrak{o})\}$ = rmin{rmin$\{\mathfrak{P}_i^+(\varrho), \mathfrak{W}_i^+(\mathfrak{o})\}$, rmin$\{\mathfrak{P}_i^+(\upsilon), \mathfrak{W}_i^+(\mathfrak{o})\}$} = rmin$\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_1$.

Also $\mathfrak{P}_i^-(\varrho-\upsilon)$ = rmax$\{\mathfrak{P}_i^-(\varrho-\upsilon), \mathfrak{W}_i^-(\mathfrak{o}-\mathfrak{o})\}$= $(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\varrho-\upsilon, \mathfrak{o}-\mathfrak{o})$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^-[(\varrho, \mathfrak{o})-(\upsilon, \mathfrak{o})]$ ≤ rmax$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\varrho, \mathfrak{o}), (\mathfrak{P}_i\times\mathfrak{W}_i)^-(\upsilon, \mathfrak{o})\}$ = rmax{rmax$\{\mathfrak{P}_i^-(\varrho), \mathfrak{W}_i^-(\mathfrak{o})\}$, rmax$\{\mathfrak{P}_i^-(\upsilon), \mathfrak{W}_i^-(\mathfrak{o})\}$} = rmax$\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_1$.

And $\mathfrak{P}_i^-(\varrho\upsilon)$ = rmax$\{\mathfrak{P}_i^-(\varrho\upsilon), \mathfrak{W}_i^-(\mathfrak{o}\mathfrak{o})\}$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\varrho\upsilon, \mathfrak{o}\mathfrak{o})$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^-[(\varrho, \mathfrak{o})(\upsilon, \mathfrak{o})]$ ≤ rmax$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\varrho, \mathfrak{o}), (\mathfrak{P}_i\times\mathfrak{W}_i)^-(\upsilon, \mathfrak{o})\}$ = rmax{rmax$\{\mathfrak{P}_i^-(\varrho), \mathfrak{W}_i^-(\mathfrak{o})\}$, rmax$\{\mathfrak{P}_i^-(\upsilon), \mathfrak{W}_i^-(\mathfrak{o})\}$} = rmax$\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_1$.

Hence $\mathfrak{P}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{O}_1$.

(ii) Let ϱ, υ be in $\mathfrak{O}_2$. Then $(\mathfrak{e}, \varrho)$ and $(\mathfrak{e}, \upsilon)$ are in $\mathfrak{O}_1\times\mathfrak{O}_2$. For all i, i = 1, 2, …, n,

$\mathfrak{W}_i^+(\varrho-\upsilon)$ = rmin$\{\mathfrak{P}_i^+(\mathfrak{e}-\mathfrak{e}), \mathfrak{W}_i^+(\varrho-\upsilon)\}$= $(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\mathfrak{e}-\mathfrak{e}, \varrho-\upsilon)$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^+[(\mathfrak{e}, \varrho)-(\mathfrak{e}, \upsilon)]$ ≥ rmin$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\mathfrak{e}, \varrho), (\mathfrak{P}_i\times\mathfrak{W}_i)^+(\mathfrak{e}, \upsilon)\}$ = rmin{rmin$\{\mathfrak{P}_i^+(\mathfrak{e}), \mathfrak{W}_i^+(\varrho)\}$, rmin$\{\mathfrak{P}_i^+(\mathfrak{e}), \mathfrak{W}_i^+(\upsilon)\}$} = rmin$\{\mathfrak{W}_i^+(\varrho), \mathfrak{W}_i^+(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_2$.

And $\mathfrak{W}_i^+(\varrho\upsilon)$ = rmin$\{\mathfrak{P}_i^+(\mathfrak{e}\mathfrak{e}), \mathfrak{W}_i^+(\varrho\upsilon)\}$= $(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\mathfrak{e}\mathfrak{e}, \varrho\upsilon)$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^+[(\mathfrak{e}, \varrho)(\mathfrak{e}, \upsilon)]$ ≥ rmin$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^+(\mathfrak{e}, \varrho), (\mathfrak{P}_i\times\mathfrak{W}_i)^+(\mathfrak{e}, \upsilon)\}$ = rmin{rmin$\{\mathfrak{P}_i^+(\mathfrak{e}), \mathfrak{W}_i^+(\varrho)\}$, rmin$\{\mathfrak{P}_i^+(\mathfrak{e}), \mathfrak{W}_i^+(\upsilon)\}$} = rmin$\{\mathfrak{W}_i^+(\varrho), \mathfrak{W}_i^+(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_2$.

