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Efficiency of Runge-Kutta-Fehlberg Method in Resolving Uncertainty: a Fuzzy Approach to Delay Differential Equations

[1,2]Ch. Subba Reddy, [3*]T. L. Yookesh
[1]Mentor in Mathematics, RGUKT,Nuzvid.
[2] Research Scholar, School of Applied Sciencies and Humanities,Deparment of Mathematics & Statistics
Vignan's Foundation for Science, Technology & Research, Guntur-522 213, INDIA
[3] Assistant Professor, School of Applied Sciencies and Humanities,Deparment of Mathematics & Statistics
Vignan's Foundation for Science, Technology & Research, Guntur-522 213, INDIA
E.Mail:*subbareddych@rguktn.ac.in anddrytl_sh@vignan.ac.in* *corresponding author

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Abstract:

This research delves into the examination of the convergence behavior of the Runge-Kutta-Fehlberg Method (RKFM) when applied to address the challenges posed by Delay Differential Equations (DDE) within the framework of fuzzy concepts. The integration of fuzzy logic is explored as a strategic means to grapple with the inherent uncertainty in obtained results, acknowledging the limitations of conventional approaches in dealing with nuanced uncertainties in DDEs. The core of the study involves a meticulous analysis of numerical outcomes, with a specific emphasis on evaluating the effectiveness and efficiency of the Runge-Kutta-Fehlberg method. The assessment includes a thorough comparison of results obtained through the RKFM against those derived from the Runge-Kutta 4th-order method, as well as benchmarking against the exact solution. The findings of this research not only validate the suitability of the RKFM for solving Delay Differential Equations but also highlight its comparative advantages over the traditional Runge-Kutta 4th-order method. The results contribute to an enhanced understanding of the performance and applicability of the RKFM in the context of fuzzy DDEs, thereby advancing the landscape of numerical methods for addressing complex dynamic systems.

Keywords: Approximate method, fuzzy number, convergence

1. Introduction

Exploring temporal dynamics across varied scenarios holds a central position in applied design research. This underscores the urgent need to enhance digital models and frameworks capable of adeptly managing frequently encountered ambiguous situations.

The utilization of fuzzy logic in this context is driven by the recognition that traditional methodologies may fall short in addressing the nuanced uncertainties that often characterize problems involving Delay Differential Equations. By introducing fuzzy logic into our analytical framework, we aim to enhance the robustness of our methodology and foster a more comprehensive understanding of the system's dynamics.The notion of a fuzzy derivative, which was originally introduced by Chang and Zadeh [7], underwent significant development through the contributions of Dubois and Prade [8], particularly in the context of the augmentation rule. Kandel and Byat [13] are credited with coining the term "Fuzzy differential equation." Subsequently, Abbasbandi and Allahviranloo [2] conducted mathematical computations to address fuzzy differential conditions related to the Seikkala derivative [19]. Jafari et al. [10] took on the challenging task of satisfying the nth requirement of fuzzy differential conditions using the variational iteration method but faced difficulties in the process.

Building on this, Allahviranloo et al. [1, 3] employed a correct indicator strategy to navigate the mathematical intricacies of differential equations with fuzziness. Driver [9] made a substantial contribution through a comprehensive book that elucidates general differential conditions and differential delay conditions. Bellen et al. [5, 6] provided mathematical solutions for Delay Differential Equations (DDE). In their work [14], Khastan et al. deliberated on the concepts of differential conditions for fuzzy changes under summed differentiability. Barzinji et al. [4] directed their focus toward linear differential differential frameworks for analyzing the stability of a coherent state.Abbasbandi and Allahwiranloo [2] laid the foundation for a numerical technique by introducing the fourth-order Runge-Kutta methodology. Their initial work focused on refining fuzzy differential conditions. Building upon this groundwork, Pederson and Sambandham [18] delved into the numerical treatment of fuzzy differential delay scenarios. They applied the Runge-Kutta method to handle these scenarios effectively. Expanding the scope, Al-Rawi et al. [21] extended the approach to address numerical challenges posed by differential carry constraints. In their study, they employed the fourth-order Runge-Kutta strategy to enhance the resolution of such challenges. In parallel investigations, K. Kanagarajan et al. [12, 11] explored alternative methods for handling fuzzy-delay differential equations. Specifically, they investigated the Runge-Kutta-Nystrom and the 5th-Runge-Kutta strategies to provide comprehensive insights into effective solutions for this class of equations. Further diversifying the application of Runge-Kutta techniques, V. Parimala et al. [17] employed the 2nd-Runge-Kutta technique to address fuzzy differential scenarios characterized by fuzzy initial conditions. This extension demonstrates the versatility of the Runge-Kutta family of methods in handling various types of fuzzy scenarios.

