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A Research on Bipolar Valued Vague Normal Subrings of a Ring

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Abstract:

This paper introduces and discusses certain properties of bipolar valued vague normal subring of a ring.

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INTRODUCTION.

In this year 1965, first, fuzzy set had been introduced by Zadeh [15]. Succeeding years, fuzzy set was grown in different ways. The following are extension of fuzzy set, they are vague set, intuitionistic fuzzy set, bipolar valued fuzzy set and etc. Vague set by W. L. Gau and D. J. Buehrer [7]; F*uzzy group by* Azriel Rosenfeld [2]; Bipolar valued fuzzy subset by W.R.Zhang[16]; Vague group by RanjitBiswas [12]; Bipolar vague set by Cicily Flora. S and Arockiarani.I [4]; Bipolar valued fuzzy subgroup by Anitha.M.S., et.al.[1]; In similar way, [3], [11], [13], [14], [5], [6], [8], [9] and [10] were useful to write this paper.

1.PRELIMINARIES.

Definition 1.1 [15] *A map* $\mathfrak{R}: \mathbb{M} \to [0,1]$ *is called a fuzzy subset*(FS) *of* $\mathbb{M}$.

Definition 1.2 [7] *The ordered structure* $\mho = \{(\mathfrak{z}, [\mho_T(\mathfrak{z}), 1 - \mho_F(\mathfrak{z})]) : \mathfrak{z} \in \mathbb{W}\}$ *is called* a *vague set*(VS) *of* $\mathbb{w}$, *where* $\mho_T: \mathbb{w} \to [0,1]$ *is a truth membership* map and $\mho_F: \mathbb{w} \to [0, 1]$ is a false membership map, such that $\mho_T(\mathfrak{z}) + \mho_F(\mathfrak{z}) \leq 1$, *for all* $\mathfrak{z}$ *in* $\mathbb{W}$.

Definition 1.3 [7] *The interval* $[\mho_T(\mathfrak{z}), 1 - \mho_F(\mathfrak{z})]$ *is called the vague value of* $\mathfrak{z}$ *in* $\mho$

and it is denoted by $\mho(\mathfrak{z})$, *i.e.*, $\mho(\mathfrak{z}) = [\mho_T(\mathfrak{z}), 1 - \mho_F(\mathfrak{z})]$.

Example 1.4. $\mho$ = { < $\mathfrak{z}$, [0.5, 0.6] >, < $\mathfrak{v}$, [0.7, 0.8] >, < $\mathfrak{n}$, [0.4, 0.9] >} is a vague set of $\mathfrak{R} = \{\mathfrak{z}, \mathfrak{v}, \mathfrak{n}\}$.

Definition 1.5 [16] *The ordered structure* $\mathfrak{T} = \{(\mathfrak{z}, \mathfrak{T}^+(\mathfrak{z}), \mathfrak{T}^-(\mathfrak{z})) : \mathfrak{z} \in \mathbb{W}\}$ *is called* a bipolar *valued fuzzy subset of* $\mathbb{w}$, *where* $\mathfrak{T}^+: \mathbb{w} \to [0,1]$ *is a positive membership* map and $\mathfrak{T}^-: \mathbb{w} \to [-1,0]$ is a negative membership map.

Definition 1.6 [4] *The ordered structure* $\mho = \{(\mathfrak{z}, [\mho_T^+(\mathfrak{z}), 1 - \mho_F^+(\mathfrak{z})], [-1 - \mho_F^-(\mathfrak{z}), \mho_T^-(\mathfrak{z})]) : \mathfrak{z} \in \mathbb{W}\}$ *is called a bipolar valued vague subset of* $\mathbb{W}$, *where* $\mho_T^+ : \mathbb{W} \to [0,1], \mho_F^+ : \mathbb{W} \to [0,1], \mho_T^- : \mathbb{W} \to [-1, 0]$, *and* $\mho_F^- : \mathbb{W} \to [-1, 0]$ *are mapping such that*

$\mho_T^+(\mathfrak{z}) + \mho_F^+(\mathfrak{z}) \leq 1, -1 \leq \mho_T^-(\mathfrak{z}) + \mho_F^-(\mathfrak{z})$, *for all* $\mathfrak{z}$ *in* $\mathbb{W}$. Bipolar valued vague subset

$\mho$ is denoted as $\mho = \{(\mathfrak{z}, \mho^+(\mathfrak{z}), \mho^-(\mathfrak{z})) : \mathfrak{z} \in \mathbb{W}\}$, where $\mho^+(\mathfrak{z}) = [\mho_T^+(\mathfrak{z}), 1 - \mho_F^+(\mathfrak{z})]$ and $\mho^-(\mathfrak{z}) = [-1 - \mho_F^-(\mathfrak{z}), \mho_T^-(\mathfrak{z})]$. It is denoted as $B_{VV}S$

Example 1.7. $\mho$ = { < $\mathfrak{z}$, [0.5, 0.75], [–0.55, –0.32] >, < $\mathfrak{v}$, [0.7, 0.8], [–0.45, –0.23] >, < $\mathfrak{n}$, [0.4, 0.9], [–0.005, –0.002] >} is a $B_{VV}S$ of $\mathfrak{R} = \{\mathfrak{z}, \mathfrak{v}, \mathfrak{n}\}$.

