Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 7s (2024)

Multi-Bipolar Fuzzy Planar Graphs

K. Rama Kishore [a, b], Ch. Ramprasad [b,*], P. L. N. Varma [c]

[a, c] Department of Mathematics and Statistics, School of Applied Sciences, VFSTR (Deemed to be University), Vadlamudi, 522 213, India

[a, b, *] Department of Mathematics, Vasireddy Venkatadri Institute of Technology, Namburu, 522 508, India

Email ids: [a]itskrk987@gmail.com, [b,*]ramprasadchegu1984@gmail.com
[c]plnvarma@gmail.com

Article History:

Received: *01-06-2024*

Revised: *03-07-2024*

Accepted: *29-07-2024*

Abstract:

Generalization of a bipolar fuzzy set helps to arrive at an m-BPF set that can be extended to understand important properties of planar and multigraph. The definition of m-BPFMGs, m-BPFPGs and m-BPFDGs are introduced, and some of their intriguing aspects are studied. An m-BPFPGs level of planarity is measured in this case using the term "degree of planarity." On the subject of planarity, some theorems have been proved. Additionally, we investigate isomorphism across m-BPFPGs.

Keywords: Graphs, Multigraph, Planar Graph, FuzzyGraphs.

Nomenclature	
m-bipolar fuzzy	m-BPF
m-bipolar fuzzy graph	m-BPFG
m-bipolar fuzzy multigraph	m-BPFMG
m- bipolar fuzzy multi line	m-BPFML
m-bipolar fuzzy planar graph	m-BPFPG
m-bipolar fuzzy complete multi graph	m-BPFCMG
m-bipolar fuzzy dual graph	m-BPFDG

1. Introduction

Ease of use and flexibility has made it possible to apply graph theory in real life situations requiring further research in this field. Ideally, problems related to traffic and power line management provide suitable areas of applications of graph theory. However, the complex nature of space organization and management through cost effective procedures is a dominant impediment to find a tangible solution, even to this date. For example, the problems of over lapping and cross-crossing power lines have traditional solutions of separating them in space and height. In printed circuit boards, this problem is solved by using layers of separation. We can try to minimize the height parameters and decrease the layers using the concept of planar graphs.

When it comes to traffic management, routes with various degrees of congestion pose a common problem in several developed cities. With the help of the graph theory an application for an information system that informs road users and drivers about levels of congestion on roads can be developed.

Electrical circuits can be represented using a graph model. In this system, minimizing no overlapping circuits is the primary goal. Rail lines, transmission lines, underground tunnels, etc. are all crucial components of city development. Accidents can happen when people are crossing. Despite the preference for routes without crossings, space constraints sometimes make them necessary. It is advisable to cross between congested and less congested routes rather than two congested ones. The definition of "congested" is ambiguous. We frequently use terms like "congested,""extremely congested,""very congested," etc. Linguistic terms are what they are, and they have certain relationship values. Strong routes are those that are congested, whereas weak routes are those that are less congested. Thus, in a city plan, crossing across strong and weak pathways may be allowed with a certain level of safety. Strong line and weak line of a fuzzy planar graph are denoted by the words "strong route" and "weak route," respectively, and the idea of fuzzy planar graph is denoted by the ability to traverse between strong and weak lines [1, 19, 23].

Zhang [32] originally put forward the concept of bipolar fuzzy sets in 1994 as an extension of fuzzy sets [30]. Fuzzy sets with a connection degree range of [-1, 1] are extended in bipolar fuzzy sets [30]. In a bipolar fuzzy set, an element's connection degree of 0 indicates that it has no bearing on the consequent property, its relationship degree of (0, 1] that it somewhat fulfills the property, and its relationship degree of [-1, 0) that it slightly satisfies the implied counter-property. Despite having a similar appearance, bipolar fuzzy sets and intuitionistic fuzzy sets are fundamentally distinct sets. It's crucial to have the ability to handle bipolar information in various fields. Positive information is known to reflect what is seen as possible, even as negative in order is known to stand for what is thought to be unfeasible. Recent fresh studies have been inspired by this area in various different areas. Bipolarity also occurs because we work with spatial information in applications such as image processing and spatial reasoning; as a result, fuzzy and possibility formalisms for bipolar information have been created [12]. For instance, positive information may be stated as a set of potential places and negative in order may be expressed as a set of unfeasible places when evaluating the position of an item in a space.

The modeling of real-time systems where the underlying amount of information fluctuates with different degrees of precision is increasingly being done using fuzzy graph theory. Fuzzy models are beginning to show promise because they aim to reduce the differences between the symbolic models used in expert systems and the conventional numerical models used in research and engineering. Kaufmann proposed the first notion of a fuzzy graph [15] based on Zadeh's fuzzy relations [31]. Cut-nodes, roadways, connectivity, trees, and cycles are only a few of the fundamental graph-theoretic concepts that Rosenfeld [22] proposed as having a fuzzy equivalent. Sunitha and Vijaya kumar [27] described fuzzy trees, while Bhattacharya [10] made some comments on fuzzy graphs.

Research in artificial intelligence and quantum computing makes it necessary to introduce concepts related to logical causality and decision-making that are integral to technical applications. This can be to some extent captured by the m-BPFG as this theory provides possibility for a ternary classification in the place at conventional binary classification.

We know that we have positive (+) and negative (-) as well-established bipolar coordinates for traditional graph analysis. However, complex developments in the field of science tell us that there are categories which are both positive and negative at the given point of time [2-6].

This can be demonstrated with the example of opinion making behavior of a country like India. Though India has a democratically elected government at the center, each state has its own democratically elected state governments. While opinions regarding many central government policies are expressed as agreements or oppositions by state governments, there are some central policies where there is a degree of both agreement and opposition. Let us explain an example from latest farm law proposed by the government of India with all 29 states taken as an m-block. In principle, this law was directly accepted by 20 states while six states opposed it. There are three states which expressed modifications in the policy as it evolved a mixed response. This is important as it influences the political system and patterns of voting in India.

However, in each of the states that have accepted the policy some sections of the farmers opposed the policy; and in the states which opposed the policy there was a call for agreement from some sections of the farmers. This provides a suitable a set for applying the concept of m-BPFGs. m-BPFMGs, m-BPFPGs, and m-BPFDGs are introduced, and some of their interesting aspects are studied. Additionally, we discuss isomorphism across m-BPFPGs [20, 33-35].

In this work, we have only utilized accepted terminology and definitions. The readers are directed to [7, 8, 11, 13, 14, 16-18, 21, 24-26, 28, 29] extra symbols, jargon, and uses not addressed in the research.

2. Preliminaries

The terms "m-BPFS," "m-BPFG," are defined in this section. For the m-BPFG to be generalized, an equivalence criterion was established. Create a relationship of equivalence from the given set V, $\leftrightarrow$ on $V \times V - \{(\iota, \iota): \iota \in V\}$ as follows: $(\iota_{1,}\kappa_1) \leftrightarrow (\iota_{2,}\kappa_2)$ if and only if either $(\iota_{1,}\kappa_1) = (\iota_{2,}\kappa_2)$ or $\iota_{1,} = \kappa_2, \kappa_1 = \iota_{2,}$.

The equivalence class containing the components (ι, κ) is represented by $\iota\kappa$ or $\kappa\iota$, while the quotient set is denoted by $\overleftrightarrow{V^2}$.

Definition 2.1: An m-BPFG of a graph $G^* = (V, E)$ is a pair $G = (V, Q, R)$ where $Q = \langle [P_h o\Psi_Q^+, P_h o\Psi_Q^-]_{h=1}^{m} \rangle$, $P_h o\Psi_Q^+ : V \to [0, 1]$ and $P_h o\Psi_Q^- : V \to [-1, 0]$ an m-BPFS is an m-BPFS on V and $R = \langle [P_h o\Psi_R^+, P_h o\Psi_R^-]_{h=1}^{m} \rangle$, $P_h o\Psi_R^+ : \overleftrightarrow{V^2} \to [0, 1]$ and $P_h o\Psi_Q^- : \overleftrightarrow{V^2} \to [-1, 0]$ in an m-BPFS in $\overleftrightarrow{V^2}$such that

$$P_h o\Psi_R^+(\iota\kappa) \leq \min\{P_h o\Psi_Q^+(\iota), P_h o\Psi_Q^+(\kappa)\},$$

$$P_h o\Psi_R^-(\iota\kappa) \geq \max\{P_h o\Psi_Q^-(\iota), P_h o\Psi_Q^-(\kappa)\}$$

for all $\iota\kappa \in \overleftrightarrow{V^2}, h = 1, 2, \cdots, m$ and $P_h o\Psi_R^+(\iota\kappa) = P_h o\Psi_R^-(\iota\kappa) = 0$ for all $\iota\kappa \in \overleftrightarrow{V^2} - E$.