In a unique approach, Narayanamoorthy et al. [15, 16] combined Runge-Kutta's third technique with the Seikkal derivative [19] to resolve fuzzy differential situations. This combination of methods showcases the innovative ways researchers are exploring to enhance the capabilities of existing numerical techniques for tackling complex fuzzy scenarios. This current article serves as an extension of the work presented in [16], aiming to further explore the effectiveness of the time lag differential condition method within a fuzzy environment.

The crux of our endeavor lies in the comprehensive analysis of numerical outcomes, where we meticulously evaluate the performance of the Runge-Kutta-Fehlberg method. Our assessment involves a meticulous comparison of results obtained using the RKFM against those derived from the Runge-Kutta 4th order method, as well as a benchmarking against the exact solution. Through this rigorous comparative analysis, we aim to unveil the effectiveness and efficiency of the Runge-Kutta-Fehlberg method specifically within the context of solving Delay Differential Equations. The numerical results obtained not only serve to validate the suitability of the RKFM but also provide insights into its comparative advantages over the traditional Runge-Kutta 4th-order method. The meticulous examination of these results contributes to advancing our understanding of the applicability and performance of the RKFM in the realm of fuzzy Delay Differential Equations, thereby enriching the landscape of numerical methods for addressing complex dynamic systems.

2. Preliminaries

Definition 1.*A membership function is a mathematical representation in fuzzy logic that assigns a degree of membership between 0 and 1 to elements in a universe of discourse, indicating the extent to which they belong to a specific fuzzy set. It quantifies the degree of truth or membership for each element, allowing for a nuanced representation of uncertainty and imprecision in fuzzy systems.In general, a fuzzy set* $\overline{\mathrm{A}}$ *in* δ *is characterized by its membership function* $\overline{\mathrm{A}} = \{(x, \mu_{\overline{\mathrm{A(x)}}}) \mid x \in \delta\}$.

Definition 2.*In fuzzy set theory, an alpha-cut is a crisp set derived from a fuzzy set by retaining only those elements whose membership grades are greater than or equal to a specified threshold value, denoted as "alpha." The fuzzy alpha-cut essentially transforms a fuzzy set into a conventional, non-fuzzy set by considering only the elements that exhibit a sufficiently high degree of membership.*

Definition 3.*The membership function of a triangular fuzzy number is typically represented by a triangle on the membership scale, reflecting the gradual decrease in membership from the modal value towards the bounds. This type of fuzzy number is often used to model uncertainty or imprecision in situations where the exact value is uncertain but can be estimated to lie within a specific range.*

Mathematically, a triangular fuzzy number A can be represented as:

$$A = \{(x, \mu_A(x)) | a \leq x \leq c\},$$

Where $\mu_A(x)$ *is the mathematical function of the triangular fuzzy number A, defined as:*

$$\mu(x) = \begin{cases} 1 - \dfrac{a-x}{\alpha} & if \quad a - \alpha < x < a, \\ 1 - \dfrac{x-a}{\beta} & if \quad a < x < a + \beta, \\ 0 & otherwise. \end{cases}$$

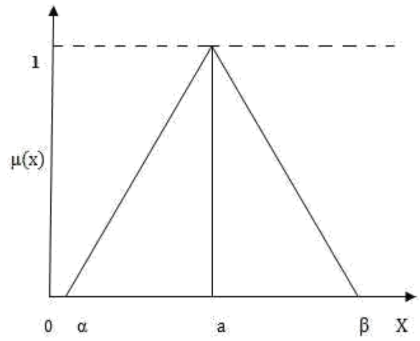

Figure 2.1: Triangular Fuzzy Number

In this representation, a, b, and c are the lower bound, modal value, and upper bound, respectively. Triangular fuzzy numbers are widely used in fuzzy logic, fuzzy systems, and decision-making under uncertainty.