Definition 1.8 [4] Let $\mho = \langle \mho^+, \mho^- \rangle$ and $\mathfrak{G} = \langle \mathfrak{G}^+, \mathfrak{G}^- \rangle$ be *two* $B_{VV}Ss$ *of a set* $\mathcal{W}$. *Then*

(i) $\mho \subset \mathfrak{G}$ *if and only if* $\mho^+(z) \leq \mathfrak{G}^+(z)$ and $\mho^-(z) \geq \mathfrak{G}^-(z)$, *for all* $z \in \mathcal{W}$.

(ii) $\mho \cap \mathfrak{G} = \{\langle z, \text{rmin}(\mho^+(z), \mathfrak{G}^+(z)), \text{rmax}(\mho^-(z), \mathfrak{G}^-(z)) \rangle / z \in \mathcal{W}\}$.

Definition 1.9 [5] *A* $B_{VV}S$ $\mathfrak{C} = \langle \mathfrak{C}^+, \mathfrak{C}^- \rangle$ *of a ring* $\mathfrak{Z}$ *is said to ba a bipolar* valued vague subring of $\mathfrak{Z}$ ($B_{VV}SR$) *if* $\mathfrak{C}$ *has*,

(i) $\mathfrak{C}^+(\mathfrak{y} - \mathfrak{w}) \geq rmin\{\mathfrak{C}^+(\mathfrak{y}), \mathfrak{C}^+(\mathfrak{w})\}$,

(ii) $\mathfrak{C}^+(\mathfrak{y}\mathfrak{w}) \geq rmin\{\mathfrak{C}^+(\mathfrak{y}), \mathfrak{C}^+(\mathfrak{w})\}$,

(iii) $\mathfrak{C}^-(\mathfrak{y} - \mathfrak{w}) \leq rmax\{\mathfrak{C}^-(\mathfrak{y}), \mathfrak{C}^-(\mathfrak{w})\}$,

(iv) $\mathfrak{C}^-(\mathfrak{y}\mathfrak{w}) \leq rmax\{\mathfrak{C}^-(\mathfrak{y}), \mathfrak{C}^-(\mathfrak{w})\}$, *for all* $\mathfrak{y}, \mathfrak{w} \in \mathfrak{Z}$,

where $rmin\{[\mathfrak{r}, \mathfrak{s}], [\mathfrak{t}, \mathfrak{u}]\} = [\min\{\mathfrak{r}, \mathfrak{t}\}, \min\{\mathfrak{s}, \mathfrak{u}\}]$ and

$rmax\{[\mathfrak{r}, \mathfrak{s}], [\mathfrak{t}, \mathfrak{u}]\} = [\max\{\mathfrak{r}, \mathfrak{t}\}, \max\{\mathfrak{s}, \mathfrak{u}\}]$.

Example 1.10. $\mathcal{W} = \{< 0, [0.006, 0.008], [-0.009, -0.006] >, < 1, [0.005, 0.007],$

$[-0.008, -0.005] >, < 2, [0.005, 0.007], [-0.008, -0.005] > \}$ *is a* $B_{VV}SR$ of the ring

$Z_3 = \{0, 1, 2\}$.

Definition 1.11 *A* $B_{VV}SR$ $\mathfrak{C} = \langle \mathfrak{C}^+, \mathfrak{C}^- \rangle$ *of a ring* $\mathfrak{Z}$ *is said to ba a bipolar* valued vague normal subring of $\mathfrak{Z}$ ($B_{VV}NSR$) *if* $\mathfrak{C}$ *has*,

(i) $\mathfrak{C}^+(\mathfrak{y}\mathfrak{w}) = \mathfrak{C}^+(\mathfrak{w}\mathfrak{y})$,

(ii) $\mathfrak{C}^-(\mathfrak{y}\mathfrak{w}) = \mathfrak{C}^-(\mathfrak{w}\mathfrak{y})$, *for all* $\mathfrak{y}, \mathfrak{w} \in \mathfrak{Z}$.