Definition 2.2: Let $Z \neq \phi$. Two functions, "count positive relationship" of $H(C_H^+)$and "count negative relationship" of $H(C_H^-)$,are used to describe an m-bipolar fuzzy multiset (m-BPFMS) H generated from Z. Here$C_H^+ : Z \to R_1$and $C_H^- : Z \to R_2$, [0,1] and [-1,0] are the intervals from which the collections of every crisp multisite, R_1 and $R_2$are chosen. The definition of the positive sponsorship sequence is as follows: $\langle P_1 o\Psi_H^+(\iota), P_2 o\Psi_H^+(\iota), \cdots, P_m o\Psi_H^+(\iota) \rangle$and $\langle P_1 o\Psi_H^-(\iota), P_2 o\Psi_H^-(\iota), \cdots, P_m o\Psi_H^-(\iota) \rangle$will be used to represent the negative relationship sequence. The following symbols stand for an m-BPFMS:

$$\{\iota: \langle [P_1 \mathrm{o} \Psi_{\mathrm{H}}^{+}(\iota), P_1 \mathrm{o} \Psi_{\mathrm{H}}^{-}(\iota)], [P_2 \mathrm{o} \Psi_{\mathrm{H}}^{+}(\iota), P_2 \mathrm{o} \Psi_{\mathrm{H}}^{-}(\iota)], \cdots, [P_m \mathrm{o} \Psi_{\mathrm{H}}^{+}(\iota), P_m \mathrm{o} \Psi_{\mathrm{H}}^{-}(\iota)] \rangle : \iota \in Z\}.$$

3. m-Bipolar Fuzzy Planar Graphs (m-BPFPGs)

Using the idea of an m-BPFMS, we first propose the idea of an m-BPFPG.

Definition 3.1: Let $V \neq \phi$ and $Q = \langle [P_h \mathrm{o} \Psi_{\mathrm{Q}}^{+}, P_h \mathrm{o} \Psi_{\mathrm{Q}}^{-}]_{\mathrm{h}=1}^{\mathrm{m}} \rangle$ be an m-BPFMS on V. Let $R = \left\{ \frac{\iota\kappa}{\langle [P_h \mathrm{o} \Psi_{\mathrm{R}}^{+}(\iota\kappa)_u, P_h \mathrm{o} \Psi_{\mathrm{R}}^{-}(\iota\kappa)_u]_{\mathrm{h}=1}^{\mathrm{m}} \rangle}, \; u = 1,2,\cdots,t \,|\, \iota\kappa \in V \times V \right\}$ be an m-BPFMS of $V \times V$ such that

$$P_h \mathrm{o} \Psi_{\mathrm{R}}^{+}(\iota\kappa)_u \leq \min\{P_h \mathrm{o} \Psi_{\mathrm{Q}}^{+}(\iota), P_h \mathrm{o} \Psi_{\mathrm{Q}}^{+}(\kappa)\},$$

$$P_h \mathrm{o} \Psi_{\mathrm{R}}^{-}(\iota\kappa)_u \geq \max\{P_h \mathrm{o} \Psi_{\mathrm{Q}}^{-}(\iota), P_h \mathrm{o} \Psi_{\mathrm{Q}}^{-}(\kappa)\}$$

for all $\iota, \kappa \in V$, $u = 1,2,\cdots,t$ and $h = 1, 2, \cdots, m$.Then$G$is called an m-BPFMG.

Remember that there could be a number of lines linking the nodes ι and κ. $[P_h \mathrm{o} \Psi_{\mathrm{R}}^{+}(\iota\kappa)_u, P_h \mathrm{o} \Psi_{\mathrm{R}}^{-}(\iota\kappa)_u]_{h=1}^{m}$ represent, respectively, the positive and negative relationship values of the line $\iota\kappa$ in G. u stands for the quantity of lines connecting the nodes. R is referred to as being m-BPFML set in m-BPFMG G. $P_h \mathrm{o} \Psi_{\mathrm{R}}^{+}(\iota\kappa)_u = 0 = P_h \mathrm{o} \Psi_{\mathrm{R}}^{-}(\iota\kappa)_u$ for all $\iota\kappa \in V \times V - E, 0 \leq P_h \mathrm{o} \Psi_{\mathrm{R}}^{+}(\iota\kappa)_u \leq 1, -1 \leq P_h \mathrm{o} \Psi_{\mathrm{R}}^{-}(\iota\kappa)_u \leq 0$ for $h = 1, 2, \cdots, m$.

Example 1: Take a multigraph of $G^* = (V, E)$ such that $V = \{\iota, \kappa, \upsilon, \tau\}$, $E = \{\iota\kappa, , \kappa\upsilon, \upsilon\tau, \iota\tau\}$. Let $Q = \langle [P_h \mathrm{o} \Psi_{\mathrm{Q}}^{+}, P_h \mathrm{o} \Psi_{\mathrm{Q}}^{-}]_{\mathrm{h}=1}^{\mathrm{m}} \rangle$ be an m-BPFS on V and let $R = \left\{ \frac{\iota\kappa}{\langle [P_h \mathrm{o} \Psi_{\mathrm{R}}^{+}(\iota\kappa)_u, P_h \mathrm{o} \Psi_{\mathrm{R}}^{-}(\iota\kappa)_u]_{\mathrm{h}=1}^{\mathrm{m}} \rangle}, \; u = 1,2,\cdots,t \,|\, \iota\kappa \in V \times V \right\}$be an m-BPFML set of $V \times V$ defined by

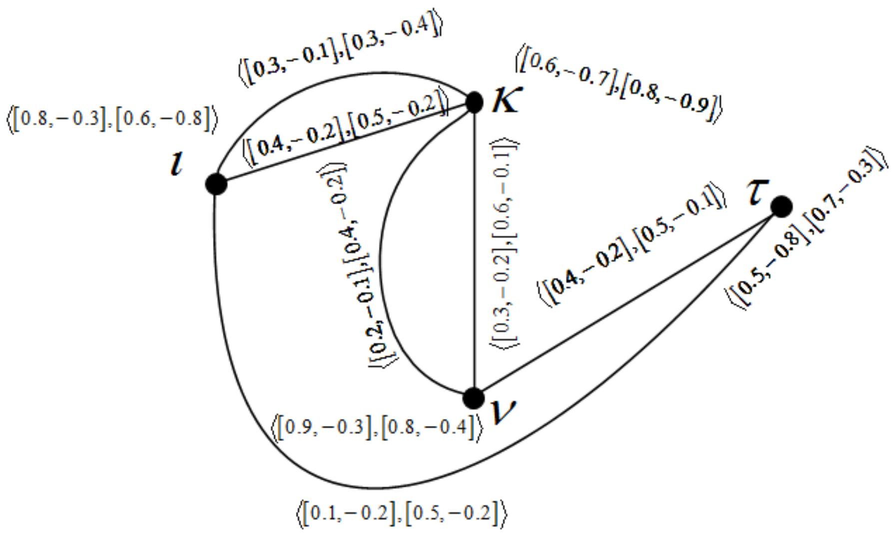

Fig. 1 m-Bipolar Fuzzy Multigraph

It is clear from Fig. 1 through simple calculations that it is an m-BPFMG.

Here $Q = \left\{\frac{\iota}{\langle[0.8,-0.3],[0.6,-0.8]\rangle}, \frac{\kappa}{\langle[0.6,-0.7],[0.8,-0.9]\rangle}, \frac{\upsilon}{\langle[0.9,-0.3],[0.8,-0.4]\rangle}, \frac{\tau}{\langle[0.5,-0.8],[0.7,-0.3]\rangle}\right\}$,

$$R = \left\{\begin{matrix} \frac{\iota\kappa}{\langle[0.4,-0.2],[0.5,-0.2]\rangle}, \frac{\iota\kappa}{\langle[0.3,-0.1],[0.3,-0.4]\rangle}, \frac{\kappa\upsilon}{\langle[0.3,-0.2],[0.6,-0.1]\rangle}, \\ \frac{\kappa\upsilon}{\langle[0.2,-0.1],[0.4,-0.2]\rangle}, \frac{\upsilon\tau}{\langle[0.4,-0.2],[0.5,-0.1]\rangle}, \frac{\iota\tau}{\langle[0.1,-0.2],[0.5,-0.2]\rangle} \end{matrix}\right\}.$$

Definition 3.2: Let $R = \left\{\frac{\iota\kappa}{\langle[P_h o\Psi_R^+(\iota\kappa)_u, P_h o\Psi_R^-(\iota\kappa)_u]_{h=1}^m\rangle}, \ u = 1,2,\cdots,t \mid \iota\kappa \in V \times V\right\}$ be an m-BPFME set in an m-BPFMG G. A multiline $\iota\kappa$ of G is strong if

$$\frac{1}{2}\min\{P_h o\Psi_Q^+(\iota), P_h o\Psi_Q^+(\kappa)\} \le P_h o\Psi_R^+(\iota\kappa)_u,$$

$$\frac{1}{2}\max\{P_h o\Psi_Q^-(\iota), P_h o\Psi_Q^-(\kappa)\} \le P_h o\Psi_R^-(\iota\kappa)_u$$

for all $u = 1,2,\cdots,t$ and $h = 1,2,\cdots,m$.