Also $\mathfrak{W}_i^-(\varrho-\upsilon)$ = rmax$\{\mathfrak{P}_i^-(\mathfrak{e}-\mathfrak{e}), \mathfrak{W}_i^-(\varrho-\upsilon)\}$= $(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\mathfrak{e}-\mathfrak{e}, \varrho-\upsilon)$ = $(\mathfrak{P}_i\times\mathfrak{W}_i)^-[(\mathfrak{e}, \varrho)-(\mathfrak{e}, \upsilon)]$ ≤ rmax$\{(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\mathfrak{e}, \varrho), (\mathfrak{P}_i\times\mathfrak{W}_i)^-(\mathfrak{e}, \upsilon)\}$ = rmax{rmax$\{\mathfrak{P}_i^-(\mathfrak{e}), \mathfrak{W}_i^-(\varrho)\}$, rmax$\{\mathfrak{P}_i^-(\mathfrak{e}), \mathfrak{W}_i^-(\upsilon)\}$} = rmax$\{\mathfrak{W}_i^-(\varrho), \mathfrak{W}_i^-(\upsilon)\}$, *for all* ϱ, υ in $\mathfrak{O}_2$.

And $\mathfrak{W}_i^-(\varrho\upsilon)$ = rmax{$\mathfrak{P}_i^-$(ee), $\mathfrak{W}_i^-(\varrho\upsilon)$}= $(\mathfrak{P}_i\times\mathfrak{W}_i)^-$(ee, $\varrho\upsilon$) = $(\mathfrak{P}_i\times\mathfrak{W}_i)^-[(\mathfrak{e},\ \varrho)(\mathfrak{e},\upsilon)]$ ≤ rmax{$(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\mathfrak{e},\ \varrho)$, $(\mathfrak{P}_i\times\mathfrak{W}_i)^-(\mathfrak{e},\upsilon)$ } = rmax{rmax{$\mathfrak{P}_i^-(\mathfrak{e})$, $\mathfrak{W}_i^-(\varrho)$}, rmax{$\mathfrak{P}_i^-(\mathfrak{e})$, $\mathfrak{W}_i^-(\upsilon)$}} = rmax{$\mathfrak{W}_i^-(\varrho)$, $\mathfrak{W}_i^-(\upsilon)$}, $for\ all\ \varrho,\upsilon$ in $\mathfrak{O}_2$.

Hence $\mathfrak{W}$ is a 𝔹𝕍𝕄𝕀𝔽𝕊ℝ of $\mathfrak{O}_2$.

Theorem 2.6. *Let* $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n$ *be the* 𝔹𝕍𝕄𝕀𝔽𝕊s of the rings $\mathfrak{O}_1, \mathfrak{O}_2, \ldots, \mathfrak{O}_n$ respectively and $\mathfrak{P}_1 \times \mathfrak{P}_2 \times \ldots \times \mathfrak{P}_n$ be a 𝔹𝕍𝕄𝕀𝔽𝕊ℝ of the ring $\mathfrak{O}_1 \times \mathfrak{O}_2 \times \ldots \times \mathfrak{O}_n$. Then the following are true;

For all i, j, k = 1, 2, …, n, if $\mathfrak{P}_{kj}^+(\mathfrak{o}) \geq \mathfrak{P}_{ij}^+(\varrho), \mathfrak{P}_{kj}^-(\mathfrak{o}) \leq \mathfrak{P}_{ij}^-(\varrho)$, for all $\varrho\in\mathfrak{O}_i$, then $\mathfrak{P}_i$ is a 𝔹𝕍𝕄𝕀𝔽𝕊ℝ of $\mathfrak{O}_i$. where $\mathfrak{e},\mathfrak{o}$ *are identity elements of* $\mathfrak{O}_i$ and $\mathfrak{O}_k$.

