Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 7s (2024)

# Common fixed point theorems for weakly compatible mappings in Multiplicative Metric Space

**Chiranjeevi Perala[1],Thirupathi Thota[2], Srinivas Veladi[3]**

[1]. Department of Mathematics, Osmania University, Hyderabad, Telangana ,India,
Email: chiranjeeviperala@gmail.com

[2].Department of Mathematics, Sreenidhi Institute of Science and Technology, Hyderabad,Telangana, India
Email: thotathirupathi1986@gmail.com

[3].Department of Mathematics, University College of Science, Osmania University, Hyderabad,Telangana, India
Email:srinivasmaths4141@gmail.com

***Article History:***

***Received:*** *04-06-2024*

***Revised:*** *03-07-2024*

***Accepted:*** *30-07-2024*

**Abstract**

The aim of this paper is to establish three common fixed point theorems in Multiplicative Metric Space (MMS). By utilizing the conditions concepts of non-continuous and non-compatible mappings.The first theoem is generated by applying the concepts of semi-compatible mapping, WCM and reciprocally continuous mappings.The second theorem is established by using the concepts of strongly semi compatible mappings,conditionally reciprocally continuous mappings and and OWC mappings. These conditions are weaker than the existing conditions like compatible mappings and cotinuous which generalizes the theorem of Afrah A. N. Abdou.

**Keywords**: Multiplicative Metric Space (MMS), fixed point, semi-compatible mapping, reciprocally continuous mappings , weakly compatible mapping, strongly semi compatible mappings, conditionally reciprocally continuous mappings and and OWC mappings.

**2020 Mathematics Subject Classification**: 54H25, 47H10, 54E50.

## 1. Introduction

Fixed point theory is one of the most exciting problems in current mathematics, and it may be an existing area of functional analysis.Furthermore, this topic has served as a platform for various researchers during the last several years. Within the subject of analysis,the theory of metric spaces has grown significantly.We know that the set of positive real numbers $\mathbb{R}^+$ is not complete in metric space. Inorder to overcome this problem the concept known as MMS was introduced by Bashirove in 2008. Further Ozavsar and Cevikel in 2017 investigated and developed the multiplicative contraction principle and proved some common fixed point results. Thereafter, the theory of a multiplicative metric space has been developed by many authors[2,3,4,5,6,7,8,9].In this process Afrah A. N. Abdou [1] poved a theorem in MMS in 2016.The purpose of this paper is to prove two common fixed point theorems on MMS utilizing the concept like semi-compatible mapping,WCM,reciprocally continuous mappings,strongly semi compatible mappings,conditionally reciprocally continuous mappings and and OWC mappings. Additionally we provided three examples to validate our results. We first provide some helpful definitions and examples before presenting our findings.

**Preliminaries**

**Definition 2.1** Let $X \neq \emptyset$ set and $d: X \times X \rightarrow R^{+}$ then $(X, d)$ is said to be MMS if it meets the requirements as below:
(i) $d(u, \mathrm{v}) \geq 1$ for all $u, \mathrm{v} \in X$ and $d(u, \mathrm{v}) = 1 \Leftrightarrow u = \mathrm{v}$
(ii) $d(u, \mathrm{v}) = d(\mathrm{v}, u)$ for all $u, \mathrm{v} \in X$
(iii) $d(u, \mathrm{v}) \leq d(u, w) \cdot d(w, \mathrm{v})$ for all $u, v, \mathrm{w} \in X$
(multiplicative triangle inequality)

**Definition 2.2** A sequence $\{u_j\}$ in a MMS $(X, d)$ is said to be
(i) multiplicative convergent sequence to $u$ if for every multiplicative open ball $B_{\delta}(u) = \{\zeta \mid d(u, v) < \delta\}, \delta > 1$, thereexists a positive integer N such that $u_j \in B_{\delta}(u)$ for all $j \geq 1$ i.e $d(u_j, u) \rightarrow 1$ as $j \rightarrow \infty$.
(ii) multiplicative cauchy sequence if for all $\delta \geq 1 \ \exists \ N \in N$ such that $d(u_j, u_k) < \delta$ for all $j, k > N$ i.e $d(u_j, u_k) \rightarrow 1$ as $j, k \rightarrow \infty$.

**Definition 2.3** The pair of mapping $(G, J)$ of a MMS$(X, d)$ is said to be
(i) Compatible if $\lim_{j\to\infty} d(GJu_j, JGu_j) = 1$, whenever a sequence $\{u_j\}$ in $X$ like that $Gu_j = Ju_j = \gamma$ for some $\gamma \in X$.
(ii) Weakly compatible if $G\gamma = J\gamma$ for some $\gamma \in X$ such that $GJ\gamma = JG\gamma$.

(iii) Occasionally weakly compatible if for $\gamma \in X$ such that $G\gamma = J\gamma$ implies that $d(GJ\gamma, JG\gamma) = 1$.
(iii) Reciprocally continuous if $\lim_{j\to\infty} d(GJu_j, G\gamma) = 1$ $\lim_{j\to\infty} d(JGu_j, J\gamma) = 1$ for a sequence $\{u_j\}$ in $X$ such that $\lim_{j\to\infty} Gu_j = \lim_{j\to\infty} Ju_j = \gamma$ for some $\gamma \in X$.

(iv) Semi compatible if $\lim_{j\to\infty} d(JGu_j, G\gamma) = 1$ whenever $\{u_j\}$ is a sequence in $X$ such that $\lim_{j\to\infty} Gu_j = \lim_{j\to\infty} Ju_j = \gamma$ for some $\gamma \in X$.

(v) Conditionally semi compatible if a sequence $\{u_j\}$ is a sequence in $X$ such that $\lim_{j\to\infty} Gu_j = \lim_{j\to\infty} Ju_j$ is non empty for a sequence $\{v_j\} \in X$ satisfying $\lim_{j\to\infty} Gv_j = \lim_{j\to\infty} Jv_j = \mu$ for some $\mu \in X$ then $\lim_{j\to\infty} d(JGu_j, G\gamma) = 1$ and $\lim_{j\to\infty} d(GJu_j, J\gamma) = 1$.

(vi) Strongly semi-compatible mappings if they satisfy OWC and conditionally semi-compatible property.

(vii) Conditionally reciprocally continuous if and only if $\lim_{j\to\infty} d(GJu_j, G\gamma) = 1$ *and* $\lim_{j\to\infty} d(JGu_j, J\gamma) = 1$ whenever a sequence $\{u_j\} \in X$ satisfying $\lim_{j\to\infty} Gu_j = \lim_{j\to\infty} Ju_j$ is non-empty there exists another sequence $\{v_j\} \in X$ satisfying $\lim_{j\to\infty} Gv_j = \lim_{j\to\infty} Jv_j = \gamma$ as $\gamma \in X$.

Now we present the example for strongly semi-compatible mappings and conditionally reciprocally continuous mappings.

**Example 2.1:** Let $X = [1,6]$ and $d: X \times X \rightarrow [1, \infty)$ be defined as $d(\beta, \eta) = e^{|\beta - \eta|}$ $\forall \beta, \eta \in X$ then $(X, d)$ is a MMS.

Define the self-mappings $G$ and $J$ as

$$G(\eta) = \begin{cases} 3+\eta^2 & \text{if } 1 \le \eta < 2 \\ \dfrac{2+\eta}{4} & \text{if } 2 \le \eta \le 6 \end{cases} \quad \text{and} \quad J(\eta) = \begin{cases} 4 & \text{if } 1 \le \eta < 2 \\ \eta - 1 & \text{if } 2 \le \eta \le 6. \end{cases}$$

Consider a sequence $\{\eta_k\}$ given by $\eta_k = 1 + \frac{1}{k}$ for $k > 0$.