Definition 1.12. [5] *Let* $\mathfrak{K} = \langle \mathfrak{K}^+, \mathfrak{K}^- \rangle$ *be* $B_{VV}S$ of the set $\mathbb{N}_1$, the strongest B_{VV} *relation on* $\mathbb{N}_1$, that is a B_{VV} *relation* on $\mathfrak{K}$ *is* $\mathfrak{W} = \{\langle(\mathfrak{y}, \zeta), \mathfrak{W}^+(\mathfrak{y}, \zeta), \mathfrak{W}^-(\mathfrak{y}, \zeta)\rangle$ / for all $\mathfrak{y}, \zeta \in \mathbb{N}_1\}$, where $\mathfrak{W}^+(\mathfrak{y}, \zeta)$ = rmin$\{\mathfrak{K}^+(\mathfrak{y}), \mathfrak{K}^+(\zeta)\}$ and $\mathfrak{W}^-(\mathfrak{y}, \zeta)$ = rmax$\{\mathfrak{K}^-(\mathfrak{y}), \mathfrak{K}^-(\zeta)\}$, for all $\mathfrak{y}, \zeta \in \mathbb{N}_1$.

Definition 1.13. [5]*Let* $\mathfrak{K} = \langle \mathfrak{K}^+, \mathfrak{K}^- \rangle$ and $\mathfrak{W} = \langle \mathfrak{W}^+, \mathfrak{W}^- \rangle$ *be* $B_{VV}S$s of the sets $\mathfrak{V}_1$ and $\mathfrak{V}_2$

respectively. The product of $\mathfrak{K}$ and $\mathfrak{W}$, denoted by $\mathfrak{K} \times \mathfrak{W}$, is defined as $\mathfrak{K} \times \mathfrak{W} = \{\langle(\varkappa, \zeta), (\mathfrak{K}\times\mathfrak{W})^+(\varkappa, \zeta), (\mathfrak{K}\times\mathfrak{W})^-(\varkappa, \zeta)\rangle$ / for all $(\varkappa, \zeta) \in \mathfrak{V}_1 \times \mathfrak{V}_2\}$, where $(\mathfrak{K}\times\mathfrak{W})^+(\varkappa, \zeta)$ = rmin$\{\mathfrak{K}^+(\varkappa), \mathfrak{W}^+(\zeta)\}$ and $(\mathfrak{K}\times\mathfrak{W})^-(\varkappa, \zeta)$ = rmax$\{\mathfrak{K}^-(\varkappa), \mathfrak{W}^-(\zeta)\}$.

Definition 1.14. $Let\ \mathfrak{W} = \langle\ \mathfrak{W}^+, \mathfrak{W}^- \rangle$ be a $B_{VV}S$ of a set $\mathbb{N}. Then\ the\ height\ of\ \mathfrak{M}\ is$

$\mathfrak{H}(\mathfrak{M}) = \langle\ \mathfrak{H}(\mathfrak{M}^+),\ \mathfrak{H}(\mathfrak{M}^-) \rangle$ which is defined as $\mathfrak{H}(\mathfrak{M}^+) = rsup\mathfrak{M}^+(\omega)$ and

$\mathfrak{H}(\mathfrak{M}^-) = rinf\mathfrak{M}^-(\omega)$ for all $\omega \in \mathbb{N}$.

Definition 1.15. $Let\ \mathfrak{W} = \langle\ \mathfrak{W}^+, \mathfrak{W}^- \rangle$ be a $B_{VV}S$ of a set $\mathcal{B}. Then\ \oplus(\mathfrak{M}) = \langle\ \oplus(\mathfrak{M}^+), \oplus(\mathfrak{M}^-) \rangle$ is defined as $\oplus(\mathfrak{M}^+)(\omega) = \mathfrak{M}^+(\omega) + [1] - \mathfrak{H}(\mathfrak{M}^+)$ and $\oplus(\mathfrak{M}^-)(\omega) = \mathfrak{M}^-(\omega) - [1] - \mathfrak{H}(\mathfrak{M}^-)$ for all $\omega \in \mathcal{B}$.

2 – THEOREMS.

Theorem 2.1. [5] $If\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ is a $B_{VV}SR$ of a ring $\mathfrak{R}_1$, then $\H{U}^+(-8) = \H{U}^+(8)$, $\H{U}^-(-8) = \H{U}^-(8), \H{U}^+(\mathfrak{o}) \geq \H{U}^+(8),\ \H{U}^-(\mathfrak{o}) \leq \H{U}^-(8), for\ all\ 8 \in \mathfrak{R}_1$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{R}_1$.