Proof. The proof follows from the theorem 2.5.

Theorem 2.7. *If* $\mathfrak{P}\times\mathfrak{W}$ *is a* 𝔹𝕍𝕄𝕀𝔽𝕊ℝ of the ring $\mathfrak{K}_1 \times \mathfrak{K}_2$, then $\mathfrak{F} = \{(\mathfrak{h},\mathfrak{z}) \in \mathfrak{K}_1 \times \mathfrak{K}_2 : (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h},\mathfrak{z}) = (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d})\ and\ (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h},\mathfrak{z}) = (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}),$ for all $i = 1,2,\ldots,n\}$ is either empty or a subring $\mathfrak{K}_1 \times \mathfrak{K}_2$, where $\mathfrak{o},\mathfrak{d}$ are first operation identity elements of $\mathfrak{K}_1$ and $\mathfrak{K}_2$.

Proof. $If\ any\ elements\ not\ satisfies\ the\ condition, then\ \mathfrak{F}\ is\ empty.$

Let $(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}$. For all $i = 1,2,\ldots,n$, then

$(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] \geq$ rmin{$(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_2,\mathfrak{z}_2)$}

$= rmin\,\{(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d}), (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d})\} = (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d}).$

Thus $(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] = (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d}), for\ all\ (\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}.$

And $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] \leq$ rmax{$(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_2,\mathfrak{z}_2)$}

$= rmax\,\{(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}), (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d})\} = (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}).$

Thus $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] = (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}), for\ all\ (\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}.$

Therefore $(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}$.

$Also$ $(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1,\mathfrak{z}_1)(\mathfrak{h}_2,\mathfrak{z}_2)] \geq$ rmin {$(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_2,\mathfrak{z}_2)$}

$= rmin\,\{(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d}), (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d})\} = (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d}).$

Thus $(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1,\mathfrak{z}_1)(\mathfrak{h}_2,\mathfrak{z}_2)] = (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{o},\mathfrak{d}), for\ all\ (\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}.$

And $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)[(\mathfrak{h}_1,\mathfrak{z}_1)(\mathfrak{h}_2,\mathfrak{z}_2)] \leq$ rmax{$(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_2,\mathfrak{z}_2)$}

$= rmax\,\{(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}), (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d})\} = (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}).$

Thus $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)\ [(\mathfrak{h}_1,\mathfrak{z}_1)(\mathfrak{h}_2,\mathfrak{z}_2)] = (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{o},\mathfrak{d}), for\ all\ (\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}.$

Therefore $(\mathfrak{h}_1,\mathfrak{z}_1)(\mathfrak{h}_2,\mathfrak{z}_2) \in \mathfrak{F}$. $Hence\ \mathfrak{F}\ is\ a\ subring\ of\ \mathfrak{K}_1 \times \mathfrak{K}_2.$

Theorem 2.8. *If* $\mathfrak{P}\times\mathfrak{W}$ *is a* 𝔹𝕍𝕄𝕀𝔽𝕊ℝ of the ring $\mathfrak{G}_1 \times \mathfrak{G}_2$, then $\mathbb{Y} = \{(\mathfrak{h},\mathfrak{z}) \in \mathfrak{G}_1 \times \mathfrak{G}_2 : (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h},\mathfrak{z}) = [1]\ and\ (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h},\mathfrak{z}) = [-1],$ for all $i = 1,2,\ldots,n\}$ is either empty or a subring $\mathfrak{G}_1 \times \mathfrak{G}_2$.

Proof. $If\ any\ elements\ not\ satisfies\ the\ condition, then\ \mathbb{Y}\ is\ empty.$

Let $(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathbb{Y}$. For all $i = 1,2,\ldots,n$, then

$(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] \geq$ rmin{$(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_2,\mathfrak{z}_2)$}

$= rmin\,\{[1],[1]\} = [1].$

Thus $(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] = [1], for\ all\ (\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathbb{Y}.$

And $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] \leq$ rmax{$(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_2,\mathfrak{z}_2)$}