$\lim_{k\to\infty} G(\eta_k) = \lim\limits_{k\to\infty} G\left(1+\frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[3+\left(1+\frac{1}{k}\right)^2\right] = 4$ and

$\lim_{k\to\infty} J(\eta_k) = \lim\limits_{k\to\infty} J\left(1+\frac{1}{k}\right) = \lim\limits_{k\to\infty} 4 = 4.$

Therefore $\lim_{k\to\infty} G(\eta) = \lim_{k\to\infty} J(\eta) = 4 = \mu_1$ is non-empty.

Consider another sequence $\{\eta_k\}$ given by $\beta_k = 2 + \frac{1}{k}$ for $k > 0$.

$\lim_{k\to\infty} G(\beta_k) = \lim\limits_{k\to\infty} G\left(2+\frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[\frac{2+\left(2+\frac{1}{k}\right)}{4}\right] = 1$ and

$\lim_{k\to\infty} J(\beta_k) = \lim\limits_{k\to\infty} J\left(2+\frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[2+\frac{1}{k}-1\right] = 1.$

Therefore $\lim_{k\to\infty} G(\eta) = \lim_{k\to\infty} J(\eta) = 1 = \mu_2$ (say) .

This gives $G\mu_2 = G(1) = 4$ and $J\mu_2 = J(1) = 4$.

Further $\lim_{k\to\infty} GJ(\beta_k) = \lim_{k\to\infty} GJ\left(2+\frac{1}{k}\right) = \lim_{k\to\infty} G\left[1+\frac{1}{k}\right] = \lim\limits_{k\to\infty}\left[3+\left(1+\frac{1}{k}\right)^2\right] = 4$ and

$$\lim_{k\to\infty} JG(\beta_k) = \lim_{k\to\infty} JG\left(2+\frac{1}{k}\right) = \lim_{k\to\infty} J\left[\frac{2+\left(2+\frac{1}{k}\right)}{4}\right] = \lim_{k\to\infty} J\left[1+\frac{1}{k}\right] = 4.$$

Therefore $\lim_{k\to\infty} d(GJ\beta_k, J\mu_2) = 1$ and$\lim_{k\to\infty} d(JG\beta_k, G\mu_2) = 1$.

This shows that the couple $(G, J)$ is conditionally semi-compatible in MMS.

Now $G(1) = J(1) = 4$ and $(2) = J(2) = 1$ , showing that 1,2 are coincidence points.

Then $GJ(2) = JG(2) = 4$ implies $d\big(GJ(2), JG(2)\big) = 1$ and $GJ(1) = \frac{3}{2}$ ,$JG(1) = 3$ implies $d\big(GJ(1), JG(1)\big) \neq 1$.

As a result, the two self maps $G$ and $J$ are OWC.

Therefore the self maps $G$ and $J$ are strongly semi-compatible mappings in MMS.

**Example 2.2:** Let $X = [0,4]$ and $d: X \times X \to [1, \infty)$ be defined as $d(\beta, \eta) = e^{|\beta - \eta|}$ $\forall\, \beta, \eta \in X$ then $(X, d)$ is a MMS.

Define the self-mappings $G$ and $I$ as

$$G(\eta) = \begin{cases} 3+\eta^2 & \text{if } 0 \le \eta \le 1 \\ 5-\eta & \text{if } 1 < \eta \le 4 \end{cases} \quad \text{and} \quad I(\eta) = \begin{cases} 3+\eta & \text{if } 0 \le \eta \le 1 \\ \eta - 3 & \text{if } 1 < \eta \le 4. \end{cases}$$

Consider a sequence sequences $\eta_k = \left\{4 - \frac{1}{k}\right\}$ where $k > 0$.

$\lim_{k\to\infty} G(\eta_k) = \lim_{k\to\infty} G\left(4 - \frac{1}{k}\right) = \lim_{k\to\infty}\left[5 - \left(4 - \frac{1}{k}\right)\right] = 1$ and

$\lim_{k\to\infty} I(\eta_k) = \lim_{k\to\infty} I\left(4 - \frac{1}{k}\right) = \lim_{k\to\infty}\left[\left(4 - \frac{1}{k}\right) - 3\right] = 1.$

Therefore $\lim_{k\to\infty} G(\eta) = \lim_{k\to\infty} I(\eta) = 1 = \mu_1$ (non-empty).

Furether $\lim_{k\to\infty} GI(\eta_k) = 4$ and $\lim_{k\to\infty} IG(\eta_k) = -2$.

Which gives $\lim_{k\to\infty} d(GI\eta_k, IG\eta_k) \neq 1$.

Hence the couple $(G, I)$ is not compatible.

Consider a sequence sequences $\beta_k = \left\{1 - \frac{1}{k}\right\}$ where $k > 0$.

$\lim_{k\to\infty} G(\beta_k) = \lim_{k\to\infty} G\left(1 - \frac{1}{k}\right) = \lim_{k\to\infty}\left[3 \pm \left(1 - \frac{1}{k}\right)^2\right] = 4$ and

$\lim_{k\to\infty} I(\beta_k) = \lim_{k\to\infty} I\left(1 - \frac{1}{k}\right) = \lim_{k\to\infty}\left[3 + 1 - \frac{1}{k}\right] = 4.$

Therefore $\lim_{k\to\infty} G(\beta_k) = \lim_{k\to\infty} I(\beta_k) = 4 = \mu_2$ (say).

Furthermore $\lim_{k\to\infty} GI\big((\beta_k)\big) = 1$ and $\lim_{k\to\infty} IG\big((\beta_k)\big) = 1$.

Also we absorve that $G\mu_2 = I\mu_2 = 1$.

Thus $\lim_{k\to\infty} d(GI\beta_k, G\mu_2) = 1$ but $\lim_{k\to\infty} d(IG\beta_k, I\mu_2) \neq 1$.

Hence two self-maps G and I are conditionally reciprocally continuous in MMS.

Afrah A.N.Abdou [1] established the following Theorem in MMS.

**Theorem 2.1**

Assume that $(X, d)$ is a complete MMS and the mappings $A, S, B$ and $T$ are defined on $X$ such that

(B1) $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$

(B2)

$$d(S\alpha, T\beta) \leq \left[\phi\left(max\left\{d(A\alpha, B\beta), \frac{d(A\alpha, S\alpha)d(B\beta, T\beta)}{1 + d(A\alpha, B\beta)}, \frac{d(A\alpha, T\beta)d(B\beta, A\alpha)}{1 + d(A\alpha, B\beta)}\right\}\right)\right]^{\lambda}$$

for all $\alpha, \beta, \in X$ ,where $0 < \lambda < 1$.

Where $\phi: [0, \infty) \to [0, \infty)$ is a continuous and monotone increasing function such that $\phi(t) < t$ for all $t > 0$.

(B3) one of $A$,$B$, $S$, and $T$is continuous

(B4) both the pairs $(A, S)$ and $(B, T)$ are compatible.

Then the four maps $A, B, S$ and $T$ share a unique common fixed point in $X$.

We now generalize Theorem 2.1 as below.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 7s (2024)

Now we proceed to our main result.

## 3. Results and Discussion

### 3.1 Theorem:

Suppose that in a complete MMS $(X,d)$, the four self-mappings $A,S,B$ and $T$ meeting the requirements

(C1) $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$

(C2)

$$d(Su,Tv) \leq \left[\phi\left(max\left\{d(Au,Bv), \frac{d(Au,Su)d(Bv,Tv)}{1+d(Au,Bv)}, \frac{d(Au,Tv)d(Bv,Au)}{1+d(Au,Bv)}\right\}\right)\right]^{\lambda}$$

for all $u,v \in X$ ,where $0 < \lambda < 1$.