Theorem 2.2. $If\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of a ring $\mathfrak{R}_1$, then $\H{U}^+(-8) = \H{U}^+(8)$, $\H{U}^-(-8) = \H{U}^-(8), \H{U}^+(\mathfrak{o}) \geq \H{U}^+(8),\ \H{U}^-(\mathfrak{o}) \leq \H{U}^-(8), for\ all\ 8 \in \mathfrak{R}_1$, where $\mathfrak{o}$ is an first operation identity element of $\mathfrak{R}_1$.

Proof. By the theorem 2.1, it can be easily shown.

Theorem 2.3. [5] $Let\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ be a $B_{VV}SR$ of the ring $\mathfrak{R}_1$. (i) If $\H{U}^+(\xi - \upsilon) = [0]$, then either $\H{U}^+(\xi) = [0]\ or\ \H{U}^+(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$; (ii) if $\H{U}^+(\xi\upsilon) = [0]$, then either $\H{U}^+(\xi) = [0] or\ \H{U}^+(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$; (iii) if $\H{U}^-(\xi - \upsilon) = [0]$, then either $\H{U}^-(\xi) = [0]\ or$ $\H{U}^-(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$; (iv) if $\H{U}^-(\xi\upsilon) = [0]$, then either $\H{U}^-(\xi) = [0]\ or\ \H{U}^-(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$.

Theorem 2.4. $Let\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ be a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of the ring $\mathfrak{R}_1$. (i) If $\H{U}^+(\xi - \upsilon) = [0]$, then either $\H{U}^+(\xi) = [0]\ or\ \H{U}^+(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$; (ii) if $\H{U}^+(\xi\upsilon) = [0]$, then either $\H{U}^+(\xi) = [0] or\ \H{U}^+(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$; (iii) if $\H{U}^-(\xi - \upsilon) = [0]$, then either $\H{U}^-(\xi) = [0]\ or$ $\H{U}^-(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$; (iv) if $\H{U}^-(\xi\upsilon) = [0]$, then either $\H{U}^-(\xi) = [0]\ or\ \H{U}^-(\upsilon) = [0], for\ all\ \xi, \upsilon \in \mathfrak{R}_1$.

Proof. By the theorem 2.3, it can be easily shown.

Theorem 2.5. [5] $If\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ is a $B_{VV}SR$ of the ring $\AA_1$, then $\mathfrak{S} = \{\mathfrak{h} \in \AA_1:$ $\H{U}^+(\mathfrak{h}) = [1] and\ \H{U}^-(\mathfrak{h}) = [-1]\}$ is either empty or a subring of $\AA_1$.

Theorem 2.6. $If\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of the ring $\AA_1$, then $\mathfrak{S} = \{\mathfrak{h} \in \AA_1:$ $\H{U}^+(\mathfrak{h}) = [1] and\ \H{U}^-(\mathfrak{h}) = [-1]\}$ is either empty or a subring of $\AA_1$.

Proof. By the theorem 2.5, it can be easily shown.

Theorem 2.7. [5] $If\ \H{U} = \langle\ \H{U}^+, \H{U}^- \rangle$ and $\mathfrak{F} = \langle\ \mathfrak{F}^+, \mathfrak{F}^- \rangle\ are\ two\ B_{VV}SRs$ of a ring $\hat{W}$, then their intersection Ű ∩ $\mathfrak{F}$ *is also a* $B_{VV}SR$ of Ŵ.

Theorem 2.8. *If* Ű $= \langle$ Ű$^{+}$, Ű$^{-} \rangle$ and $\mathfrak{F} = \langle \mathfrak{F}^{+}, \mathfrak{F}^{-} \rangle$ *are two* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}s$ of a ring Ŵ, then their intersection Ű ∩ $\mathfrak{F}$ *is also a* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Ŵ.

Proof. Let $\varkappa, \upsilon$ be in $\mathfrak{R}_1$. Let Ű ∩ $\mathfrak{F} = \mathfrak{U}$. *By Theorem* 2.7, $\mathfrak{K} \cap \mathfrak{W}$ *is also a* $B_{VV}SR$ of $\mathfrak{R}_1$. Then $\mathfrak{U}^{+}(\varkappa\upsilon)$ = rmin{Ű$^{+}(\varkappa\upsilon)$, $\mathfrak{F}^{+}(\varkappa\upsilon)$} = rmin{Ű$^{+}(\upsilon\varkappa)$, $\mathfrak{F}^{+}(\upsilon\varkappa)$} = $\mathfrak{U}^{+}(\upsilon\varkappa)$, ∀ $\varkappa, \upsilon$ in $\mathfrak{R}_1$. And $\mathfrak{U}^{-}(\varkappa\upsilon)$ = rmax{Ű$^{-}(\varkappa\upsilon)$, $\mathfrak{F}^{-}(\varkappa\upsilon)$} = rmax{Ű$^{-}(\upsilon\varkappa)$, $\mathfrak{F}^{-}(\upsilon\varkappa)$} = $\mathfrak{U}^{-}(\upsilon\varkappa)$, ∀ $\varkappa$, υ in $\mathfrak{R}_1$. Hence Ű ∩ $\mathfrak{F} = \mathfrak{U}$ *is also a* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of $\mathfrak{R}_1$.