Definition 3.3: Let $R = \left\{\frac{\iota\kappa}{\langle[P_h o\Psi_R^+(\iota\kappa)_u, P_h o\Psi_R^-(\iota\kappa)_u]_{h=1}^m\rangle}, \ u = 1,2,\cdots,t \mid \iota\kappa \in V \times V\right\}$ be an m-BPFME set in an m-BPFMG G. An m-BPFMG G is complete if

$$P_h o\Psi_R^+(\iota\kappa)_u = \min\{P_h o\Psi_Q^+(\iota), P_h o\Psi_Q^+(\kappa)\},$$

$$P_h o\Psi_R^-(\iota\kappa)_u = \max\{P_h o\Psi_Q^-(\iota), P_h o\Psi_Q^-(\kappa)\}$$

for all $\iota,\kappa \in V$, $u = 1,2,\cdots,t$ and $h = 1,2,\cdots,m$.

Example 2: Consider an m-BPFMG $G = (V, Q, R)$ as exposed in Fig. 2. It is obvious from Fig. 2 by simple calculations that it is an m-BPFCMG.

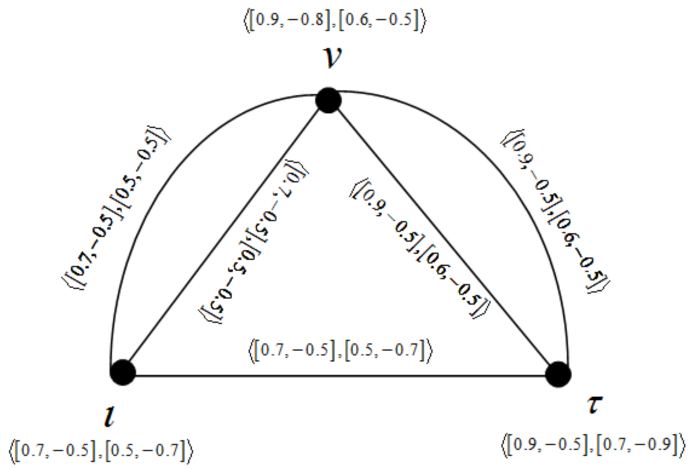

Fig. 2. 2-Bipolar Fuzzy Complete Multigraph

Here $Q = \left\{\frac{\upsilon}{\langle[0.9,-0.8],[0.6,-0.5]\rangle}, \frac{\iota}{\langle[0.7,-0.5],[0.5,-0.7]\rangle}, \frac{\tau}{\langle[0.9,-0.5],[0.7,-0.9]\rangle}\right\}$,

$$R = \left\{\begin{matrix} \frac{\upsilon\iota}{\langle[0.7,-0.5],[0.5,-0.5]\rangle}, \frac{\upsilon\iota}{\langle[0.7,-0.5],[0.5,-0.5]\rangle}, \frac{\upsilon\tau}{\langle[0.9,-0.5],[0.6,-0.5]\rangle}, \\ \frac{\upsilon\tau}{\langle[0.9,-0.5],[0.6,-0.5]\rangle}, \frac{\iota\tau}{\langle[0.7,-0.5],[0.5,-0.7]\rangle} \end{matrix}\right\}.$$

Definition 3.4: Energy of an m-BPFE $\iota\kappa$ can be calculated by the value $E_{\iota\kappa} = \langle [P_h\text{o}E_{\iota\kappa}^{+}, P_h\text{o}E_{\iota\kappa}^{-}]_{h=1}^{m} \rangle = \langle \left[\frac{P_h\text{o}\Psi_R^{+}(\iota\kappa)_u}{\min\{P_h\text{o}\Psi_Q^{+}(\iota), P_h\text{o}\Psi_Q^{+}(\kappa)\}}, \frac{P_h\text{o}\Psi_R^{-}(\iota\kappa)_u}{\max\{P_h\text{o}\Psi_Q^{-}(\iota), P_h\text{o}\Psi_Q^{-}(\kappa)\}} \right]_{h=1}^{m} \rangle$.

Definition 3.5: Let G be an m-BPFMG. A line $\iota\kappa$ should be an m-BPF strong if $P_h\text{o}E_{\iota\kappa}^{+} \geq 0.5$ or $P_h\text{o}E_{\iota\kappa}^{-} \leq 0.5$, for $h = 1, 2, \cdots, m$ otherwise weak.

Definition 3.6: Let G be an m-BPFMG and let R includes two lines $\frac{\iota\kappa}{\langle [P_h\text{o}\Psi_R^{+}(\iota\kappa)_u, P_h\text{o}\Psi_R^{-}(\iota\kappa)_u]_{h=1}^{m} \rangle}$ and $\frac{\upsilon\tau}{\langle [P_h\text{o}\Psi_R^{+}(\upsilon\tau)_s, P_h\text{o}\Psi_R^{-}(\upsilon\tau)_s]_{h=1}^{m} \rangle}$ which are intersect at a location Y, here u and s are permanent integers. The intersecting value at Y is defined as $\chi_Y = \langle [P_h\text{o}\chi_Y^{+}, P_h\text{o}\chi_Y^{-}]_{h=1}^{m} \rangle$, where $P_h\text{o}\chi_Y^{+} = \frac{P_h\text{o}E_{\iota\kappa}^{+} + P_h\text{o}E_{\upsilon\tau}^{+}}{2}$, $P_h\text{o}\chi_Y^{-} = \frac{P_h\text{o}E_{\iota\kappa}^{-} + P_h\text{o}E_{\upsilon\tau}^{-}}{2}$ for $h = 1, 2, \cdots, m$.

Planarity falls as the amount of points of junction in an m-BPFMG rises. With this in consideration, we now presented the notion of an m-BPFPG below.

Definition 3.7: Let G be an m-BPFMG for a particular geometrical representation and define Y_1, $Y_2, \cdots, Y_r$ as the points where the lines are intersect. Consequently, G is described as an m-BPFPG with a m-BPF planarity value

$$\Gamma = \langle [P_h\text{o}\Gamma^{+}, P_h\text{o}\Gamma^{-}]_{h=1}^{m} \rangle = \langle \left[\frac{1}{1+\{P_h\text{o}\chi_{Y_1}^{+} + P_h\text{o}\chi_{Y_2}^{+} + \cdots + P_h\text{o}\chi_{Y_r}^{+}\}}, \frac{-1}{1+\{P_h\text{o}\chi_{Y_1}^{-} + P_h\text{o}\chi_{Y_2}^{-} + \cdots + P_h\text{o}\chi_{Y_r}^{-}\}} \right]_{h=1}^{m} \rangle.$$

It is obvious that Γ is bounded, since $0 < P_h\text{o}\Gamma^{+} \leq 1$ and $-1 \leq P_h\text{o}\Gamma^{-} \leq 0$ for all $h = 1, 2, \cdots, m$. The m-BPF planarity value of a given geometrical representation of an m-BPFPG is $\langle [-1, 1]_{h=1}^{m} \rangle$ if there is no point of junction for such representation. In this instance, the crisp planar graph serves as the underlying crisp graph of this m-BPFG.

Example 3: Take a multigraph $G^* = (V, E)$ of $G = (V, Q, R)$ such that $V = \{\iota, \kappa, \upsilon, \tau\}$, $E = \{\iota\kappa, \iota\tau, \upsilon\tau, \upsilon\kappa\}$.

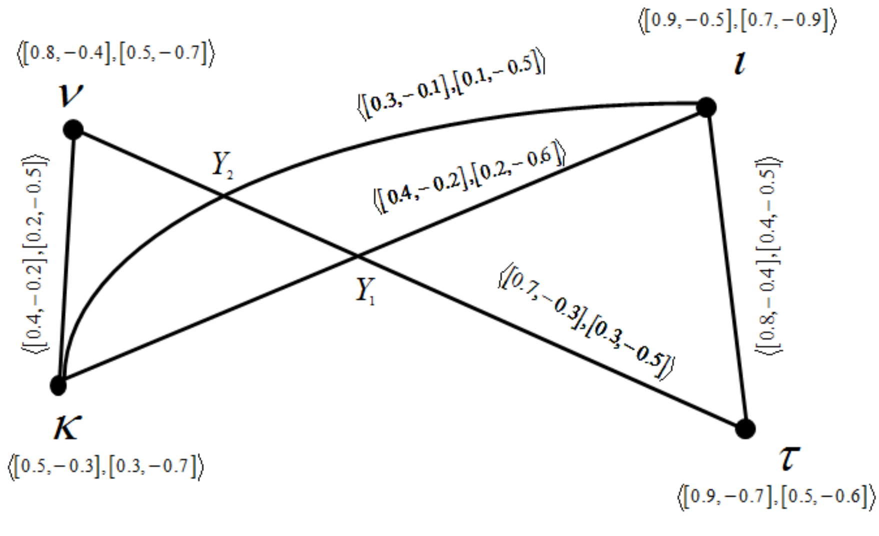

Fig. 3. 2-Bipolar Fuzzy Planar Graph

Let $Q = \{\frac{\iota}{\langle[0.9,-0.5],[0.7,-0.9]\rangle}, \frac{\kappa}{\langle[0.5,-0.3],[0.3,-0.7]\rangle}, \frac{\upsilon}{\langle[0.8,-0.4],[0.5,-0.7]\rangle}, \frac{\tau}{\langle[0.9,-0.7],[0.5,-0.6]\rangle}\}$be an m-BPF node set of V and $R = \left\{\begin{array}{c} \frac{\iota\kappa}{\langle[0.4,-0.2],[0.2,-0.6]\rangle}, \frac{\iota\kappa}{\langle[0.3,-0.1],[0.1,-0.5]\rangle}, \frac{\upsilon\tau}{\langle[0.7,-0.3],[0.3,-0.5]\rangle}, \\ \frac{\iota\tau}{\langle[0.8,-0.4],[0.4,-0.5]\rangle}, \frac{\upsilon\kappa}{\langle[0.4,-0.2],[0.2,-0.5]\rangle} \end{array}\right\}$.