Where $\phi: [0,\infty) \to [0,\infty)$ is a continuous and monotone increasing function such that $\phi(t) < t$ for all $t > 0$.

(C3) the pair $(B,T)$ is reciprocally continuous and semi-compatible

(C4) the pair $(A,S)$ is weakly compatible mappings.

Then there exists a unique common fixed point for the above mappings.

**Proof:** Since By $S(X) \subseteq B(X)$, we can consider a point $u_0 \in X$ there exists $u_1 \in X$ such that $Su_0 = Bu_1 = v_0$.At this $u_1 \in X$ ∃ a point $u_2$ in $X \ni Tu_1 = Au_2 = v_1$ and so on.

Likewise, we are able to define $Bu_{2j+1} = Su_{2j} = v_{2j}$; $Au_{2j+2} = Tu_{2j+1} = v_{2j+1}$ for $j = 0,1,2, ....$

Now it is possible to establish that th e sequence $\{v_j\}$

$$d(v_{2j}, v_{2j+1}) = d(Su_{2j}, Tu_{2j+1})$$

$$\leq \left[\phi\left(max\left\{d(Au_{2j}, B\beta), \frac{d(Au_{2j}, Su_{2j})d(Bu_{2j+1}, Tu_{2j+1})}{1+d(Au_{2j}, Bu_{2j+1})}, \frac{d(Au_{2j}, Tu_{2j+1})d(Bu_{2j+1}, Au_{2j})}{1+d(Au_{2j}, Bu_{2j+1})}\right\}\right)\right]^{\lambda}$$

$$\leq \left[\phi\left(max\{d(v_{2j-1}, v_{2j}), d(v_{2j}, v_{2j+1}), d(v_{2j-1}, v_{2j+1})\}\right)\right]^{\lambda}$$

$$\leq \left[\phi\left(max\{d(v_{2j-1}, v_{2j}). d(v_{2j}, v_{2j+1})\}\right)\right]^{\lambda}$$

$$\leq \left[d(v_{2j-1}, v_{2j})\right]^{\lambda}.\left[d(v_{2j}, v_{2j+1})\right]^{\lambda}$$

which implies that

$$d(v_{2j}, v_{2j+1}) \leq \left[d(v_{2j-1}, v_{2j})\right]^{\frac{\lambda}{1-\lambda}} = \left[d(v_{2j-1}, v_{2j})\right]^{h}$$

Where $h = \frac{\lambda}{1-\lambda} \in (0,1)$. Similaly,we have

$$d(v_{2j+2}, v_{2j+1}) = d(Su_{2j+2}, Tu_{2j+1})$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 7s (2024)

$$\leq \left[\phi\left(max\left\{d\left(Au_{2j+2},Bu_{2j+1}\right),\frac{d\left(Au_{2j+2},Su_{2j+2}\right)d\left(Bu_{2j+1},Tu_{2j+1}\right)}{1+d\left(Au_{2j+2},Bu_{2j+1}\right)},\frac{d\left(Au_{2j+2},Tu_{2j+1}\right)d\left(Bu_{2j+1},Au_{2j+2}\right)}{1+d\left(Au_{2j+2},Bu_{2j+1}\right)}\right.\right.$$

$$\leq \left[\phi\left(max\left\{d\left(v_{2j},v_{2j+1}\right),d\left(v_{2j+2},v_{2j+1}\right),d\left(v_{2j},v_{2j+1}\right)\right\}\right)\right]^{\lambda}$$

$$\leq \left[\phi\left(max\left\{d\left(v_{2j},v_{2j+1}\right).d\left(v_{2j+2},v_{2j+1}\right)\right\}\right)\right]^{\lambda}$$

$$\leq \left[d\left(v_{2j},v_{2j+1}\right)\right]^{\lambda}.\left[d\left(v_{2j+2},v_{2j+1}\right)\right]^{\lambda}$$

which implies that

$$d\left(v_{2j+1},v_{2j+2}\right) \leq \left[d\left(v_{2j},v_{2j+2}\right)\right]^{\frac{\lambda}{1-\lambda}} = \left[d\left(v_{2j},v_{2j+1}\right)\right]^{h}$$

Thus it follows that, for all $j \geq 1$,

$$d\left(v_j,v_{j+1}\right) \leq \left[d\left(v_{j-1},v_j\right)\right]^{h} \leq \left[d\left(v_{j-2},v_{j-1}\right)\right]^{h^2} \leq \cdots [d(v_0,v_1)]^{h^j}$$

Therefore, $j,k \in N$ with $j < k$, by the multiplicative triangle inequality , we obtain

$$d\left(v_j,v_k\right) \leq d\left(v_j,v_{j+1}\right).d\left(v_{j+1},v_{j+2}\right) \ldots..d(v_{k-1},v_k)$$

$$\leq [d(v_0,v_1)]^{h^j}.[d(v_0,v_1)]^{h^{j+1}} \ldots \ldots.[d(v_0,v_1)]^{h^{k-1}}$$

$$\leq [d(v_0,v_1)]^{\frac{h^j}{1-h}}.$$

Which means that $d\left(v_j,v_k\right) \to 1$ as $j,k \to \infty$.

Hence $\{v_j\}$ is multiplicative cauchy sequence.

By the completeness of $X \exists z \in X \ni v_j \to \psi$ as $j \to \infty$.

Accordingly, the sequences $Au_{2j},Su_{2j},Tu_{2j+1},Bu_{2j+1} \to \psi$ as $j \to \infty$.

By (C3) the couple $(B,T)$ is reciprocally continuous $\lim_{j\to\infty} d\left(BTu_j,B\psi\right) = 1$
$\lim_{j\to\infty} d\left(TBu_j,T\psi\right) = 1$. (1)

Also the couple $(B,T)$ is semi-compatible, we have $\lim_{j\to\infty} d\left(BTu_j,T\psi\right) = 1$ (2)

From (1) and (2) we get $B\psi = T\psi$ (3)

Since $T(X) \subseteq A(X)$which gives then thereexists $\gamma \in X$ such that $A\gamma = Au_j$ since $Au_j \to \psi$ as $j \to \infty$.

Which implies $A\gamma = \psi$ (4)

Now we prove that $A\gamma = S\gamma = \psi$

Putting $u = \gamma$ and $v = u_j$ in (C2) we have

$$d\left(S\gamma,Tu_j\right) \leq \left[\phi\left(max\left\{d\left(A\gamma,Bu_j\right),\frac{d(A\gamma,S\gamma)d\left(Bu_j,Tu_j\right)}{1+d\left(A\gamma,Bu_j\right)},\frac{d\left(A\gamma,Tu_j\right)d\left(Bu_j,A\gamma\right)}{1+d\left(A\gamma,Bu_j\right)}\right\}\right)\right]^{\lambda}$$

$$d(S\gamma,\psi) \le \left[\phi\left(max\left\{d(S\gamma,\psi),\frac{d(S\gamma,S\gamma)d(\psi,\psi)}{1+d(S\gamma,\psi)},\frac{d(S\gamma,\psi)d(\psi,S\gamma)}{1+d(S\gamma,\psi)}\right\}\right)\right]^{\lambda}$$

$$d(S\gamma,\psi) \le \left[\phi\big(d(S\gamma,\psi)\big)\right]^{\lambda}$$

$$d(S\gamma,\psi) \le \left[\phi\big(d(S\gamma,\psi)\big)\right]^{\lambda}$$

Which gives $S\gamma=\psi$.

Therefore $A\gamma=S\gamma=\psi$ .