Theorem 2.9. [5] *If* $\mathfrak{P}_1$, $\mathfrak{P}_2$, … and $\mathfrak{P}_m$ *are* $B_{VV}SR$s of the ring Å$_1$, then their intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \ldots \cap \mathfrak{P}_m$ *is also a* $B_{VV}SR$ of Å$_1$.

Theorem 2.10. *If* $\mathfrak{P}_1$, $\mathfrak{P}_2$, … and $\mathfrak{P}_m$ *are* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$s of the ring Å$_1$, then their intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \ldots \cap \mathfrak{P}_m$ *is also a* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å$_1$.

Proof. By the theorem 2.9, it can be easily shown.

Theorem 2.11. [5] *If* $\mathfrak{P}_1$, $\mathfrak{P}_2$, … *are* $B_{VV}SR$s of the ring Å$_1$, then their intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \ldots$ *is also a* $B_{VV}SR$ of Å$_1$.

Theorem 2.12. *If* $\mathfrak{P}_1$, $\mathfrak{P}_2$, … *are* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$s of the ring Å$_1$, then their intersection $\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \ldots$ *is also a* $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å$_1$.

Proof. By the theorem 2.11, it can be easily shown.

Theorem 2.13. [5] *Let* $\mathfrak{P}$ *be a* $B_{VV}S$ of a ring Å and $\mathfrak{M}$ be the stronget $\mathbb{B}_{\mathbb{VV}}$ relation of Å. Then $\mathfrak{P}$ is a $B_{VV}SR$ of Å if and only if $\mathfrak{M}$ is a $B_{VV}SR$ of Å×Å.

Theorem 2.14. *Let* $\mathfrak{P}$ *be a* $B_{VV}S$ of a ring Å and $\mathfrak{M}$ be the stronget B_{VV} relation of Å. Then $\mathfrak{P}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å if and only if $\mathfrak{M}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å×Å.

Proof. Let $\varrho, \upsilon, \zeta, \xi$ be in Å. Then (ϱ, ζ) and (υ, ξ) are in Å×Å. By Theorem 2.13, $\mathfrak{M}$ is a $B_{VV}SR$ of Å×Å, then $\mathfrak{W}^{+}[(\varrho, \zeta)(\upsilon, \xi)] = \mathfrak{W}^{+}(\varrho\upsilon, \zeta\xi)$ = rmin{$\mathfrak{P}^{+}(\varrho\upsilon)$, $\mathfrak{P}^{+}(\zeta\xi)$} = rmin {$\mathfrak{P}^{+}(\upsilon\varrho)$, $\mathfrak{P}^{+}(\xi\zeta)$}= $\mathfrak{W}^{+}(\upsilon\varrho, \xi\zeta) = \mathfrak{W}^{+}[(\upsilon, \xi)(\varrho, \zeta)]$, ∀ $(\varrho, \zeta), (\upsilon, \xi)$∈Å×Å. And $\mathfrak{W}^{-}[(\varrho, \zeta)(\upsilon, \xi)] = \mathfrak{W}^{-}(\varrho\upsilon, \zeta\xi)$ = rmax{$\mathfrak{P}^{-}(\varrho\upsilon)$, $\mathfrak{P}^{-}(\zeta\xi)$} = rmax{$\mathfrak{P}^{-}(\upsilon\varrho)$, $\mathfrak{P}^{-}(\xi\zeta)$} = $\mathfrak{W}^{-}(\upsilon\varrho, \xi\zeta)$ =$\mathfrak{W}^{-}[(\upsilon, \xi)(\varrho, \zeta)]$, ∀ $(\varrho, \zeta), (\upsilon, \xi)$∈ Å×Å. Hence $\mathfrak{W}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å×Å.