$= rmax\,\{[-1],[-1]\} = [-1].$

Thus $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)[(\mathfrak{h}_1,\mathfrak{z}_1) - (\mathfrak{h}_2,\mathfrak{z}_2)] = [-1], for\ all\ (\mathfrak{h}_1,\mathfrak{z}_1), (\mathfrak{h}_2,\mathfrak{z}_2) \in \mathbb{Y}.$

Therefore $(\mathfrak{h}_1, \mathfrak{z}_1) - (\mathfrak{h}_2, \mathfrak{z}_2) \in \mathbb{Y}$.
Also $(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1, \mathfrak{z}_1)(\mathfrak{h}_2, \mathfrak{z}_2)] \geq \text{rmin}\,\{(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_1, \mathfrak{z}_1), (\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)(\mathfrak{h}_2, \mathfrak{z}_2)\}$
$= rmin\,\{[1], [1]\} = [1]$.
Thus $(\mathfrak{P}_i^+ \times \mathfrak{W}_i^+)[(\mathfrak{h}_1, \mathfrak{z}_1)(\mathfrak{h}_2, \mathfrak{z}_2)] = [1], for\ all\ (\mathfrak{h}_1, \mathfrak{z}_1), (\mathfrak{h}_2, \mathfrak{z}_2) \in \mathbb{Y}$.
And $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)[(\mathfrak{h}_1, \mathfrak{z}_1)(\mathfrak{h}_2, \mathfrak{z}_2)] \leq \text{rmax}\{(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_1, \mathfrak{z}_1), (\mathfrak{P}_i^- \times \mathfrak{W}_i^-)(\mathfrak{h}_2, \mathfrak{z}_2)\}$
$= rmax\,\{[-1], [-1]\} = [-1]$.
Thus $(\mathfrak{P}_i^- \times \mathfrak{W}_i^-)\,[(\mathfrak{h}_1, \mathfrak{z}_1)(\mathfrak{h}_2, \mathfrak{z}_2)] = [-1], for\ all\ (\mathfrak{h}_1, \mathfrak{z}_1), (\mathfrak{h}_2, \mathfrak{z}_2) \in \mathbb{Y}$.
Therefore $(\mathfrak{h}_1, \mathfrak{z}_1)(\mathfrak{h}_2, \mathfrak{z}_2) \in \mathbb{Y}. Hence\ \mathbb{Y}\ is\ a\ subring\ of\ \mathfrak{G}_1 \times \mathfrak{G}_2.$
Theorem 2.9. *Let* $\mathfrak{P}$ *be a* $\mathbb{BVMIFS}$ of a ring $\mathfrak{Z}$ and $\mathfrak{M}$ be the stronget $\mathbb{BVMIF}$ relation of $\mathfrak{Z}$. Then $\mathfrak{P}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{Z}$ if and only if $\mathfrak{M}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{Z}\times\mathfrak{Z}$.
Proof. Let ϱ, υ be in $\mathfrak{Z}$ and ζ, ξ be in $\mathfrak{Z}$. Then (ϱ, ζ) and (υ, ξ) are in $\mathfrak{Z}\times\mathfrak{Z}$. For all i, i = 1, 2, …, n, if $\mathfrak{P}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{Z}$, then
$\mathfrak{W}_i^+[(\varrho, \zeta)-(\upsilon, \xi)] = \mathfrak{W}_i^+(\varrho-\upsilon, \zeta-\xi) = \text{rmin}\{\mathfrak{P}_i^+(\varrho-\upsilon), \mathfrak{P}_i^+(\zeta-\xi)\} \geq \text{rmin}\{\text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\upsilon)\}, \text{rmin}\{\mathfrak{P}_i^+(\zeta), \mathfrak{P}_i^+(\xi)\}\} = \text{rmin}\{\text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\zeta)\}, \text{rmin}\{\mathfrak{P}_i^+(\upsilon), \mathfrak{P}_i^+(\xi)\}\} = \text{rmin}\{\mathfrak{W}_i^+(\varrho, \zeta), \mathfrak{W}_i^+(\upsilon, \xi)\}$, for all $(\varrho, \zeta), (\upsilon, \xi)$ in $\mathfrak{Z}\times\mathfrak{Z}$.
And $\mathfrak{W}_i^+[(\varrho, \zeta)(\upsilon, \xi)] = \mathfrak{W}_i^+(\varrho\upsilon, \zeta\,\xi) = \text{rmin}\{\mathfrak{P}_i^+(\varrho\upsilon), \mathfrak{P}_i^+(\zeta\,\xi)\} \geq \text{rmin}\{\text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\upsilon)\}, \text{rmin}\{\mathfrak{P}_i^+(\zeta), \mathfrak{P}_i^+(\xi)\}\} = \text{rmin}\{\text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\zeta)\}, \text{rmin}\{\mathfrak{P}_i^+(\upsilon), \mathfrak{P}_i^+(\xi)\}\} = \text{rmin}\,\{\mathfrak{W}_i^+(\varrho, \zeta), \mathfrak{W}_i^+(\upsilon, \xi)\}$, for all $(\varrho, \zeta), (\upsilon, \xi)$ in $\mathfrak{Z}\times\mathfrak{Z}$.