Since the couple $(A\ ,S)$ is WCM and $\gamma$ is coincidence point then $AS\gamma=SA\gamma$ we have $A\psi=S\psi$. (5)

Putting $u=\psi$ and $v=v_j$ in (C2) we have

$$d(S\psi,Tv_j) \le \left[\phi\left(max\left\{d(A\psi,Bv_j),\frac{d(A\psi,S\psi)d(Bv_j,Tv_j)}{1+d(A\psi,Bv_j)},\frac{d(A\psi,Tv_j)d(Bv_j,A\psi)}{1+d(A\psi,Bv_j)}\right\}\right)\right]^{\lambda}$$

$$d(S\psi,\psi) \le \left[\phi\left(max\left\{d(S\psi,\psi),\frac{d(S\psi,S\psi)d(\psi,\psi)}{1+d(S\psi,\psi)},\frac{d(S\psi,\psi)d(\psi,S\psi)}{1+d(S\psi,\psi)}\right\}\right)\right]^{\lambda}$$

$$d(S\psi,\psi) \le \left[\phi\big(d(S\psi,\psi)\big)\right]^{\lambda}$$

$$d(S\psi,\psi) \le [d(S\psi,\psi)]^{\lambda}$$

Which implies $\psi=S\psi$ (6)

From (5) and (6) we obtain $A\psi=S\psi=\psi$. (7)

In the inequality (C2) putting $u=\gamma$ and $v=\psi$

$$d(S\gamma,T\psi) \le \left[\phi\left(max\left\{d(A\gamma,B\psi),\frac{d(A\gamma,S\gamma)d(B\psi,T\psi)}{1+d(A\gamma,B\psi)},\frac{d(A\gamma,T\psi)d(B\psi,A\gamma)}{1+d(A\gamma,B\psi)}\right\}\right)\right]^{\lambda}$$

$$d(\psi,T\psi) \le \left[\phi\left(max\left\{d(\psi,T\psi),\frac{d(\psi,\psi)d(T\psi,T\psi)}{1+d(\psi,T\psi)},\frac{d(\psi,T\psi)d(T\psi,\psi)}{1+d(\psi,T\psi)}\right\}\right)\right]^{\lambda}$$

$$d(\psi,T\psi) \le \left[\phi\big(d(\psi,T\psi)\big)\right]^{\lambda}$$

$$d(\psi,T\psi) \le [d(\psi,T\psi)]^{\lambda}$$

Which implies $\psi=T\psi$ (8)

From (3) and (8) it gives

$\psi=T\psi=B\psi$ (9)

From (7) and (9) we have $A\psi=S\psi=T\psi=B\psi=\psi$.

This demonstrates that the common fixed point for the maps above is $\psi$.

**For Uniqueness:** Assume that $\rho$ be another fixed point then $\rho=S\rho=A\rho=T\rho=B\rho$.

Putting $u = \psi$ and $v = \rho$ in $(C2)$

$$d(S\psi, T\rho) \leq \left[\phi\left(max\left\{d(A\psi, B\rho), \frac{d(A\psi, S\psi)d(B\rho, T\rho)}{1 + d(A\psi, B\rho)}, \frac{d(A\psi, T\rho)d(B\rho, A\psi)}{1 + d(A\psi, B\rho)}\right\}\right)\right]^{\lambda}$$

$$d(\psi, \rho) \leq \left[\phi\left(max\left\{d(\psi, \rho), \frac{d(\psi, \psi)d(\rho, \rho)}{1 + d(\psi, \rho)}, \frac{d(\psi, \rho)d(\rho, \psi)}{1 + d(\psi, \rho)}\right\}\right)\right]^{\lambda}$$

$$d(\psi, \rho) \leq \left[\phi\big(d(\psi, \rho)\big)\right]^{\lambda}$$

$d(\psi, \rho) \leq [d(\psi, \rho)]^{\lambda}$, a contradiction

which implies $\psi = \rho$.

This demonstrates $\psi$ is the unique common fixed point of four self-mappings.

**3.1 Example:**

Suppose $X = [0,1]$ be defined in MMS $(X, d)$ with $d(\beta, \eta) = e^{|\beta - \eta|}$ $\forall\, \beta, \eta \in X$.

We define self-mappings $A, S, B$ and $T$ as

$$S(\eta) = \begin{cases} \dfrac{1-\eta}{3} & \text{if } 0 \leq \eta \leq \dfrac{1}{4} \\ \dfrac{1+\eta}{3} & \text{if } \dfrac{1}{4} < \eta \leq 1 \end{cases}; \qquad B(\eta) = \begin{cases} 1 - 12\eta^2 & \text{if } 0 \leq \eta \leq \dfrac{1}{4} \\ \eta & \text{if } \dfrac{1}{4} < \eta \leq 1 \end{cases};$$

$$T(\eta) = \begin{cases} 4\eta^2 - 4\eta + 1 & \text{if } 0 \leq \eta \leq \frac{1}{4} \\ \frac{1+\eta}{5} & \text{if } \frac{1}{4} < \eta \leq 1 \end{cases} \quad \text{and } A(\eta) = \begin{cases} 1 - 3\eta & \text{if } 0 \leq \eta \leq \frac{1}{4} \\ \frac{1+2\eta}{6} & \text{if } \frac{1}{4} < \eta \leq 1. \end{cases}$$

Clearly $S(X) = \left[\frac{1}{4}, \frac{1}{3}\right] \cup \left(\frac{1}{4}, 1\right] = \left(\frac{1}{4}, 1\right] \subseteq B(X) = \left[\frac{1}{4}, 1\right] \cup \left(\frac{5}{12}, \frac{2}{3}\right] = \left[\frac{1}{4}, 1\right]$ and

$T(X) = \left[\frac{1}{4}, 1\right] \cup \left(\frac{1}{4}, \frac{2}{5}\right] = \left[\frac{1}{4}, 1\right] \subseteq A(X) = \left[\frac{1}{4}, 1\right] \cup \left(\frac{1}{4}, \frac{1}{2}\right] = \left[\frac{1}{4}, 1\right]$

so that condition $(C1)$ is fulfilled.

Suppose we have a sequence $\eta_k = \left\{\frac{1}{4} - \frac{1}{k}\right\}$ for $k > 0$.

Now $\lim_{k\to\infty} A(\eta_k) = \lim_{k\to\infty} A\left(\frac{1}{4} - \frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[1 - 3\left(\frac{1}{4} - \frac{1}{k}\right)\right] = \frac{1}{4}$ and

$\lim_{k\to\infty} S(\eta_k) = \lim_{k\to\infty} S\left(\frac{1}{4} - \frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[\frac{1-\left(\frac{1}{4}-\frac{1}{k}\right)}{3}\right] = \frac{1}{4}$.

This gives $\lim_{k\to\infty} A(\eta_k) = \lim_{k\to\infty} S(\eta_k) = \frac{1}{4}$.

Also $\lim_{k\to\infty} B(\eta_k) = \lim_{k\to\infty} B\left(\frac{1}{4} - \frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[1 - 12\left(\frac{1}{4} - \frac{1}{k}\right)^2\right] = \frac{1}{4}$ and

$\lim_{k\to\infty} T(\eta_k) = \lim_{k\to\infty} T\left(\frac{1}{4} - \frac{1}{k}\right) = \lim\limits_{k\to\infty}\left[4\left(\frac{1}{4} - \frac{1}{k}\right)^2 - 4\left(\frac{1}{4} - \frac{1}{k}\right) + 1\right] = \frac{1}{4}$.

This gives $\lim_{k\to\infty} B(\eta_k) = \lim_{k\to\infty} T(\eta_k) = \frac{1}{4}$.