Definition 4.*For arbitraryfuzzynumbers* $\psi = \left(\overline{\psi}, \underline{\psi}\right)$, $\vartheta = \left(\overline{\vartheta}, \underline{\vartheta}\right)$ *the quality*

$$D(\psi, \vartheta) = \left\{ \sup_{0 \leq \gamma \leq 1} \left|\underline{\psi}(\gamma) - \underline{\vartheta}(\gamma)\right|, \sup_{0 \leq \gamma \leq 1} \left|\overline{\psi}(\gamma) - \overline{\vartheta}(\gamma)\right| \right\}$$ *is the distance between* ψ *and* ϑ.

3. Examining The Convergence Behavior Of Fuzzy Delay Differential Equations

On considering the first-order differential equation [16],

$$\begin{cases} \mathcal{T}'(g) = h(g, \mathcal{T}(g)), g \in [g_0, G] \\ \mathcal{T}(g) = \mathcal{T}_0 \end{cases}$$

where $\mathcal{T}$ is a crisp function of $g, h(g, \mathcal{T})$. We represent the fuzzy function as $\mathcal{T} = [\underline{\mathcal{T}}, \overline{\mathcal{T}}]$. We have

$$[h(g,m)] = [\underline{h}(g,m), \overline{h}(g,m)]$$

where,

$$\underline{h}(g,m) = H[g, \underline{m}, \overline{m}];\ \overline{h}(g,m) = P[g, \underline{m}, \overline{m}]$$

By using the extension principle [16],

$$h(g, m(g))(s) = \sup\{m(g)(\tau) \backslash s = h(g,\tau)\}, s \in \mathbb{R}$$

From this it follows that

$$[h(g, m(g))]_\alpha = [\underline{h}(g, m(g); \alpha); \overline{h}(g, m(g); \alpha)], \alpha \in [0,1]$$

Where

$$\underline{h}(g, m(g); \alpha) = min\{h(g,v) \backslash v \in [m(g)]_\alpha\}; \overline{h}(g, m(g); \alpha) = max\{h(g,v) \backslash v \in [m(g)]_\alpha\} \tag{3.1}$$

Theorem 3.1.

Let $|h(g, \underline{v}) - h(g, \overline{v})| \leq p(g, |\underline{v}, \overline{v}|), g \geq 0, (\underline{v}, \overline{v}) \in \mathbb{R}$represent a continuous function that fulfills conditions $p: \mathbb{R} \to \mathbb{R}and\alpha \to p(g, \alpha)$. This guarantees the existence of a unique solution to the initial value problem on the domain, with initial condition being the sole solution to the differential equation (3.1).

Lemma 3.2. The numbers should be arranged in sequential order$\{C_i\}_{i=0}^{I}$ satisfy

$|C_{i+1}| \leq A|C_i| + B, 0 \leq i \leq I - 1$,for some given constants A and B. Then

$|C_i| \leq A^i|C_0| + B\frac{A^{i-1}}{A-1}, 0 \leq i \leq I.$

Lemma3.3: The numbers should be arranged in the following manner$\{C_i\}_{i=0}^{I}$ and$\{D_i\}_{i=0}^{I}$satisfy

$$|C_{i+1}| \leq |C_i| + A\min\{|C_i|, |D_i|\} + B,$$

$$|D_{i+1}| \leq |D| + A\max\{|C_i|, |D_i|\} + B$$

for some given positive constants A and B, then$V_i = |C_i| + |D_i|$where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$.

4. Runge-Kutta-Fehlberg Method

In this current discussion, the approach proposed for addressing the fuzzy delay differential equation entails integrating the Runge-Kutta-Fehlberg Method (RKFM) within a fuzzy framework. Let $Q = [Q_1, Q_2]$ represent the exact solution, with $q = [q_1, q_2]$ serving as an approximation for the ambiguous initial value problem. The resulting solution materializes as the computed phase points.