Conversely, assume that $\mathfrak{W}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å×Å. By Theorem 2.13, $\mathfrak{P}$ is a $B_{VV}SR$ of Å, rmin{$\mathfrak{P}^{+}(\varrho\upsilon)$, $\mathfrak{P}^{+}(\zeta\xi)$} = $\mathfrak{W}^{+}(\varrho\upsilon, \zeta\xi) = \mathfrak{W}^{+}[(\varrho, \zeta)(\upsilon, \xi)] = \mathfrak{W}^{+}[(\upsilon, \xi)(\varrho, \zeta)] = \mathfrak{W}^{+}(\upsilon\varrho, \xi\zeta)$ = rmin{$\mathfrak{P}^{+}(\upsilon\varrho)$, $\mathfrak{P}^{+}(\xi\zeta)$}, put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of Å, then $\mathfrak{P}^{+}(\varrho\upsilon) = \mathfrak{P}^{+}(\upsilon\varrho)$, ∀ ϱ, υ∈ Å. And rmax{$\mathfrak{P}^{-}(\varrho\upsilon)$, $\mathfrak{P}^{-}(\zeta\xi)$} = $\mathfrak{W}^{-}(\varrho\upsilon, \zeta\xi) = \mathfrak{W}^{-}[(\varrho, \zeta)(\upsilon, \xi)] = \mathfrak{W}^{-}[(\upsilon, \xi)(\varrho, \zeta)] = \mathfrak{W}^{-}(\upsilon\varrho, \xi\zeta)$ = rmax{$\mathfrak{P}^{-}(\upsilon\varrho)$, $\mathfrak{P}^{-}(\xi\zeta)$}, put $\zeta = \mathfrak{o}$ and $\xi = \mathfrak{o}$, where $\mathfrak{o}$ is an first operation identity element of Å, then $\mathfrak{P}^{-}(\varrho\upsilon) = \mathfrak{P}^{-}(\upsilon\varrho)$, ∀ ϱ, υ∈ Å. Hence $\mathfrak{P}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of Å.

Theorem 2.15. [5] $Let\ \mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_m$ be $B_{VV}Ss$ of a ring Å and $\mathfrak{M}$ be the strongest B_{VV} n-dimensional relation of Å. Then $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_m$ are $B_{VV}SR\ of$ Å $if\ and\ only\ if\ \mathfrak{M}\ is\ a\ B_{VV}SR\ of$ Å×Å…×Å (m times).

Theorem 2.16. $Let\ \mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_m$ be $\mathbb{B}_{\mathbb{VV}}\mathbb{S}s$ of a ring Å and $\mathfrak{M}$ be the strongest B_{VV} n-dimensional relation of Å. Then $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_m$ are $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}\ of$ Å $if\ and\ only\ if\ \mathfrak{M}\ is\ a\ \mathbb{B}_{\mathbb{VV}}\mathbb{NSR}\ of$ Å×Å…×Å (m times).

Proof. By the theorem 2.15, it can be easily shown.

Theorem 2.17. [5] If $\mathfrak{P}$ and $\mathfrak{W}$ are $B_{VV}SR$s of the rings $Å_1$ and $Å_2$ respectively, then $\mathfrak{P} \times \mathfrak{W}$ is a $B_{VV}SR$ of the ring $Å_1 \times Å_2$.

Theorem 2.18. If $\mathfrak{P}$ and $\mathfrak{W}$ are $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$s of the rings $Å_1$ and $Å_2$ respectively, then $\mathfrak{P} \times \mathfrak{W}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of the ring $Å_1 \times Å_2$.

Proof. Let ϱ, υ be in $Å_1$ and ζ, ξ be in $Å_2$. Then (ϱ, ζ), $(\upsilon, \xi) \in Å_1 \times Å_2$. By Theorem 2.17, $\mathfrak{P} \times \mathfrak{W}$ is a $B_{VV}SR$ of the ring $Å_1 \times Å_2$, then $(\mathfrak{P}\times\mathfrak{W})^+[(\varrho, \zeta)(\upsilon, \xi)] = (\mathfrak{P}\times\mathfrak{W})^+(\varrho\upsilon, \zeta\xi)$ = rmin$\{\mathfrak{P}^+(\varrho\upsilon), \mathfrak{W}^+(\zeta\xi)\}$= rmin$\{\mathfrak{P}^+(\upsilon\varrho), \mathfrak{W}^+(\xi\zeta)\} = (\mathfrak{P}\times\mathfrak{W})^+(\upsilon\varrho, \xi\zeta) = (\mathfrak{P}\times\mathfrak{W})^+[(\upsilon, \xi)(\varrho, \zeta)]$, $\forall\ (\varrho, \zeta), (\upsilon, \xi) \in Å_1 \times Å_2$. And $(\mathfrak{P}\times\mathfrak{W})^-[(\varrho, \zeta)(\upsilon, \xi)] = (\mathfrak{P}\times\mathfrak{W})^-(\varrho\upsilon, \zeta\xi)$ = rmax$\{\mathfrak{P}^-(\varrho\upsilon), \mathfrak{W}^-(\zeta\xi)\}$ = rmax$\{\mathfrak{P}^-(\upsilon\varrho), \mathfrak{W}^-(\xi\zeta)\} = (\mathfrak{P}\times\mathfrak{W})^-(\upsilon\varrho, \xi\zeta) = (\mathfrak{P}\times\mathfrak{W})^-[(\upsilon, \xi)(\varrho, \zeta)]$, $\forall\ (\varrho, \zeta), (\upsilon, \xi) \in Å_1 \times Å_2$. Hence $\mathfrak{P}\times\mathfrak{W}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of $Å_1 \times Å_2$**.**