Therefore $\lim_{k\to\infty} A(\eta_k) = \lim_{k\to\infty} S(\eta_k) = \lim_{k\to\infty} B(\eta_k) = \lim_{k\to\infty} T(\eta_k) = \frac{1}{4} = \mu$.

Also $B\left(\frac{1}{4}\right) = T\left(\frac{1}{4}\right) = \frac{1}{4}$.

This implies

$\lim_{k\to\infty} BT\left(\frac{1}{4}-\frac{1}{k}\right) = \lim_{k\to\infty} B\left(4\left(\frac{1}{4}-\frac{1}{k}\right)^2 - 4\left(\frac{1}{4}-\frac{1}{k}\right)+1\right) = \lim_{k\to\infty} B\left[\frac{1}{4}+\frac{4}{k^2}+\frac{2}{k}\right] = \lim_{k\to\infty}\left[\frac{1}{4}+\frac{4}{k^2}+\frac{2}{k}\right] = \frac{1}{4}$

and

$\lim_{k\to\infty} TB(\eta_k) = \lim_{k\to\infty} TB\left(\frac{1}{4}-\frac{1}{k}\right) = \lim_{k\to\infty} T\left[1-12\left(\frac{1}{4}-\frac{1}{k}\right)^2\right] = \frac{1}{4}-\frac{12}{k^2}+\frac{6}{k}\lim_{k\to\infty} T\left[\frac{1}{4}-\frac{12}{k^2}+\frac{6}{k}\right] = \lim_{k\to\infty}\left[4\left(\frac{1}{4}-\frac{12}{k^2}+\frac{6}{k}\right)^2 - 4\left(\frac{1}{4}-\frac{12}{k^2}+\frac{6}{k}\right)+1\right] = \frac{1}{4}$.

This implies $\lim_{k\to\infty} d(BT\eta_k, B\mu) = d\left(\frac{1}{4},\frac{1}{4}\right) = 1$ and$\lim_{k\to\infty} d(TB\eta_k, T\mu) = d\left(\frac{1}{4},\frac{1}{4}\right) = 1$.

This proves that the couple $(B,T)$ is reciprocally continuous and semi-compatible in MMS.

Further $A\left(\frac{1}{4}\right) = S\left(\frac{1}{4}\right) = \frac{1}{4}$ which imliies that $\frac{1}{4}$ is the coincidence point of $A$ and $S$.

Hence $AS\left(\frac{1}{4}\right) = A\left(\frac{1}{4}\right) = \frac{1}{4}$ and $SA\left(\frac{1}{4}\right) = S\left(\frac{1}{4}\right) = \frac{1}{4}$ which gives $d\left(AS\left(\frac{1}{4}\right), SA\left(\frac{1}{4}\right)\right) = 1$

Hence the couple $(A,S)$ is weakly compatible mappings.

But $\lim_{k\to\infty} AS(\eta_k) = \lim_{k\to\infty} AS\left(\frac{1}{4}-\frac{1}{k}\right) = \lim_{k\to\infty} A\left[\frac{1-\left(\frac{1}{4}-\frac{1}{k}\right)}{3}\right] = \lim_{k\to\infty} A\left[\frac{1}{4}+\frac{1}{3k}\right]$

$= \lim_{k\to\infty}\left[\frac{4\left(\frac{1}{4}+\frac{1}{3k}\right)^2 - 3\left(\frac{1}{4}+\frac{1}{3k}\right)+2}{6}\right] = \frac{1}{4}$ and

$\lim_{k\to\infty} SA(\eta_k) = \lim_{k\to\infty} SA\left(\frac{1}{4}-\frac{1}{k}\right) = \lim_{k\to\infty} S\left[\frac{1-4\left(\frac{1}{4}-\frac{1}{k}\right)^2}{3}\right] = \lim_{k\to\infty} S\left[\frac{1}{4}+\frac{1}{3}\left(\frac{2}{k}-\frac{4}{k^2}\right)\right] = \lim_{k\to\infty}\left[\frac{7-4\left(\frac{1}{4}+\frac{1}{3}\left(\frac{2}{k}-\frac{4}{k^2}\right)\right)}{18}\right] = \frac{4}{9}$.

Similarly $\lim_{k\to\infty} BT\left(\frac{1}{4}-\frac{1}{k}\right) = \lim_{k\to\infty} B\left(\frac{3-4\left(\frac{1}{4}-\frac{1}{k}\right)}{8}\right) = \lim_{k\to\infty} B\left[\frac{1}{4}+\frac{1}{2k}\right] = \lim_{k\to\infty}\left[\frac{4\left(\frac{1}{4}+\frac{1}{2k}\right)^2+1}{15}\right] = \frac{1}{12}$

and

$$\lim_{k\to\infty} TB(\eta_k) = \lim_{k\to\infty} TB\left(\frac{1}{4}-\frac{1}{k}\right) = \lim_{k\to\infty} T\left[\frac{2+4\left(\frac{1}{4}-\frac{1}{k}\right)}{12}\right] = \lim_{k\to\infty} T\left[\frac{1}{4}-\frac{1}{3k}\right] = \lim_{k\to\infty}\left[\frac{3-4\left(\frac{1}{4}-\frac{1}{3k}\right)}{8}\right]$$
$$= \frac{1}{4}$$

Therefore $\lim_{k\to\infty} d(AS\eta_k, SA\eta_k) = d\left(\frac{1}{4},\frac{4}{9}\right) \neq 1$ and $\lim_{k\to\infty} d(BT\eta_k, TB\eta_k) = d\left(\frac{1}{12},\frac{1}{4}\right) \neq 1$.

Demonstrating that the compatibility condition is not satisfied.

Also $A\left(\frac{1}{4}\right) = S\left(\frac{1}{4}\right) = B\left(\frac{1}{4}\right) = T\left(\frac{1}{4}\right) = \frac{1}{4}$.

It has been observed that the unique common fixed point to four self-mapping $A, S, B$ and $T$ is $\frac{1}{4}$.

We now demonstrate another theorem on MMS.

**3.2 Theorem:** Suppose that in a complete MMS $(X, d)$, the four self-mappings $A, S, B$ and $T$ meeting the requirements

(D1) $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$

(D2)

$$d(Su, Tv) \leq \left[\phi\left(max\left\{d(Au, Bv), \frac{d(Au, Su)d(Bv, Tv)}{1 + d(Au, Bv)}, \frac{d(Au, Tv)d(Bv, Au)}{1 + d(Au, Bv)}\right\}\right)\right]^{\lambda}$$

for all $u, v \in X$ ,where $0 < \lambda < 1$.

Where $\phi: [0,\infty) \to [0,\infty)$ is a continuous and monotone increasing function such that $\phi(t) < t$ for all $t > 0$.

(D3) the pair $(B, T)$ is strongly semi-compatible and conditinally reciprocally continuous

(D4) the pair $(A, S)$ is OWC.

Then there exists a unique common fixed point for the above mappings.

**Proof:** As in theorem (3.1), $\{vj\}$ is cauchy sequence.

By the completeness of $X \exists z \in X \ni v_j \to \psi$ as $j \to \infty$.

Accordingly, the sequences $Au_{2j}, Su_{2j}, Tu_{2j+1}, Bu_{2j+1} \to \psi$ as $j \to \infty$.