$$\rho = \frac{D - d_0}{M}, d_1 = d_0 + i\rho, 0 \leq i \leq M$$

Then we obtain the,

$$\underline{q}(l+1) = min\left(q_l + \frac{25}{216}l_1 + \frac{1408}{2565}l_3 + \frac{2197}{4104}l_4 - \frac{1}{5}l_5\right);$$
$$\overline{q}(l+1) = max\left(q_l + \frac{25}{216}l_1 + \frac{1408}{2565}l_3 + \frac{2197}{4104}l_4 - \frac{1}{5}l_5\right)$$

where

$$l_1(d,\alpha) = \{min\,\kappa(f(d_l,q_l));\ max\,\kappa(f(d_l,q_l))\}$$

$$l_2(d,\alpha) = \left\{min\,\kappa\left(f\left(d_l + \frac{1}{4}\kappa, q_l + \frac{1}{4}l_1\right)\right);\ max\,\kappa\left(f\left(d_l + \frac{1}{4}\kappa, q_l + \frac{1}{4}l_1\right)\right)\right\}$$

$$l_3(d,\alpha) = \left\{min\,\kappa\left(f\left(d_l + \frac{3}{8}\kappa, q_l + \frac{3}{32}l_1 + \frac{9}{32}l_2\right)\right);\ max\,\kappa\left(f\left(d_l + \frac{3}{8}\kappa, q_l + \frac{3}{32}l_1 + \frac{9}{32}l_2\right)\right)\right\}$$

$$l_4(d,\alpha) = \left\{min\,\kappa\left(f\left(d_l + \frac{12}{13}\kappa, q_l + \frac{1932}{2197}l_1 - \frac{7200}{2197}l_2 + \frac{72916}{2197}l_3\right)\right);\ max\,\kappa\left(f\left(d_l + \frac{12}{13}\kappa, q_l + \frac{1932}{2197}l_1 - \frac{7200}{2197}l_2 + \frac{72916}{2197}l_3\right)\right)\right\}$$

$$l_5(d,\alpha) = \left\{min\,\kappa\left(f\left(d_l + \kappa, q_l + \frac{439}{216}l_1 - 8l_2 + \frac{3680}{513}l_3 - \frac{845}{4104}l_4\right)\right);\ max\,\kappa\left(f\left(d_l + \kappa, q_l + \frac{439}{216}l_1 - 8l_2 + \frac{3680}{513}l_3 - \frac{845}{4104}l_4\right)\right)\right\}$$

$$l_6(d,\alpha) = \left\{min\,\kappa\left(f\left(d_l + \frac{1}{2}\kappa, q_l - \frac{8}{27}l_1 + 2l_2 - \frac{3544}{2565}l_3 + \frac{1859}{4104}l_4 - \frac{11}{40}l_5\right)\right);\ max\,\kappa\left(f\left(d_l + \frac{1}{2}\kappa, q_l - \frac{8}{27}l_1 + 2l_2 - \frac{3544}{2565}l_3 + \frac{1859}{4104}l_4 - \frac{11}{40}l_5\right)\right)\right\}$$

The parameter α governing observance is held constant, following which the precise and approximate solutions for d_M are expressed as follows:

$[Q(d_m)]_\alpha = [Q_1(d_m;\alpha);\ Q_2(d_m;\alpha)]$

$[q(d_m)]_\alpha = [q_1(d_m;\alpha);\ q_2(d_m;\alpha)], 0 \le m \le M$ (4.1)

In light of this, we observe the convergence$[q_1(d_m;\alpha);\ q_2(d_m;\alpha)]$converges to $[Q_1(d_m;\alpha);\ Q_2(d_m;\alpha)]$with respect to$\alpha \to 0$.

Theorem 4.1.

Assuming the form (4.1) with $\alpha\epsilon[0,1]$, then $\lim_{f\to 0}\left[\underline{q}(d_m)\right]_\alpha = \underline{Q}(d_m;\ \alpha)$; $\lim_{f\to 0}[\overline{q}(d_m)]_\alpha = \overline{Q}(d_m;\ \alpha)$.

5. Illustrative Example

Consider the a linear fuzzy delay differential equation as with initial condition $u(0) = 1$,

$q'(t) = \frac{1}{2}e^{t/2}q\left(\frac{t}{2}\right) + \frac{1}{2}q(t)$ (5.1) with$q(0) = (r, 2 - r)$.