Theorem 2.19. [5] $If\ \mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_m\ are\ B_{VV}SR$s of the rings $Å_1, Å_2, \dots, Å_m$

respectively, then $\mathfrak{P}_1 \times \mathfrak{P}_2 \times \dots \times \mathfrak{P}_m$ is a $B_{VV}SR$ of the ring $Å_1 \times Å_2 \times \dots \times Å_m$.

Theorem 2.20. $If\ \mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_m\ are\ \mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$s of the rings $Å_1, Å_2, \dots, Å_m$

respectively, then $\mathfrak{P}_1 \times \mathfrak{P}_2 \times \dots \times \mathfrak{P}_m$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of the ring $Å_1 \times Å_2 \times \dots \times Å_m$.

Proof. By the theorem 2.19, it can be easily shown.

Theorem 2.21. [5] If $\mathfrak{P}$ is a $B_{VV}SR\ of\ the\ ring\ Å_1$, then $\oplus(\mathfrak{P})$ is a $B_{VV}SR$ of $Å_1$.

Theorem 2.22. If $\mathfrak{P}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}\ of\ the\ ring\ Å_1$, then $\oplus(\mathfrak{P})$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of $Å_1$.

Proof. Let ϱ, υ be in $Å_1$. By Theorem 2.21, $\oplus(\mathfrak{P})$ is a $B_{VV}SR$ of $Å_1$, $\oplus(\mathfrak{P}^+)(\varrho\upsilon) = \mathfrak{P}^+(\varrho\upsilon)+[1]-\mathfrak{H}(\mathfrak{P}^+) = \mathfrak{P}^+(\upsilon\varrho)+[1]-\mathfrak{H}(\mathfrak{P}^+) = \oplus(\mathfrak{P}^+)(\upsilon\varrho)$, $\forall\ \varrho, \upsilon \in Å_1$. And $\oplus(\mathfrak{P}^-)(\varrho\upsilon) = \mathfrak{P}^-(\varrho\upsilon)+[1]-\mathfrak{H}(\mathfrak{P}^-) = \mathfrak{P}^-(\upsilon\varrho)+[1]-\mathfrak{H}(\mathfrak{P}^-) = \oplus(\mathfrak{P}^-)(\upsilon\varrho)$, $\forall\ \varrho, \upsilon \in Å_1$. Hence $\oplus(\mathfrak{P})$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}$ of $Å_1$.

Theorem 2.23. [5] Let $\mathfrak{P}$ is a $B_{VV}SR\ of\ the\ ring\ Å_1$.

(i) Then $\oplus(\mathfrak{P}^+)(\mathfrak{o}) = [1]\ and\ \oplus(\mathfrak{P}^-)(\mathfrak{o}) = [-1], where\ \mathfrak{o}\ is\ an\ identity\ of\ Å_1$.

(ii) $\mathfrak{P}^+(\mathfrak{o}) = [1]\ and\ \mathfrak{P}^-(\mathfrak{o}) = [-1] \Leftrightarrow \oplus(\mathfrak{P}^+)(\mathfrak{w}) = \mathfrak{P}^+(\mathfrak{w})\ and$

$\oplus(\mathfrak{P}^-)(\mathfrak{w}) = \mathfrak{P}^-(\mathfrak{w}), for\ all\ \mathfrak{w} \in Å_1, where\ \mathfrak{o}\ is\ an\ identity\ of\ Å_1$.