By (D3) the couple $(B, T)$ is strongly semi-compatible whenever $B(u_{2j}) = T(u_{2j}) \to \psi$ (non-empty) implies $\{vj\}$ is a sequence in $X$ so that $B(vj) = T(vj) = \gamma$ forsome $\gamma \in X$ then

$d(BTvj, B\gamma) = 1$ and $d(TBvj, T\gamma) = 1$ as $j \to \infty$. (1)

Also the coulple $(B, T)$ is conditinally reciprocally continuous implies

$d(BTvj, T\gamma) = 1$ and $d(TBvj, B\gamma) = 1$ as $j \to \infty$. (2)

From (1) and (2) it gives $d(T\gamma, B\gamma) = 1$ (3)

Since $T(X) \subseteq A(X)$ implies $\gamma_1 \in X$ such that $d(T\gamma, A\gamma_1) = 1$.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 7s (2024)

Therefore $T\gamma = B\gamma = A\gamma_1$.    (4)

Now we show that $S\gamma_1 = A\gamma_1$.

In the inequality $(D2)$ putting $u = \gamma_1$ and $v = \gamma$

$$d(S\gamma_1, T\gamma) \le \left[\phi\left(max\left\{d(A\gamma_1, B\,\gamma), \frac{d(A\gamma_1, S\gamma_1)d(B\,\gamma, T\,\gamma)}{1 + d(A\gamma_1, B\,\gamma)}, \frac{d(A\gamma_1, T\,\gamma)d(B\,\gamma, A\gamma_1)}{1 + d(A\gamma_1, B\gamma)}\right\}\right)\right]^\lambda$$

$$d(S\gamma_1, A\gamma_1) \le \left[\phi\left(max\left\{d(A\gamma_1, A\gamma_1), \frac{d(A\gamma_1, S\gamma_1)d(A\gamma_1, A\gamma_1)}{1 + d(A\gamma_1, A\gamma_1)}, \frac{d(A\gamma_1, A\gamma_1)d(A\gamma_1, A\gamma_1)}{1 + d(A\gamma_1, A\gamma_1)}\right\}\right)\right]^\lambda$$

$$d(S\gamma_1, A\gamma_1) \le \left[\phi\big(d(A\gamma_1, S\gamma_1)\big)\right]^\lambda$$

$$d(S\gamma_1, A\gamma_1) \le [d(A\gamma_1, S\gamma_1)]^\lambda$$

Which implies $S\gamma_1 = A\gamma_1$        (5)

From (4) and (5) it gives  $T\gamma = B\gamma = A\gamma_1 = S\gamma_1 = \psi_1$ (say).        (6)

But the couple $(B, T)$ is strongly semi-compatible mappings $d(B\gamma, T\gamma) = 1$ and $d(BT\gamma, TB\gamma) = 1$

Implies $B\psi_1 = T\psi_1$.

Since the couple $(A, S)$ is OWC implies $A\gamma_1 = S\gamma_1$ which gives $AS\gamma_1 = SA\gamma_1$

implies $A\psi_1 = S\psi_1$.

Now we show that $S\psi_1 = \psi_1$.

In the inequality (D2) putting $u = \psi_1$ and $v = \gamma$

$$d(S\psi_1, T\gamma) \le \left[\phi\left(max\left\{d(A\psi_1, B\gamma), \frac{d(A\psi_1, S\psi_1)d(B\gamma, T\gamma)}{1 + d(A\gamma, B\gamma)}, \frac{d(A\psi_1, T\gamma)d(B\gamma, A\psi_1)}{1 + d(A\psi_1, B\gamma)}\right\}\right)\right]^\lambda$$

as $j \to \infty$  this implies

$$d(S\psi_1, \psi_1) \le \left[\phi\left(max\left\{d(S\psi_1, \psi_1), \frac{d(S\psi_1, S\psi_1)d(\psi_1, \psi_1)}{1 + d(S\gamma, \gamma)}, \frac{d(S\psi_1, \gamma)d(\gamma, S\psi_1)}{1 + d(S\psi_1, \psi_1)}\right\}\right)\right]^\lambda$$

$$d(S\psi_1, \psi_1) \le \left[\phi\big(d(S\psi_1, \psi_1)\big)\right]^\lambda$$

$$d(S\psi_1, \psi_1) \le [d(S\psi_1, \psi_1)]^\lambda$$

implies $S\psi_1 = \psi_1$.

Therefore $A\psi_1 = S\psi_1 = \psi_1$.       (7)

Now we show that $T\psi_1 = \psi_1$

In the inequality (D2) putting $u = \psi_1$ and $v = \psi_1$

$$d(S\psi_1, T\psi_1) \leq \left[\phi\left(max\left\{d(A\psi_1, B\psi_1), \frac{d(A\psi_1, S\psi_1)d(B\psi_1, T\psi_1)}{1+d(A\psi_1, B\psi_1)}, \frac{d(A\psi_1, T\psi_1)d(B\psi_1, A\psi_1)}{1+d(A\psi_1, B\psi_1)}\right\}\right)\right]^\lambda$$

as $j \to \infty$ this implies

$$d(\psi_1, T\psi_1) \leq \left[\phi\left(max\left\{d(\psi_1, T\psi_1), \frac{d(\psi_1, \psi_1)d(T\psi_1, T\psi_1)}{1+d(\psi_1, T\psi_1)}, \frac{d(\psi_1, T\psi_1)d(T\psi_1, \psi_1)}{1+d(\psi_1, T\psi_1)}\right\}\right)\right]^\lambda$$

$$d(\psi_1, T\psi_1) \leq \left[\phi\big(d(\psi_1, T\psi_1)\big)\right]^\lambda$$

$$d(\psi_1, T\psi_1) \leq [d(\psi_1, T\psi_1)]^\lambda$$

implies $\psi_1 = T\psi_1$.

Therefore $B\psi_1 = T\psi_1 = \psi_1$. (8)

From (7) and (8) we get $A\psi_1 = S\psi_1 = B\psi_1 = T\psi_1 = \psi_1$.

Thus $\psi_1$ is a fixed point that is common to all four self-maps $A, S, B$ and $T$.

**For Uniqueness:** Assume that $\rho$ be another fixed point then $\rho = S\rho = A\rho = T\rho = B\rho$.

Putting $u = \psi$ and $v = \rho$ in $(C2)$

$$d(S\psi, T\rho) \leq \left[\phi\left(max\left\{d(A\psi, B\rho), \frac{d(A\psi, S\psi)d(B\rho, T\rho)}{1+d(A\psi, B\rho)}, \frac{d(A\psi, T\rho)d(B\rho, A\psi)}{1+d(A\psi, B\rho)}\right\}\right)\right]^\lambda$$

$$d(\psi, \rho) \leq \left[\phi\left(max\left\{d(\psi, \rho), \frac{d(\psi, \psi)d(\rho, \rho)}{1+d(\psi, \rho)}, \frac{d(\psi, \rho)d(\rho, \psi)}{1+d(\psi, \rho)}\right\}\right)\right]^\lambda$$

$$d(\psi, \rho) \leq \left[\phi\big(d(\psi, \rho)\big)\right]^\lambda$$

$d(\psi, \rho) \leq [d(\psi, \rho)]^\lambda$, a contradiction

which implies $\psi = \rho$.

This demonstrates $\psi$ is the unique common fixed point of four self-mappings.

**3.2 Example:**

Suppose $X = [0,1]$ be defined in MMS $(X, d)$ with $d(\beta, \eta) = e^{|\beta-\eta|}$ $\forall$ $\beta, \eta \in X$.