be a m-BPFML set of $V \times V$ is defined by an m-BPFMG as exposed in Fig. 3 has 2 point of junctions Y_1 and Y_2 . Y_1 is a point between the lines $\frac{\iota\kappa}{\langle[0.4,-0.2],[0.2,-0.6]\rangle}$ and $\frac{\upsilon\tau}{\langle[0.7,-0.3],[0.3,-0.5]\rangle}$ and Y_2 is between $\frac{\iota\kappa}{\langle[0.3,-0.1],[0.1,-0.5]\rangle}$ and $\frac{\upsilon\tau}{\langle[0.7,-0.3],[0.3,-0.5]\rangle}$. For the line $\frac{\iota\kappa}{\langle[0.4,-0.2],[0.2,-0.6]\rangle}$, $E_{\iota\kappa} = \langle[0.8, 0.67], [0.67, 0.86]\rangle$. For the line$\frac{\iota\kappa}{\langle[0.3,-0.1],[0.1,-0.5]\rangle}$, $E_{\iota\kappa} = \langle[0.6, 0.33], [0.33, 0.71]\rangle$ and for the line$\frac{\upsilon\tau}{\langle[0.7,-0.3],[0.3,-0.5]\rangle}$, $\mathrm{E}_{\upsilon\tau} = \langle[0.88, 0.75], [0.6, 0.83]\rangle$. For the point of junction Y_1 , intersecting value $\chi_{Y_1} = \langle[0.84, 0.71], [0.63, 0.85]\rangle$ and that for the next point of junction Y_2, $\chi_{Y_2} = \langle[0.74, -0.54], [0.47, 0.77]\rangle$. Therefore, the m-BPF planarity value for the m-BPFMG show in Fig. 3. is $\langle[0.39, -0.44], [0.48, -0.38]\rangle$.

From the next theorem, we can compute an m-BPF planarity value for an m-BPFCMG.

Theorem 3.1: Let G be an m-BPFCMG. An m-BPF planarity value, $\Gamma = \langle[P_h\mathrm{o}\Gamma^+, P_h\mathrm{o}\Gamma^-]_{h=1}^m\rangle$ of G is given by $P_h\mathrm{o}\Gamma^+ = \frac{1}{1+\mathrm{d_s}}$ and $P_h\mathrm{o}\Gamma^- = \frac{-1}{1+\mathrm{d_s}}$, $h = 1, 2, \cdots, m$, where $\mathrm{d_s}$ is the number of point of junctions between the lines in G.

Proof: As G is complete, we get$P_h\mathrm{o}\Psi_{\mathrm{R}}^+(\iota\kappa)_u = \min\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\kappa)\}$,

$P_h\mathrm{o}\Psi_{\mathrm{R}}^-(\iota\kappa)_u = \max\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\kappa)\}$ for all $\iota, \kappa \in V$, $h = 1, 2, \cdots, m$ and $u = 1,2, \cdots, t$.

Let $Y_1, Y_2, \cdots, Y_r$ be the points of junction connecting the lines in G.

For a line $\iota\kappa$ in G, $\langle[P_h\mathrm{o}E_{\iota\kappa}^+, P_h\mathrm{o}E_{\iota\kappa}^-]_{h=1}^m\rangle = \langle\left[\frac{P_h\mathrm{o}\Psi_R^+(\iota\kappa)_u}{\min\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\kappa)\}}, \frac{P_h\mathrm{o}\Psi_R^-(\iota\kappa)_u}{\max\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\kappa)\}}\right]_{h=1}^m\rangle = \langle[1, 1]_{h=1}^m\rangle$.

Consequently, for the point Y_1 which is the point of junctions connecting the lines $\iota\kappa$ and $\upsilon\tau$, the junction value is $\chi_{Y_1} = \langle[1, 1]_{h=1}^m\rangle$.

Hence, $\chi_{Y_h} = \langle[1, 1]_{h=1}^m\rangle$ for $h = 1, 2, \cdots, m$.

Now for $h = 1, 2, \cdots, m, \langle[P_h\mathrm{o}\Gamma^+, P_h\mathrm{o}\Gamma^-]\rangle$

$$= \langle\left[\frac{1}{1 + \{P_h\mathrm{o}\chi_{Y_1}^+ + P_h\mathrm{o}\chi_{Y_2}^+ + \cdots + P_h\mathrm{o}\chi_{Y_r}^+\}}, \quad \frac{-1}{1 + \{P_h\mathrm{o}\chi_{Y_1}^- + P_h\mathrm{o}\chi_{Y_2}^- + \cdots + P_h\mathrm{o}\chi_{Y_r}^-\}}\right]\rangle$$

$$= \langle\left[\frac{1}{1 + \{1 + 1 + \cdots + 1\}}, \quad \frac{-1}{1 + \{1 + 1 + \cdots + 1\}}\right]\rangle = \langle\left[\frac{1}{1 + \mathrm{d_s}}, \quad \frac{-1}{1 + \mathrm{d_s}}\right]\rangle.$$

Therefore, m-BPF planarity Γ is given by$\langle[P_h\mathrm{o}\Gamma^+, P_h\mathrm{o}\Gamma^-]_{h=1}^m\rangle$where

$P_h\mathrm{o}\Gamma^+ = \frac{1}{1+\mathrm{d_s}}$and $P_h\mathrm{o}\Gamma^- = \frac{-1}{1+\mathrm{d_s}}$, $h = 1, 2, \cdots, m$.

Theorem 3.2: Let G be an m-BPFPG with m-BPF planarity value $\langle [P_h o\Gamma^+, P_h o\Gamma^-]_{h=1}^{m} \rangle$ is such that $P_h o\Gamma^+ \geq 0.5$, or $P_h o\Gamma^- \leq 0.5$ for $h = 1, 2, \cdots, m$. Then the number of points of junction between m-BPF strong lines in G is at most one.

Proof: Let G be a strong m-BPFPG. Assume that G has at least two point of junctions Y_1 and Y_2 between the two strong lines in G. For any strong line $\left(\frac{\iota\kappa}{\langle [P_h o\Psi_R^+(\iota\kappa)_u, P_h o\Psi_R^-(\iota\kappa)_u]_{h=1}^{m} \rangle}\right)$

$P_h o\Psi_R^+(\iota\kappa)_u \geq \frac{1}{2}\min\{P_h o\Psi_Q^+(\iota), P_h o\Psi_Q^+(\kappa)\}, P_h o\Psi_R^-(\iota\kappa)_u \leq \frac{1}{2}\max\{P_h o\Psi_Q^-(\iota), P_h o\Psi_Q^-(\kappa)\}$.

This proves that $P_h o\chi_Y^+ \geq 0.5$ or $P_h o\chi_Y^- \leq 0.5$ for $h = 1, 2, \cdots, m$. Thus for 2 intersecting strong lines $\left(\frac{\iota\kappa}{\langle [P_h o\Psi_R^+(\iota\kappa)_u, P_h o\Psi_R^-(\iota\kappa)_u]_{h=1}^{m} \rangle}\right)$ and $\left(\frac{\upsilon\tau}{\langle [P_h o\Psi_R^+(\upsilon\tau)_s, P_h o\Psi_R^-(\upsilon\tau)_s]_{h=1}^{m} \rangle}\right)$,

$\frac{P_h oE_{\iota\kappa}^+ + P_h oE_{\upsilon\tau}^+}{2} \geq 0.5$, $\frac{P_h oE_{\iota\kappa}^- + P_h oE_{\upsilon\tau}^-}{2} \leq 0.5$, for $h = 1, 2, \cdots, m$.

That is, $P_h o\chi_{Y_1}^+ \geq 0.5, P_h o\chi_{Y_1}^- \leq 0.5$. Similarly$P_h o\chi_{Y_2}^+ \geq 0.5, P_h o\chi_{Y_2}^- \leq 0.5$.

This implies that$1 + P_h o\chi_{Y_1}^+ + P_h o\chi_{Y_2}^+ \geq 2, 1 + P_h o\chi_{Y_1}^- + P_h o\chi_{Y_2}^- \leq 2$.

Therefore, $P_h o\Gamma^+ = \frac{1}{1 + P_h o\chi_{Y_1}^+ + P_h o\chi_{Y_2}^+} \leq 0.5$, $P_h o\Gamma^- = \frac{-1}{1 + P_h o\chi_{Y_1}^- + P_h o\chi_{Y_2}^-} \geq -0.5$ for $h = 1, 2, \cdots, m$.