(iii) $For\ \mathfrak{w} \in Å_1, \mathfrak{P}^+(\mathfrak{w}) = \mathfrak{P}^+(\mathfrak{o})\ and\ \mathfrak{P}^-(\mathfrak{w}) = \mathfrak{P}^-(\mathfrak{o}) \Leftrightarrow$

$\oplus(\mathfrak{P}^+)(\mathfrak{w}) = [1]\ and\ \oplus(\mathfrak{P}^-)(\mathfrak{w}) = [-1], where\ \mathfrak{o}\ is\ an\ identity\ of\ \text{Å}_1.$

(iv) $For\ \mathfrak{w} \in \text{Å}_1$, if $\mathfrak{P}^+(\mathfrak{w}) = [1] and\ \mathfrak{P}^-(\mathfrak{w}) = [-1], then \oplus (\mathfrak{P}^+)(\mathfrak{w}) = [1]\ and$ $\oplus(\mathfrak{P}^-)(\mathfrak{w}) = [-1].$

(v) $\oplus(\oplus(\mathfrak{P})) = \oplus(\mathfrak{P}).$

(vi) $\oplus(\mathfrak{P})\ is\ a\ \mathbb{B}_{\mathbb{VNV}}\mathbb{SR}\ of\ the\ ring\ \text{Å}_1 containing\ \mathfrak{P}.$

(vii) $\mathfrak{P}\ is\ a\ \mathbb{B}_{\mathbb{VNV}}\mathbb{SR}\ of\ the\ ring\ \text{Å}_1 \Leftrightarrow \oplus(\mathfrak{P}) = \mathfrak{P}.$

(viii) $If\ \oplus(\mathfrak{P}) \subseteq \mathfrak{W}, \mathfrak{W}\ is\ a\ \ B_{VV}SR\ of\ the\ ring\ \text{Å}_1, then\ \mathfrak{W}\ is\ a\ \mathbb{B}_{\mathbb{VNV}}\mathbb{SR}.$

Theorem 2.24. Let $\mathfrak{P}$ is a $\mathbb{B}_{\mathbb{VV}}\mathbb{NSR}\ of\ the\ ring\ \text{Å}_1.$

(i) Then $\oplus(\mathfrak{P}^+)(\mathfrak{o}) = [1\ ] and\ \oplus(\mathfrak{P}^-)(\mathfrak{o}) = [-1], where\ \mathfrak{o}\ is\ an\ identity\ of\ \text{Å}_1.$

(ii) $\mathfrak{P}^+(\mathfrak{o}) = [1] and\ \mathfrak{P}^-(\mathfrak{o}) = [-1] \Leftrightarrow \oplus(\mathfrak{P}^+)(\mathfrak{w}) = \mathfrak{P}^+(\mathfrak{w})\ and$

$\oplus(\mathfrak{P}^-)(\mathfrak{w}) = \mathfrak{P}^-(\mathfrak{w}), for\ all\ \mathfrak{w} \in \text{Å}_1, where\ \mathfrak{o}\ is\ an\ identity\ of\ \text{Å}_1.$

(iii) $For\ \mathfrak{w} \in \text{Å}_1, \mathfrak{P}^+(\mathfrak{w}) = \mathfrak{P}^+(\mathfrak{o})\ and\ \mathfrak{P}^-(\mathfrak{w}) = \mathfrak{P}^-(\mathfrak{o}) \Leftrightarrow$ $\oplus(\mathfrak{P}^+)(\mathfrak{w}) = [1]\ and\ \oplus(\mathfrak{P}^-)(\mathfrak{w}) = [-1], where\ \mathfrak{o}\ is\ an\ identity\ of\ \text{Å}_1.$

(iv) $For\ \mathfrak{w} \in \text{Å}_1$, if $\mathfrak{P}^+(\mathfrak{w}) = [1] and\ \mathfrak{P}^-(\mathfrak{w}) = [-1], then \oplus (\mathfrak{P}^+)(\mathfrak{w}) = [1]\ and$ $\oplus(\mathfrak{P}^-)(\mathfrak{w}) = [-1].$

(v) $\oplus(\oplus(\mathfrak{P})) = \oplus(\mathfrak{P}).$

(vi) $\oplus(\mathfrak{P})\ is\ a\ \mathbb{B}_{\mathbb{VNV}}\mathbb{NSR}\ of\ the\ ring\ \text{Å}_1 containing\ \mathfrak{P}.$

(vii) $\mathfrak{P}\ is\ a\ \mathbb{B}_{\mathbb{VNV}}\mathbb{NSR}\ of\ the\ ring\ \text{Å}_1 \Leftrightarrow \oplus(\mathfrak{P}) = \mathfrak{P}.$

(viii) $If\ \oplus(\mathfrak{P}) \subseteq \mathfrak{W}, \mathfrak{W}\ is\ a\ \mathbb{B}_{\mathbb{VV}}\mathbb{NSR}\ of\ the\ ring\ \text{Å}_1, then\ \mathfrak{W}\ is\ a\ \mathbb{B}_{\mathbb{VNV}}\mathbb{NSR}.$

Proof. By the theorem 2.23, it can be easily shown.

CONCLUSION

Using the above theorems, we can find more results. It can be extended into different types of algebra.

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