We define self-mappings $G, J, H$ and $I$ as

$$S(\eta) = \begin{cases} \dfrac{2\eta+1}{6} & \text{if } 0 \leq \eta \leq \dfrac{1}{4} \\ \dfrac{sin^2(2\pi\eta) - 2\sin 2\pi\eta + 1}{4} & \text{if } \dfrac{1}{4} < \eta \leq 1 \end{cases}; \quad B(\eta)$$

$$= \begin{cases} \dfrac{\sin 2\pi\eta - \cos 4\pi\eta + 1}{12} & \text{if } 0 \leq \eta \leq \dfrac{1}{4} \\ \dfrac{1-\eta}{3} & \text{if } \dfrac{1}{4} < \eta \leq 1 \end{cases};$$

$$T(\eta)=\begin{cases}\frac{1+4\eta}{8} & \text{if } 0\le\eta\le\frac{1}{4}\\ \frac{2\sin 2\pi\eta+\cos 2\pi\eta-1}{4} & \text{if } \frac{1}{4}<\eta\le 1\end{cases} \quad \text{and} \quad A(\eta)=\begin{cases}\frac{2\eta+1}{6} & \text{if } 0\le\eta\le\frac{1}{4}\\ \frac{\sin 2\pi\eta-\cos 2\pi\eta+1}{2} & \text{if } \frac{1}{4}<\eta\le 1.\end{cases}$$

Clearly $S(X)=\left[\frac{1}{6},\frac{1}{4}\right]\cup\left\{\frac{1}{4}\right\}=\left[\frac{1}{6},\frac{1}{4}\right]\subseteq B(X)=\left[0,\frac{1}{4}\right]\cup\left[0,\frac{1}{4}\right)=\left[0,\frac{1}{4}\right]$ and

$T(X)=\left[\frac{1}{8},\frac{1}{4}\right]\cup\left[0,\frac{1}{4}\right)=\left[0,\frac{1}{4}\right]\subseteq A(X)=\left[\frac{1}{6},\frac{1}{4}\right]\cup[0,1)=[0,1).$

so that condition (D1) is fulfilled.

Suppose we have a sequence $\eta_k=\left\{\frac{1}{4}-\frac{1}{k}\right\}$ for $k>0$.

Now $\lim_{k\to\infty}B(\eta_k)=\lim_{k\to\infty}B\left(\frac{1}{4}-\frac{1}{k}\right)=\lim\limits_{k\to\infty}\left[\frac{\sin 2\pi\left(\frac{1}{4}-\frac{1}{k}\right)-\cos 4\pi\left(\frac{1}{4}-\frac{1}{k}\right)+1}{12}\right]=\frac{1}{4}$ and

$\lim_{k\to\infty}T(\eta_k)=\lim_{k\to\infty}T\left(\frac{1}{4}-\frac{1}{k}\right)=\lim\limits_{k\to\infty}\left[\frac{1+4\left(\frac{1}{4}-\frac{1}{k}\right)}{8}\right]=\frac{1}{4}.$

This gives $\lim_{k\to\infty}B(\eta_k)=\lim_{k\to\infty}T(\eta_k)=\frac{1}{4}=\mu_1$ and $T\mu_1=B\mu_1=\frac{1}{4}$.

Further $\lim_{k\to\infty}BT(\eta_k)=\lim\limits_{k\to\infty}BT\left(\frac{1}{4}-\frac{1}{k}\right)=\lim_{k\to\infty}B\left(\frac{1+4\left(\frac{1}{4}-\frac{1}{k}\right)}{8}\right)=$

$\lim_{k\to\infty}B\left[\frac{1}{4}-\frac{1}{2k}\right]=\lim\limits_{k\to\infty}\left[\frac{\sin 2\pi\left(\frac{1}{4}-\frac{1}{2k}\right)-\cos 4\pi\left(\frac{1}{4}-\frac{1}{2k}\right)+1}{12}\right]=\frac{1}{4}$ and

$$\lim_{k\to\infty}TB(\eta_k)=\lim_{k\to\infty}TB\left(\frac{1}{4}-\frac{1}{k}\right)$$

$$=\lim_{k\to\infty}T\left[\frac{\sin 2\pi\left(\frac{1}{4}-\frac{1}{2k}\right)-\cos 4\pi\left(\frac{1}{4}-\frac{1}{2k}\right)+1}{12}\right]=\lim_{k\to\infty}\left[\frac{1+4\left(\frac{1+cos\left(2\pi\left(\frac{1}{4}-\frac{1}{k}\right)\right)}{4}\right)}{8}\right]=\frac{1}{4}.$$

$\lim_{k\to\infty}d(BT\eta_k,T\mu_1)=\lim\limits_{k\to\infty}d(TB\eta_k,B\mu_1)=d\left(\frac{1}{4},\frac{1}{4}\right)=1.$

Now take another sequence $\beta_k=\left\{\frac{1}{4}+\frac{1}{k}\right\}$ for $k>0$.

Now $\lim_{k\to\infty}B(\eta_k)=\lim_{k\to\infty}B\left(\frac{1}{4}+\frac{1}{k}\right)=\lim\limits_{k\to\infty}\left[\frac{1-\left(\frac{1}{4}+\frac{1}{k}\right)}{3}\right]=\frac{1}{4}$ and

$\lim_{k\to\infty}T(\eta_k)=\lim_{k\to\infty}T\left(\frac{1}{4}+\frac{1}{k}\right)=\lim\limits_{k\to\infty}\left[\frac{2\sin 2\pi\left(\frac{1}{4}+\frac{1}{k}\right)+\cos 2\pi\left(\frac{1}{4}+\frac{1}{k}\right)-1}{4}\right]=\frac{1}{4}.$

This gives $\lim_{k\to\infty} B(\eta_k) = \lim_{k\to\infty} T(\eta_k) = \frac{1}{4} = \mu_2$ and $T\mu_2 = B\mu_2 = \frac{1}{4}$.

Further $\lim_{k\to\infty} BT(\eta_k) = \lim_{k\to\infty} BT\left(\frac{1}{4}+\frac{1}{k}\right) = \lim_{k\to\infty} B\left(\frac{2\sin 2\pi\left(\frac{1}{4}+\frac{1}{k}\right)+\cos 2\pi\left(\frac{1}{4}+\frac{1}{k}\right)-1}{4}\right) =$
$\lim_{k\to\infty} B\left(\frac{2\sin 2\pi\left(\frac{1}{4}+\frac{1}{k}\right)+\cos 2\pi\left(\frac{1}{4}+\frac{1}{k}\right)-1}{4}\right) =$
$\lim_{k\to\infty} B\left[\frac{1}{4}-\frac{1}{2k}\right] = \lim_{k\to\infty} B\left[\frac{1}{4}-\frac{1}{2k}\right] = \lim_{k\to\infty}\left[\frac{\sin 2\pi\left(\frac{1}{4}-\frac{1}{2k}\right)-\cos 4\pi\left(\frac{1}{4}-\frac{1}{2k}\right)+1}{12}\right] = \frac{1}{4}$ and
$\lim_{k\to\infty} TB(\eta_k) = \lim_{k\to\infty} TB\left(\frac{1}{4}-\frac{1}{k}\right) =$
$\lim_{k\to\infty} T\left[\frac{\sin 2\pi\left(\frac{1}{4}-\frac{1}{2k}\right)-\cos 4\pi\left(\frac{1}{4}-\frac{1}{2k}\right)+1}{12}\right] = \lim_{k\to\infty}\left[\frac{1+4\left(\frac{1+cos\left(2\pi\left(\frac{1}{4}-\frac{1}{k}\right)\right)}{4}\right)}{8}\right] = \frac{1}{4}$.

Also

$\lim_{k\to\infty} d(BT\eta_k, B\mu_2) = d\left(\frac{1}{4},\frac{1}{4}\right) = 1$ and $\lim_{k\to\infty} d(TB\eta_k, T\mu_2) = d\left(\frac{1}{4},\frac{1}{4}\right) = 1$

This shows that the self-maps $T$ and $B$ are conditionally reciprocally continuous in MMS.