It contradicts the assumption that the m-BPFG is strong m-BPFPG. Therefore, there cannot be two points where strong the lines are intersected. Naturally, the m-BPF planarity value drops as the number of strong m-BPF line junction points rises. Similar to this, if there is just one point where 2 strong lines cross, an m-BPF planarity values are $P_h o\Gamma^+ < 0.5$, and $P_h o\Gamma^- > -0.5$ for $h = 1, 2, \cdots, m$. A strong m-BPFPG is any m-BPFPG that has no crossing between the lines. Thus, we get the conclusion that there is only one position where the strong lines of G can connect.

Theorem 3.3: Let G be an m-BPFPG with an m-BPF planarity value $\Gamma = \langle [P_h o\Gamma^+, P_h o\Gamma^-]_{h=1}^{m} \rangle$.

If $P_h o\Gamma^+ \geq 0.67, P_h o\Gamma^- \leq -0.67$, then G do not have any point of junction between at two strong lines.

Proof: If possible, let Y be a point of junction between two m-BPF strong lines $\left(\frac{\iota\kappa}{\langle [P_h o\Psi_R^+(\iota\kappa)_u, P_h o\Psi_R^-(\iota\kappa)_u]_{h=1}^{m} \rangle}\right)$ and $\left(\frac{\upsilon\tau}{\langle [P_h o\Psi_R^+(\upsilon\tau)_s, P_h o\Psi_R^-(\upsilon\tau)_s]_{h=1}^{m} \rangle}\right)$.

For any m-BPF strong line $\left(\frac{\iota\kappa}{\langle [P_h o\Psi_R^+(\iota\kappa)_u, P_h o\Psi_R^-(\iota\kappa)_u]_{h=1}^{m} \rangle}\right)$, we have $P_h oE_{\iota\kappa}^+ \geq 0.5$, $P_h oE_{\iota\kappa}^- \leq 0.5$ for $h = 1, 2, \cdots, m$.For the minimum value of $P_h oE_{\iota\kappa}^+$and $P_h oE_{\upsilon\tau}^+$, $P_h o\chi_Y^+ = 0.5$ for $h = 1, 2, \cdots, m$.

Then, $P_h o\Gamma^+ = \frac{1}{1+0.5} < 0.67$.

Similarly, $P_h o\Gamma^- > -0.67$ for$h = 1, 2, \cdots, m$, a contradiction.

As a result, G lacks a spot where two m-BPF strong lines connect. Next, strong m-BPFPG is defined as follows.

Definition 3.8: An m-BPFPG G is called strong m-BPFPG if the m-BPF planarity value $\langle[P_h\mathrm{o}\Gamma^+, P_h\mathrm{o}\Gamma^-]_{h=1}^m\rangle$ of the graph is $P_h\mathrm{o}\Gamma^+ \geq 0.67, P_h\mathrm{o}\Gamma^- \leq -0.67$.

Theorem 3.4: Any complete m-BPFG of 5 nodes or complete bipartite m-BPFG of 6 nodes are not strong m-BPFPG.

Proof: Let $G = (V, Q, R\ )$ be a complete m-BPFG of 5 nodes, here $V = \{\iota, \kappa, \alpha, \upsilon, \tau\ \}$ and

$R = \left\{\left(\frac{\iota\kappa}{\langle[P_h\mathrm{o}\Psi_{\mathrm{R}}^+(\iota\kappa)\ , P_h\mathrm{o}\Psi_{\mathrm{R}}^-(\iota\kappa)]_{\mathrm{h=1}}^{\mathrm{m}}\rangle}\right), |\iota\kappa \in V \times V\right\}.$

$\forall,\ \iota, \kappa \in V,$ we get , $P_h\mathrm{o}\Psi_{\mathrm{R}}^+(\iota\kappa) = \min\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\kappa)\}$,

$P_h\mathrm{o}\Psi_{\mathrm{R}}^-(\iota\kappa) = \max\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\kappa)\}$ for $h = 1, 2, \cdots, m$.

A complete m-BPFG's m-BPF planarity value, according to Theorem 1, is $\Gamma = \langle[P_h\mathrm{o}\Gamma^+, P_h\mathrm{o}\Gamma^-]_{h=1}^m\rangle$where $P_h\mathrm{o}\Gamma^+ = \frac{1}{1+\mathrm{d_s}}$ and $P_h\mathrm{o}\Gamma^- = \frac{-1}{1+\mathrm{d_s}}$, $h = 1, 2, \cdots, m$, $\mathrm{d_s}$ is the number of point of junctions connecting the lines in G.

We know that the geometric representation of the underlying crisp graph of an m-BPF complete graph with 5 nodes is non-planar, and that no representation can avoid one point of junction.

So, $\langle[P_h\mathrm{o}\Gamma^+, P_h\mathrm{o}\Gamma^-]_{h=1}^m\rangle = \langle[05, -0.5]_{h=1}^m\rangle$.

Hence, G is not astrong m-BPFPG.

The complete bipartite m-BPFG of 6 nodes cannot be shown to constitute a strong m-BPFPG, however.

4. Faces of an m-Bipolar fuzzy planar graph

The parameter of importance is the face of the m-BPFPG. A region enclosed by m-BPF lines is known as the face of an m-BPFPG. Each m-BPF face has m-BPF lines that define its boundaries. Crisp face is created when all of the lines in the boundary of an m-BPF face have relationship values of $[1, -1]$. The existence of the m-BPF face is negated if one of these lines is eliminated. Therefore, the minimum energy of m-BPF lines in a border determines whether an m-BPF face exists. Below is a definition of an m-BPF face and its relationship values in an m-BPFG.

Definition 4.1: Let G be an m-BPFPG and$R = \left\{\left(\frac{\iota\kappa}{\langle[P_h\mathrm{o}\Psi_{\mathrm{R}}^+(\iota\kappa)_u\ , P_h\mathrm{o}\Psi_{\mathrm{R}}^-(\iota\kappa)_u]_{\mathrm{h=1}}^{\mathrm{m}}\rangle}\right), u = 1,2, \cdots, t|\iota\kappa \in V \times V\right\}$. An m-BPF face of G is a region, bordered by the set of m-BPF lines $E' \subset V \times V$, of a geometric sign of G. The positive and negative values of the m-BPF face is $\langle[P_h\mathrm{o\ F}^+, P_h\mathrm{o\ F}^-]_{h=1}^m\rangle$, where $P_h\mathrm{o\ F}^+ = \min\left\{\left(\frac{P_h\mathrm{o}\Psi_R^+(\iota\kappa)_u}{\min\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^+(\kappa)\}}\right), u = 1,2, \cdots, t|\iota\kappa \in E'\right\}, P_h\mathrm{o\ F}^- = \max\left\{\left(-\frac{P_h\mathrm{o}\Psi_R^-(\iota\kappa)_u}{\max\{P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\iota), P_h\mathrm{o}\Psi_{\mathrm{Q}}^-(\kappa)\}}\right), u = 1,2, \cdots, t|\iota\kappa \in E'\right\}, h = 1, 2, \cdots, m.$

Definition 4.2: If an m-BPF face has a positive relationship value $P_h\mathrm{o\ F}^+ > 0.5$ or a negative relationship value $P_h\mathrm{o\ F}^+ >$ -0.5, for $h = 1, 2, \cdots, m$, it is said to be a strong m-BPF face; otherwise, it is called to be a weak face. The outer m-BPF face is an infinite region found in every m-BPFPG. Remaining faces are call inner m-BPF faces.

Example 4: Take an m-BPFPG $G = (V, Q, R)$as shown in Fig. 4.where

$$Q = \begin{Bmatrix} \frac{q_1}{\langle[0.8,-0.7],[0.7,-0.65]\rangle}, \frac{q_2}{\langle[0.9,-0.52],[0.8,-0.7]\rangle}, \\ \frac{q_3}{\langle[0.7,-0.9],[0.5,-0.7]\rangle}, \frac{q_4}{\langle[0.63,-0.51],[0.79,-0.54]\rangle} \end{Bmatrix},$$

$$R = \begin{Bmatrix} \frac{q_1q_2}{\langle[0.75,-0.51],[0.65,-0.54]\rangle}, \frac{q_1q_3}{\langle[0.62,-0.63],[0.49,-0.53]\rangle}, \frac{q_2q_3}{\langle[0.69,-0.5],[0.49,-0.65]\rangle}, \\ \frac{q_1q_4}{\langle[0.23,-0.14],[0.6,-0.2]\rangle}, \frac{q_3q_4}{\langle[0.59,-0.49],[0.47,-0.52]\rangle} \end{Bmatrix}.$$

Then m-BPFPG has the following faces:

- m-BPF face β_1 is bounded by the lines $\frac{q_1q_4}{\langle[0.23,-0.14],[0.6,-0.2]\rangle}, \frac{q_3q_4}{\langle[0.59,-0.49],[0.47,-0.52]\rangle}$, $\frac{q_1q_3}{\langle[0.62,-0.63],[0.49,-0.53]\rangle}$,
- m-BPF face β_2 is surrounded by the lines $\frac{q_1q_2}{\langle[0.75,-0.51],[0.65,-0.54]\rangle}$, $\frac{q_1q_3}{\langle[0.62,-0.63],[0.49,-0.53]\rangle}$, $\frac{q_2q_3}{\langle[0.69,-0.5],[0.49,-0.65]\rangle}$,
- outer m-BPF face β_3 surrounded by the lines $\frac{q_1q_2}{\langle[0.75,-0.51],[0.65,-0.54]\rangle}$, $\frac{q_2q_3}{\langle[0.69,-0.5],[0.49,-0.65]\rangle}$, $\frac{q_1q_4}{\langle[0.23,-0.14],[0.6,-0.2]\rangle}$, $\frac{q_3q_4}{\langle[0.59,-0.49],[0.47,-0.52]\rangle}$.