Under the fuzzy condition the exact solution of (5.1) are

$$\min q(t,r) = re^t;$$

$\max q(t,r) = (2-r)e^t$ (5.2)

In employing the Runge-Kutta Fuzzy Method (RKFM) to approximate the solution for the provided equation (5.1), the resulting fuzzified form, as per Definition 3, is expressed as follows.

$$\min q'(t) = \frac{1}{2}e^{t/2}\underline{q}\left(\frac{t}{2}\right) + \frac{1}{2}\underline{q}(t);$$

$\max q'(t) = \frac{1}{2}e^{t/2}\overline{q}\left(\frac{t}{2}\right) + \frac{1}{2}\overline{q}(t)$ (5.3)

with the initial conditionmin $q(0) = r$; max $\mathrm{q}(0) = 2 - r)$.

The numerical results derived from the approximate method are presented in the table below for $t = 1$,

Table 5.1: Approximate Values

	Runge-Kutta Fehlberg method		Runge-Kutta fourth order method	
r	$\min q(t,r)$	$\max q(t,r)$	$\min q(t,r)$	$\max q(t,r)$
0	1.71879	3.71879	1.71827	3.71827
0.2	1.91879	3.51879	1.91827	3.51827
0.4	2.11879	3.31879	2.11827	3.31827
0.6	2.31879	3.11879	2.31827	3.11827
0.8	2.51879	2.91879	2.51827	2.91827
1	2.71879	2.71879	2.71827	2.71827

Table 5.2: Error Table

	Exact Solution		**RKFM**		**RK 4th**	
r	$\min Q(t,r)$	$\max Q(t,r)$	$\min q(t,r)$	$\max q(t,r)$	$\min q(t,r)$	$\max q(t,r)$
0	0	5.43657	1.71902	1.71756	1.71828	1.71829
0.2	0.54366	4.89291	1.37536	1.37390	1.37463	1.37463
0.4	1.087316	4.34924	1.03170	1.03024	1.03109	1.03010
0.6	1.63097	3.80560	0.68094	0.68659	0.68732	0.68732
0.8	2.17467	3.26194	0.33729	0.34293	0.34366	0.34366
1	2.71829	2.71829	0.00072	0.00072	0.00095	0.00095

The validity of the approximate values obtained through the RKFM (Runge-Kutta Fehlberg Method) and RH-4th (Runge-Hutta 4th Order) is substantiated in Tables 5.1, wherein these values are contrasted with the exact values presented in Table 5.2. Furthermore, a visual representation of the comparisons is depicted in Figures 5.1 and 5.2.

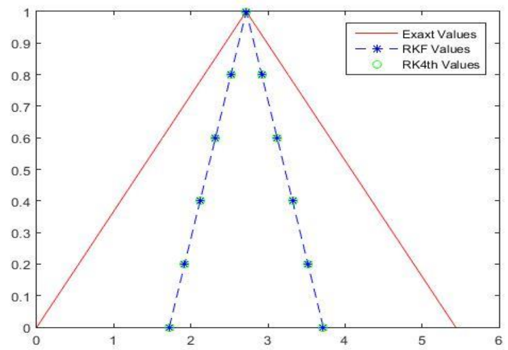

Figure 5.1: Comparison Between Approximate and Exact Solution

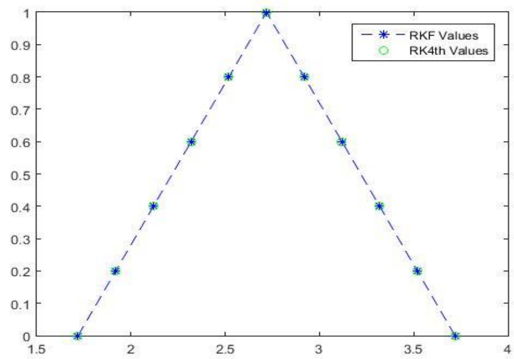

Figure 5.2: Comparision Between RKF and RK4th order Methods

6. Conclusion

In this investigation, we scrutinize the integration of the proposed Runge-Kutta Fuzzy Method (RKFM) for solving fuzzy delay difference equations. The assessment involves a numerical example that demonstrates the efficacy of the implemented approximate method. A comparative evaluation is conducted against the exact Runge-Kutta 4th-order system, elucidating the accuracy of the proposed RKFM. This rigorous analysis serves to validate and establish the stability of the RKFM in the context of solving fuzzy delay difference equations.

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