Also $\mathfrak{W}_i^-[(\varrho, \zeta)-(\upsilon, \xi)] = \mathfrak{W}_i^-(\varrho-\upsilon, \zeta-\xi) = \text{rmax}\{\mathfrak{P}_i^-(\varrho-\upsilon), \mathfrak{P}_i^-(\zeta-\xi)\} \leq \text{rmax}\{\text{rmax}\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\upsilon)\}, \text{rmax}\{\mathfrak{P}_i^-(\zeta), \mathfrak{P}_i^-(\xi)\}\} = \text{rmax}\{\text{rmax}\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\zeta)\}, \text{rmax}\{\mathfrak{P}_i^-(\upsilon), \mathfrak{P}_i^-(\xi)\}\} = \text{rmax}\{\mathfrak{W}_i^-(\varrho, \zeta), \mathfrak{W}_i^-(\upsilon, \xi)\}$, for all $(\varrho, \zeta), (\upsilon, \xi)$ in $\mathfrak{Z}\times\mathfrak{Z}$.
And $\mathfrak{W}_i^-[(\varrho, \zeta)(\upsilon, \xi)] = \mathfrak{W}_i^-(\varrho\upsilon, \zeta\xi) = \text{rmax}\{\mathfrak{P}_i^-(\varrho\upsilon), \mathfrak{P}_i^-(\zeta\xi)\} \leq \text{rmax}\{\text{rmax}\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\upsilon)\}, \text{rmax}\{\mathfrak{P}_i^-(\zeta), \mathfrak{P}_i^-(\xi)\}\} = \text{rmax}\{\text{rmax}\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\zeta)\}, \text{rmax}\{\mathfrak{P}_i^-(\upsilon), \mathfrak{P}_i^-(\xi)\}\} = \max\{\mathfrak{W}_i^-(\varrho, \zeta), \mathfrak{W}_i^-(\upsilon, \xi)\}$, for all $(\varrho, \zeta), (\upsilon, \xi)$ in $\mathfrak{Z}\times\mathfrak{Z}$.
Hence $\mathfrak{W}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{Z}\times\mathfrak{Z}$**.**
Conversely, assume that $\mathfrak{W}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{Z}\times\mathfrak{Z}$**.**
For all i = 1, 2, …, n,
$\text{rmin}\{\mathfrak{P}_i^+(\varrho-\upsilon), \mathfrak{P}_i^+(\zeta-\xi)\} = \mathfrak{W}_i^+(\varrho-\upsilon, \zeta-\xi) = \mathfrak{W}_i^+[(\varrho, \zeta)-(\upsilon, \xi)] \geq \text{rmin}\{\mathfrak{W}_i^+(\varrho, \zeta), \mathfrak{W}_i^+(\upsilon, \xi)\} = \text{rmin}\{\text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\zeta)\}, \text{rmin}\{\mathfrak{P}_i^+(\upsilon), \mathfrak{P}_i^+(\xi)\}\}$,
put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{Z}$, then
$\mathfrak{P}_i^+(\varrho-\upsilon) \geq \text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\upsilon)\}$, for all ϱ, υ in $\mathfrak{Z}$.
And $\text{rmin}\{\mathfrak{P}_i^+(\varrho\upsilon), \mathfrak{P}_i^+(\zeta\xi)\} = \mathfrak{W}_i^+(\varrho\upsilon, \zeta\xi) = \mathfrak{W}_i^+[(\varrho, \zeta)(\upsilon, \xi)] \geq \text{rmin}\{\mathfrak{W}_i^+(\varrho, \zeta), \mathfrak{W}_i^+(\upsilon, \xi)\} = \text{rmin}\{\text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\zeta)\}, \text{rmin}\{\mathfrak{P}_i^+(\upsilon), \mathfrak{P}_i^+(\xi)\}\}$,
put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{Z}$, then
$\mathfrak{P}_i^+(\varrho\upsilon) \geq \text{rmin}\{\mathfrak{P}_i^+(\varrho), \mathfrak{P}_i^+(\upsilon)\}$, for all ϱ, υ in $\mathfrak{Z}$.
Also $\text{rmax}\{\mathfrak{P}_i^-(\varrho-\upsilon), \mathfrak{P}_i^-(\zeta-\xi)\} = \mathfrak{W}_i^-(\varrho-\upsilon, \zeta-\xi) = \mathfrak{W}_i^-[(\varrho, \zeta)-(\upsilon, \xi)] \leq \text{rmax}\{\mathfrak{W}_i^-(\varrho, \zeta), \mathfrak{W}_i^-(\upsilon, \xi)\} = \text{rmax}\{\text{rmax}\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\zeta)\}, \text{rmax}\{\mathfrak{P}_i^-(\upsilon), \mathfrak{P}_i^-(\xi)\}\}$,
put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{Z}$, then
$\mathfrak{P}_i^-(\varrho-\upsilon) \leq \text{rmax}\{\mathfrak{P}_i^-(\varrho), \mathfrak{P}_i^-(\upsilon)\}$, for all ϱ, υ in $\mathfrak{Z}$.