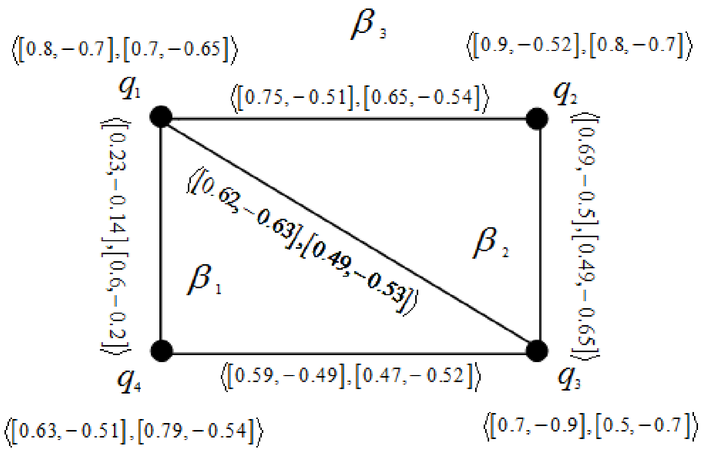

Fig. 4 Faces in m-Bipolar Fuzzy Planar Graph

Clearly the relationship value of m-BPF face $\beta_1$is$\langle[0.37, -0.27], [0.86, -0.37]\rangle$. The relationship value of m-BPF face $\beta_3$is$\langle[0.37, -0.27], [0.86, -0.37]\rangle$. Thus $\beta_1$and β_3 are weak m-BPF faces and β_2 is a strong m-BPF face with the relationship value$\langle[0.89, -0.90], [0.93, -0.82]\rangle$.

We are now introducing dual of m-BPFPG. Each m-BPF line between two nodes corresponds to each line in the border between two faces of the m-BPFPG, and in the m-BPFDG, nodes correspond to the strong m-BPF faces of the m-BPFPG. Below is the formal definition.

Definition 4.3: Let G be an m-BPFPG and let $R = \left\{\left(\frac{\iota\kappa}{\langle [P_h o\Psi_R^{+}(\iota\kappa)_u, P_h o\Psi_R^{-}(\iota\kappa)_u]_{h=1}^{m}\rangle}\right), u = 1,2,\cdots,t | \iota\kappa \in V \times V\right\}$.

Let $\beta_1, \beta_2, \cdots, \beta_r$be the strong m-BPF faces of G. An m-BPFDG of G is an m-BPFPG $G^d = (V^d, Q^d, R^d)$, where $V^d = \{q_i, i = 1, 2, \cdots, r\}$, and the node $q_i$of G^d is measured for the face $\beta_i$of G. The positive and negative relationship values of nodes are taken by the mapping $Q^d = \langle [P_h o\Psi_{Q^d}^{+}, P_h o\Psi_{Q^d}^{-}]_{h=1}^{m}\rangle$, $P_h o\Psi_{Q^d}^{+}: V^d \to [0, 1]$ and $P_h o\Psi_{Q^d}^{-}: V^d \to [-1, 0]$ such that $P_h o\Psi_{Q^d}^{+}(q_i) = \max\{P_h o\Psi_{R^d}^{+}(\iota\kappa)_u, u = 1,2,\cdots,t \mid \iota\kappa \text{ is a line of the boundary of the strong } m-BPF \text{ face } \beta_i\}$, $P_h o\Psi_{Q^d}^{-}(q_i) = \min\{P_h o\Psi_{R^d}^{-}(\iota\kappa)_u, u = 1,2,\cdots,t | \iota\kappa \text{ is a line of the boundary of the strong } m-BPF \text{ face } \beta_i\}$.

There may exist excess of one common line corresponding to two faces β_i and β_j of G. Therefore, there may be in excess of one line corresponding to two nodesq_i and $q_j$in m-BPFDG G^d.Let $\langle [P_h o\Psi_R^{+^r}(q_i q_j), P_h o\Psi_R^{-^r}(q_i q_j)]_{h=1}^{m}\rangle$ denotes the positive and negative relationship values of the rthline between $q_i$and q_j, the positive and negative relationship values of the m-BPF lines of the m-BPFDG are given by$P_h o\Psi_{R^d}^{+^r}(q_i q_j)_r = P_h o\Psi_R^{+^r}(\iota\kappa)_u$,$P_h o\Psi_{R^d}^{-^r}(q_i q_j)_r = P_h o\Psi_R^{-^r}(\iota\kappa)_u$, where$(\iota\kappa)_u$is an line in the boundary corresponding 2 strong m-BPF faces β_i and $\beta_j$and $r = 1,2,\cdots,s$,where $s$is the number of common lines in the boundary between β_i and β_j or the number of lines between q_i and q_j.If there be any strong pendant line in the m-BPFPG, then there will be a self loop in G^d corresponding to this pendant line. The line positive and negative relationship value of the self loop is equal to the positive and negative relationship values of the pendant line. m-BPFDG of m-BPFPG does not contain point of junction of the lines for a certain representation, so it is m-BPFPG with planarity Value $\langle [1, -1]_{h=1}^{m}\rangle$. Hence the m-BPF face of m-BPFDG can be similarly described as in m-BPFPGs.

Example 5: Consider an m-BPFPG $G = (V, Q, R)$of $G^* = (V, E)$as given away in Fig. 5 such that $V = \{p, q, r, s\}$, $E = \{pq, qr, rs, sp, pr\}$

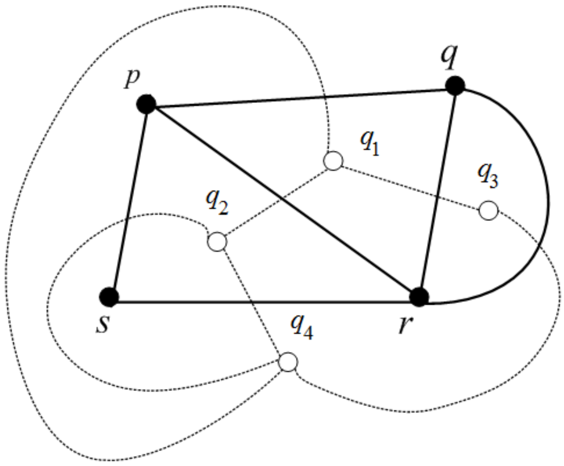

Fig. 5.2-BPFG and it’s 2-BPFDG

$Q = \left\{\frac{p}{\langle[0.9,-0.7],[0.8,-0.5]\rangle}, \frac{q}{\langle[0.7,-0.8],[0.8,-0.9]\rangle}, \frac{r}{\langle[0.5,-0.23],[0.7,-0.9]\rangle}, \frac{s}{\langle[0.6,-0.8],[0.7,-0.35]\rangle}\right\}$,

$$R = \left\{\begin{matrix} \frac{pq}{\langle[0.65,-0.69],[0.75,-0.45]\rangle}, \frac{qr}{\langle[0.49,-0.21],[0.65,-0.75]\rangle}, \frac{rs}{\langle[0.49,-0.2],[0.61,-0.32]\rangle}, \\ \frac{sp}{\langle[0.55,-0.45],[0.69,-0.23]\rangle}, \frac{pr}{\langle[0.41,-0.2],[0.66,-0.49]\rangle}, \frac{qr}{\langle[0.46,-0.2],[0.65,-0.81]\rangle} \end{matrix}\right\}.$$

The m-BPFPG has the following faces:

- m-BPF face β_1 is bounded by $\frac{pq}{\langle[0.65,-0.69],[0.75,-0.45]\rangle}$, $\frac{pr}{\langle[0.41,-0.2],[0.66,-0.49]\rangle}$, $\frac{qr}{\langle[0.49,-0.21],[0.65,-0.75]\rangle}$
- m-BPF face β_2 bounded by the lines $\frac{pr}{\langle[0.41,-0.2],[0.66,-0.49]\rangle}$, $\frac{sp}{\langle[0.55,-0.45],[0.69,-0.23]\rangle}$, $\frac{rs}{\langle[0.49,-0.2],[0.61,-0.32]\rangle}$
- m-BPF face β_3 is bounded by the lines $\frac{qr}{\langle[0.49,-0.21],[0.65,-0.75]\rangle}$, $\frac{qr}{\langle[0.46,-0.2],[0.65,-0.81]\rangle}$ and
- outer m-BPF β_4 is surrounded by $\frac{pq}{\langle[0.65,-0.69],[0.75,-0.45]\rangle}$, $\frac{qr}{\langle[0.46,-0.2],[0.65,-0.81]\rangle}$, $\frac{rs}{\langle[0.49,-0.2],[0.61,-0.32]\rangle}$, $\frac{sp}{\langle[0.55,-0.45],[0.69,-0.23]\rangle}$.

According to routine calculations, all faces are strong m-BPF faces. We consider into an account a node for the m-BPFDG for each strong m-BPF face.So the node set$V^d = \{q_1, q_2, q_3, q_4\}$, where the node q_j is given consequent to the strong m-BPF face β_j , $j = 1,2,3,4$. Thus, $Q^d = \left\{\begin{matrix} \frac{q_1}{\langle[0.65,-0.69],[0.75,-0.75]\rangle}, \frac{q_2}{\langle[0.55,-0.45],[0.69,-0.49]\rangle}, \\ \frac{q_3}{\langle[0.49,-0.21],[0.65,-0.81]\rangle}, \frac{q_4}{\langle[0.65,-0.69],[0.75,-0.81]\rangle} \end{matrix}\right\}$,

The faces $\beta_2$and β_4 in Gshare two common lines, ps and rs. Consequently, there are two lines in the m-BPFDG of G between the nodes q_2 and q_4. That line's positive and negative relationship values are given by

$\left\{\frac{q_2q_4}{\langle[0.49,-0.2],[0.61,-0.32]\rangle}, \frac{q_2q_4}{\langle[0.55,-0.45],[0.69,-0.23]\rangle}\right\}$.

The other lines of the m- BPFDG's of positive and negative relationship values are computed as follows.

$\left\{\frac{q_1q_3}{\langle[0.49,-0.21],[0.65,-0.75]\rangle}, \frac{q_1q_2}{\langle[0.41,-0.2],[0.66,-0.49]\rangle}, \frac{q_1q_4}{\langle[0.65,-0.69],[0.75,-0.45]\rangle}, \frac{q_3q_4}{\langle[0.46,-0.2],[0.65,-0.81]\rangle}\right\}$.

Thus, the line set of m-BPFDG is

$$R^d = \left\{\begin{matrix} \frac{q_2q_4}{\langle[0.49,-0.2],[0.61,-0.32]\rangle}, \frac{q_2q_4}{\langle[0.55,-0.45],[0.69,-0.23]\rangle}, \frac{q_1q_3}{\langle[0.49,-0.21],[0.65,-0.75]\rangle}, \\ \frac{q_1q_2}{\langle[0.41,-0.2],[0.66,-0.49]\rangle}, \frac{q_1q_4}{\langle[0.65,-0.69],[0.75,-0.45]\rangle}, \frac{q_3q_4}{\langle[0.46,-0.2],[0.65,-0.81]\rangle} \end{matrix}\right\}.$$

In Fig. 5, the m-BPFDG $G^d$of G is drawn by dotted line.

In m-BPFDGs, weak lines in planar graphs are not taken into an account for any computations.

Theorem 4.1: Let G be an m-BPFPG whose number of nodes, the number of m-BPF lines and number of strong faces are denoted by x, y, z respectively. Let G^d be the m-BPFDG of G. Then:

i. no. of nodes of $G^d = x$,
ii. no. of lines of $G^d = y$,
iii. no. of m-BPF faces of $G^d = z$.

Proof: The definition of m-BPFDG provides evidence for (i) (ii), and (iii).

Theorem 4.2: Let G be a strong m-BPFPG without weak lines and the m-BPFDG of G be G^d. Then the positive relationship and negative relationship values of m-BPF lines of G^d are equal to positive relationship and negative relationship values of m-BPF lines of G.

Proof: The dual graph G^d of G is a strong m-BPFPG as there is no point of junction between any lines. Let $\{\beta_1, \beta_2, \cdots, \beta_r\}$ be the set of strong faces of G.

By the definition of m-BPFDG we know that $P_h o\Psi^{+r}_{R^d}(q_i q_j) = P_h o\Psi^{+r}_{R}(\iota\kappa)$, $P_h o\Psi^{-r}_{R^d}(q_i q_j) = P_h o\Psi^{-r}_{R}(\iota\kappa)$ where $\iota\kappa$ is a common line between 2 strong m-BPF faces β_i and β_j and $r = 1,2,\cdots,s$, where s being the no. of common lines in the boundary between β_i and β_j. The no. of m-BPF lines of two graphs G and G^d are same as G has no weak lines. So, for each m-BPF line of G there is an m-BPF line in G^d with the same relationship value.

We're now studying at isomorphism between m-BPFPGs.

Definition 4.4: Let $G_a = (V_a, Q_a, R_a)$ and $G_b = (V_b, Q_b, R_b)$ be two m-BPFPGs of the graphs $G_a^* = (V_a, E_a)$ and $G_b^* = (V_b, E_b)$ respectively.

(i) An isomorphism between G_a and G_b is a bijective transformation $\zeta: V_a \to V_b$ such that for each $h = 1, 2, \cdots, m$

(a) $P_h o\Psi^{+}_{\mathrm{Q_a}}(\iota_a) = P_h o\Psi^{+}_{\mathrm{Q_b}}(\zeta(\iota_a)), P_h o\Psi^{-}_{\mathrm{Q_a}}(\iota_a) = P_h o\Psi^{-}_{\mathrm{Q_b}}(\zeta(\iota_a))$ for all $\iota_a \in V_a$

(b) $P_h o\Psi^{+}_{\mathrm{R_a}}(\iota_a\kappa_a) = P_h o\Psi^{+}_{\mathrm{R_b}}(\zeta(\iota_a)\zeta(\kappa_a)), P_h o\Psi^{-}_{\mathrm{R_a}}(\iota_a\kappa_a) = P_h o\Psi^{-}_{\mathrm{R_b}}(\zeta(\iota_a)\zeta(\kappa_a))$ for all $\iota_a\kappa_a \in \overleftrightarrow{V_a^2}$.

(ii) A weak isomorphism between G_a and G_b is a bijective transformation $\zeta: V_a \to V_b$ such that for each $h = 1, 2, \cdots, m$

(a) ζ is a homomorphism

(b) $P_h o\Psi^{+}_{\mathrm{Q_a}}(\iota_a) = P_h o\Psi^{+}_{\mathrm{Q_b}}(\zeta(\iota_a)), P_h o\Psi^{-}_{\mathrm{Q_a}}(\iota_a) = P_h o\Psi^{-}_{\mathrm{Q_b}}(\zeta(\iota_a))$ for all $\iota_a \in V_a$.

(iii) A co-weak isomorphism between G_a and G_b is a bijective transformation $\zeta: V_a \to V_b$ such that for each $h = 1, 2, \cdots, m$

(a) ζ is a homomorphism

(b) $P_h o\Psi^{+}_{\mathrm{R_a}}(\iota_a\kappa_a) = P_h o\Psi^{+}_{\mathrm{R_b}}(\zeta(\iota_a)\zeta(\kappa_a)), P_h o\Psi^{-}_{\mathrm{R_a}}(\iota_a\kappa_a) = P_h o\Psi^{-}_{\mathrm{R_b}}(\zeta(\iota_a)\zeta(\kappa_a))$ for all $\iota_a\kappa_a \in \overleftrightarrow{V_a^2}$.

That isomorphism between two m-BPFPGs is an equivalence relation is simple to confirm.

However, if two m-BPFGs are isomorphic so that one is m-BPFPG, the other will be m-BPFG. As evidence, consider the following.

Theorem 4.3: Let G_a be an m-BPFPG and let G_b be an m-BPFG. If there exists an isomorphism$\zeta: G_a \to G_b, G_b$ can be represented as m-BPFPG with the same planarity value of G_a.

Proof: Line and node weights are preserved through isomorphism. Additionally, isomorphic m-BPFGs maintain their order and size. G_b will therefore have the same size and order as G_a. After then, G_b can be represented similarly to G_a.

As a result, G_b will have the same number of junctions where lines cross and the same planarity value as G_a. Accordingly, $G_b$may be represented as m-BPFPG with the same planarity value as G_a.

Theorem 4.4: Let G_b be the m-BPFDG of m-BPFDG of a strong m-BPFDG G without weak lines. Then there exists a co-weak isomorphism connecting G and G_b.

Proof: Assume that $G_b$is an m-BPFDG of $G_a$and that $G_a$is an m-BPFDG of G. The no. of nodes of G_b are equal to the strong m-BPF faces of $G_a$and the no. of strong m-BPF faces in $G_a$is equal to the no. of nodes in G. The no. of nodes in G_b and G are therefore equal. Additionally, an m-BPFPG and its dual have the same number of lines. The line relationship value of a line in a dual graph is identical to the line relationship value of a line in m-BPFG, according to the definition of m-BPFDG. The co-weak isomorphism between G and $G_b$can be created. Hence the outcome.

With the same number of nodes, two m-BPFPGs may be isomorphic. However, the following relationships may exist between the m-BPF planarity values of two m-BPFPGs. **Theorem 4.5:** Let $G_a$and $G_b$be two isomorphic m-BPFGs with the corresponding m-BPF planarity values $\Gamma_{G_a}$and Γ_{G_b} . Then$\Gamma_{G_a} = \Gamma_{G_b}$.

Proof: Obvious.

We then make the following claims without providing any proof.