And rmax{$\mathfrak{P}_i^-(\varrho\upsilon)$, $\mathfrak{P}_i^-(\zeta\xi)$} = $\mathfrak{W}_i^-(\varrho\upsilon,\ \zeta\xi)$ = $\mathfrak{W}_i^-[(\varrho,\ \zeta)(\upsilon,\xi)]$ ≤ rmax{$\mathfrak{W}_i^-(\varrho,\ \zeta)$, $\mathfrak{W}_i^-(\upsilon,\xi)$} = rmax{rmax{$\mathfrak{P}_i^-(\varrho)$, $\mathfrak{P}_i^-(\zeta)$}, rmax{$\mathfrak{P}_i^-(\upsilon)$, $\mathfrak{P}_i^-(\xi)$}},
put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{Z}$, then
$\mathfrak{P}_i^-(\varrho\upsilon)$ ≤ rmax{$\mathfrak{P}_i^-(\varrho)$, $\mathfrak{P}_i^-(\upsilon)$}, for all ϱ, υ in $\mathfrak{Z}$.
Hence $\mathfrak{P}$ is a $\mathbb{BVMIFSR}$ of $\mathfrak{Z}$.

Theorem 2.10. *Let* $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_m$ be $\mathbb{BVMIFS}s$ of a ring $\mathfrak{Z}$ and $\mathfrak{M}$ be the strongest $\mathbb{BVMIF}$ n-dimensional relation of $\mathfrak{Z}$. Then $\mathfrak{P}_1$, $\mathfrak{P}_2$, …, $\mathfrak{P}_m$ are $\mathbb{BVMIFSR}$ *of* $\mathfrak{Z}$ *if and only if* $\mathfrak{M}$ *is a* $\mathbb{BVMIFSR}$ *of* $\mathfrak{Z}\times\mathfrak{Z}\ldots\times\mathfrak{Z}$ (m times).

Proof. *From the Theorem* 2.9, *the proof is trivial.*

3. Conclusion

Properties of transformations of $\mathbb{BVMIFSR}$of a ring have been discussed. The above concepts can be extended into bipolar valued multi I-fuzzy subfield of a field, bipolar interval valued multi fuzzy subspace of a linear space and any other algebraic system.

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