Theorem 4.6: Let $G_a$and $G_b$be two weak isomorphic m-BPFGs with the corresponding m-BPF planarity values of $\Gamma_{G_a}$and Γ_{G_b}. If the line positive relationship and a line negative relationship values of the respective intersecting lines are the same, then $\Gamma_{G_a} = \Gamma_{G_b}$.

Theorem 4.7: Let $G_a$and G_b be two co-weak isomorphic m-BPFGs with the corresponding m-BPF planarity values of $\Gamma_{G_a}$and Γ_{G_b}. If the terminal nodes of the respective intersecting lines have the same minimum, positive relationship and maximum negative relationship values, then $\Gamma_{G_a}=\Gamma_{G_b}$.

Conclusions

Unpredictable behavior and opinions of voters leads to vagueness and uncertainty when traditional graph theory is used for analyzing the voter sentiment and shifting loyalties. We already know that bipolar fuzzy sets are capable of handling fuzzy sets with vague and uncertainty data to some extent. By modifying them to include multiple parameters on bipolar. In this research, we apply the idea of m-bipolar fuzzy set to multigraph and planar graph because m-bipolar fuzzy sets have shown advantages over bipolar fuzzy sets in addressing ambiguity and uncertainty to some extent. We can generate m-BPFGs to provide better analysis. Recent developments in the fields of Artificial Intelligence, Electrical Engineering and Quantum Mechanics expand the scope and the application of m-BPFGs.

References

[1] Abdul-Jabbar, N., Naoom, J.H., Ouda, E.H.: Fuzzy dual graph. J. Al-Nahrain Univ. 12(4),168-171 (2009)

[2] Akram, M.: Bipolar fuzzy graphs. Inf. Sci. 181, 5548-5564 (2011)

[3] Akram, M.: Bipolar fuzzy graphs with applications. Knowl. -Based Syst. 39, 1-8 (2013)

[4] Akram, M., Dudek, W.A.: Regular bipolar fuzzy graphs. Neural Comput. Appl. 21, 197-205 (2012)

[5] Akram, M., Karunambigai, M.G.: Metric in bipolar fuzzy graphs. World Appl. Sci. J. 14, 1920-1927 (2011)

[6] Akram, M., Dudek, W.A., Sarwar, S.: Properties of bipolar fuzzy hypergraphs. Ital. J. Pure Appl. Math. 31. 426-458 (2013)

[7] Akram, M., Alshehri, N., Davvaz, B., Ashraf, A.: Bipolar fuzzy digraphs in decision support systems. J. Multi-Valued Logic Soft Comput. 1-21 (2016) (in press)

[8] Akram, M., Samantha, S., Pal, M.: Application of Bipolar Fuzzy Sets in Planar Graphs. Int. J. Appl. Comput. Math. 3, (2017)

[9] Alshehri, N., Akram, M.: Intuitionistic fuzzy planar graphs. Discrete Dyn. Nat. Soc. Article ID 397823

[10] Bhattacharya, P.: Some remarks on fuzzy graphs. Pattern Recognit. Lett. 6, 297-302 (1987)

[11] Cerruti, U.: Graphs and Fuzzy Graphs. Fuzzy Information and Decision Processes, pp.123-131 (1982)

[12] Dubois, D., Kaci, S., Prade, H.: Bipolarity in reasoning and decision, an introduction. In: International Conference on Information Processing and Management of Uncertainty, IPMU'04, pp. 959-966 (2004)

[13] Eslahchi, C., Onaghe, B.N.: Vertex strength of fuzzy graphs. Int. J. Math. Math. Sci. 1-9. Article ID 43614

[14] Ghorai, G., Pal, M.: Faces and dual of m-polar fuzzy planar graphs. Journal of Intelligent & Fuzzy Systems, 31(3), 2043-2049 (2016)

[15] Kauffman, A.: Introduction a la Theorie des Sous-emsemblesFlous, vol. 1. Masson et Cie, Paris (1973)

[16] Mathew, S., Sunitha, M. S.: Types of arcs in a fuzzy graph. Inf. Sci. 179(11), 1760-1768 (2009)

[17] Mordeson, J.N., Peng, C. S.: Operations on fuzzy graphs. Inf. Sci. 79, 159-170 (1994)

[18] Mordeson, J.N., Nair, P.S.: Fuzzy Graphs and Fuzzy Hypergraphs, Second Edition 2001. Physica Verlag, Heidelberg (1998)

[19] Pal,A., Samantha, S., Pal, M.: Concept of fuzzy planar graphs. In: Proceedings of Science and Information Conference 2013, October 7-9, 2013, London, UK, pp. 557-563 (2013)

[20] Prem Kumar Singh: Bipolarity in multi-way fuzzy context and its analysis using m-way granulation: Granular computing, March, 2021.

[21] Riaz, F., Ali, K.M.: Applications of graph theory in computer science. In: Cicsyn, 2011 Third International Conference on Computational Intelligence, Communication Systems and Networks, pp. 142-145 (2011)

[22] Rosenfeld, A.: Fuzzy graphs. In: Zadeh, L.A., Fu, K.S., Shimura, M. (eds) Fuzzy sets and Their Applications, pp, 77-95. Academic Press, New York (1975)

[23] Samantha, S., Pal, M., Pal, A.: New concepts of fuzzy planar graph. Int. J. Adv. Res.Artif. Intell. 3(1), 52-59 (2014)

[24] Samantha, S., Pal, M.: Bipolar fuzzy hypergraphs. Int. J. Fuzzy Logic Syst. 2(1), 17-28 (2012)

[25] Samantha, S., Pal, M.: Fuzzy Planar Graphs. IEEE Transactions on Fuzzy Systems. 23, 1936-1942 (2015)

[26] Shinoj, T.K., John, S.J.: Intuitionistic fuzzy multisets and its application in medical diagnosis. World Acad. Sci. Eng. Technol. 6(1), 1418-1421 (2012)

[27] Sunitha, M.S., Vijayakumar, A.: Complement of a fuzzy graph. Indian J. Pure Appl. Math. 33, 1451-1464 (2002)

[28] Wang, S., Zhang, Y., Yang, X., Sun, P., Dong, Z., Liu, A., Yuan, T.: Pathological brain detection by a novel image feature-fractional Fourier entropy. Entropy 17(12), 8278- 8296 (2015)

[29] Yager, R.R.: On the theory of bags. Int. J. Gen Syst. 13, 23-37 (1986)

[30] Zadeh, L.A.: Fuzzy sets. Inf. Control 8, 338-353 (1965)

[31] Zadeh, L.A.: Similarity relations and fuzzy orderings. Inf. Sci. 3(2), 177-200 (1971)

[32] Zhang, W.R.: Bipolar fuzzy sets. In: Proceedings of FUZZ-IEEE, pp. 835-840 (1998)

[33] Zhang. W.R.: From Equilibrium-Based Business Intelligence to Information Conservational Quantum-Fuzzy Cryptography-A cellular transformation of Bipolar Fuzzy Sets to Quantum Intelligence Machinery. IEEE Transactions on Fuzzy Systems (2017)

[34] Zhang. W.R.: Science Vs Sophistry- A historical debate on bipolar fuzzy sets and equilibrium-based mathematics for AI&QI. Journal of Intelligent & Fuzzy Systems. 41, 6781-6799 (2021)

[35] Zhang. W.R.: Ground-0 Axioms vs. First Principles and Second Law: From the Geometry of Light and Logic of Photon to Mind-Light-Matter Unity-AI&QI. IEEE/CAA Journal of AutomaticaSinica. 8(3), 534-553 (2021)

[36] Zhang, Y., Chen, S., Wang, S., Yang, J., Philips, P.: Magnetic resonance brain image classification based on weighted-type fractional Fourier transform and nonparallel support vector machine. Int. J. Imaging Syst. Technol. 24(4), 317-327 (2015).

[37] G. Srinivasa Rao, D. Madhusudhanarao and P. Siva Prasad, **Simple Ternary Semi-rings**, The Global Journal of Mathematics & Mathematical Sciences, 9(2) (2016), 185-196.

[38] D. Madhusudhana Rao, G. Srinivasa Rao, **Special Elements in ternary semi rings**, International Journal of Engineering Research and Applications, 4(11) (2014), 123-130.

[39] G. Srinivasa Rao, D. Madhusudhana Rao, **A Study on Ternary Semi rings**, Int. J. of Math. Archive, 5(12) (2014), 24-30.

[40] G. Srinivasa Rao, D. Madhusudhana Rao, **Characteristics of Ternary Semi rings**, Int.J. of Engg. Res. and Mgt., 2(1) (2015), 3-6.

[41] G. Srinivasa Rao, A. Nagamalleswara Rao, P.L.N. Varma, D.Madhusudhana Rao, Ch. Ramprasad, **Prime Bi-interior ideals in TGSR**, Malaya Journal of Mathematika, Vol.9, No.1, pp:542-546, 2021.

[42] G. Srinivasa Rao, A. Nagamalleswara Rao, P.L.N. Varma, D. Madhusudhana Rao, Ch. Ramprasad, **Bi-interior ideals in TGSR**, Advances in Mathematics Scientific Journal, 10 (2021), No.3, pp: 1183-1195.