Similarly $\lim_{k\to\infty} d(BT\eta_k, T\mu_2) = d\left(\frac{1}{4},\frac{1}{4}\right) = 1$ and $\lim_{k\to\infty} d(TB\eta_k, B\mu_2) = d\left(\frac{1}{4},\frac{1}{4}\right) = 1$.

This shows that the self-maps $T$ and $B$ are conditionally semi compatible in MMS.

Moreover $\frac{1}{4}$ and 1 are coincidence points $d\left(B\left(\frac{1}{4}\right), T\left(\frac{1}{4}\right)\right) = 1$ implies $d\left(BT\left(\frac{1}{4}\right), TB\left(\frac{1}{4}\right)\right) = 1$,but

$d\big(B(1), T(1)\big) = 1$ and $d\big(BT(1), TB(1)\big) \neq 1$.

As a result, the couple $(B, T)$ satisfies OWC.

Therefore the self-maps $B$ and T are strongly semi compatible mappings in MMS.

But

$\lim_{k\to\infty} AS(\eta_k) = \lim_{k\to\infty} AS\left(\frac{1}{2}-\frac{1}{k}\right) = \lim_{k\to\infty} A\left[\frac{3-2\left(\frac{1}{2}-\frac{1}{k}\right)}{4}\right] = \lim_{k\to\infty} A\left[\frac{1}{2}+\frac{1}{2k}\right] =$
$\lim_{k\to\infty}\left[\frac{2+sin\pi\left(\left(\frac{1}{2}+\frac{1}{2k}\right)\right)}{4}\right] = \frac{3}{4}$ and

$$\lim_{k\to\infty} SA(\eta_k) = \lim_{k\to\infty} SA\left(\frac{1}{2}-\frac{1}{k}\right) = \lim_{k\to\infty} S\left[\frac{1+3\left(\frac{1}{2}-\frac{1}{k}\right)}{5}\right] = \lim_{k\to\infty} S\left[\frac{1}{2}-\frac{3}{5k}\right] = \lim_{k\to\infty}\left[\frac{3-2\left(\frac{1}{2}-\frac{3}{5k}\right)}{4}\right] = \frac{1}{2}.$$

Similarly $\lim_{k\to\infty} BT\left(\frac{1}{2}-\frac{1}{k}\right) = \lim_{k\to\infty} B\left(\frac{1+2\left(\frac{1}{2}-\frac{1}{k}\right)}{4}\right) = \lim_{k\to\infty} B\left[\frac{1}{2}-\frac{1}{2k}\right] = \lim_{k\to\infty}\left[1-\left(\frac{1}{2}-\frac{1}{2k}\right)\right] = \frac{1}{2}$ and

$$\lim_{k\to\infty} TB(\eta_k) = \lim_{k\to\infty}\left(\frac{1}{2}-\frac{1}{k}\right) = \lim_{k\to\infty} T\left[1-\left(\frac{1}{2}-\frac{1}{k}\right)\right] = \lim_{k\to\infty} T\left[\frac{1}{2}+\frac{1}{k}\right] = \lim_{k\to\infty}\left[\frac{1+sin2\pi\left(\left(\frac{1}{2}+\frac{1}{k}\right)\right)}{4}\right]$$

$$=\frac{1}{4}$$

Therefore $\lim_{k\to\infty} d(AS\eta_k, SA\eta_k) = d\left(\frac{3}{4},\frac{1}{2}\right) \neq 1$ and $\lim_{k\to\infty} d(BT\eta_k, TB\eta_k) = d\left(\frac{1}{2},\frac{1}{4}\right) \neq 1$.

Demonstrating that the compatibility condition is not satisfied.

Further $A\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = \frac{1}{2}$ which imliies that $\frac{1}{2}$ is the coincidence point of $A$ and $S$.

$B\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}$ which imliies that $\frac{1}{2}$ is the coincidence point of $B$ and $T$.

Hence $AS\left(\frac{1}{2}\right) = A\left(\frac{1}{2}\right) = \frac{1}{2}$ and $SA\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = \frac{1}{2}$ which gives $d\left(AS\left(\frac{1}{2}\right), SA\left(\frac{1}{2}\right)\right) = 1$

also $BT\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right) = \frac{1}{2}$ and $TB\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}$ which gives $d\left(BT\left(\frac{1}{2}\right), TB\left(\frac{1}{2}\right)\right) = 1$.

Hence the couple $(A, S)$ and $(B, T)$ are weakly compatible mappings but not compatible.

Also $A\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}$.

It has been observed that the unique common fixed point to four self-mapping $A, S, B$ and $T$ is $\frac{1}{2}$.

## 4. Conclusion

In this paper the concepts of semi-compatible mapping, WCM , reciprocally continuous , conditionally reciprocally continuous mappings and and OWC mappings are used for obtaining the generalization of existing common fixed point theorems proved in [1] which are weaker than those classes of compatible mappings and continuous mappings. Further, these results are also substantiated with appropriate examples.

## References:

[1] Abdou, Afrah AN. "Common fixed point results for compatible-type mappings in multiplicative metric spaces." *J. Nonlinear Sci. Appl* 9 (2016): 2244-2257.

[2] B Vijayabaskerreddy and V Srinivas Fixed point results on multiplicative semi-metric space. Journal of Scientific Research, 12(3):341–348, 2020. http://dx.doi.org/10.3329/jsr.v12i3.44754

[3] Kidane KOYAS, Alemayehu GEBRE, and Aynalem KASSAYE. A common fixed point theorem for generalized weakly contractive mappings in multiplicative metric spaces. Advances in the Theory of Nonlinear Analysis and its Application, 4(1):1–13, 2020. https://doi.org/10.31197/atnaa.573903

[4] B.Vijayabaskerreddy and V. Srinivas. Some results in multiplicative metric space using absorbing mappings. Indian Journal of Science and Technology, 13(39): 4161–4167, 2020. https://doi.org/10.17485/IJST/v13i39.1629

[5] V.Srinivas and K.Mallaiah. A result on multiplicative metric space. J. Math. Comput. Sci., 10(5):1384–1398, 2020. https://doi.org/10.28919/jmcs/4628

[6] K.Satyanna and V.Srinivas. Some results of conditionally sequential absorbing and pseudo reciprocally continuous mappings in probabilistic 2-metric space. International Journal of Analysis and Applications, 20:12–12, 2022. https://doi.org/10.28924/2291-8639-20-2022-12

[7] V. Srinivas, T. Tirupathi, and K. Mallaiah. A fixed point theorem using ea property on multiplicative metric space. J. Math. Comput. Sci., 10(5):1788–1800, 2020. https://doi.org/10.28919/jmcs/4752.

[8] Thirupathi, T., and V. Srinivas. "Some Results on Conditionally Sequential Absorbing Maps in Multiplicative Metric Space." International Journal of Analysis and Applications 21 :1-12 , 2023. https://doi.org/10.28924/2291-8639-21-2023-13

[9] K. Mallaiah, V. Srinivas, Common Fixed Point of Four Maps in Sm-Metric Space, Int. J. Anal. Appl., 19 (6) (2021), 915-928. DOI: https://doi.org/10.28924/2291-8639-19-2021-915

[10] Hussain, Baqir, et al. "Some coupled fixed point theorems on multiplicative metric spaces with an application." *AIMS Mathematics* 7.8 (2022): 14631-14651. DOI:10.3934/math.2022805.

[11] M.Verma, P. Kumar, and N. Hooda. Common fixed point theorems using weakly compaible mapsin multiplicative metric spaces. Electronic Journal of Mathematical Analysis and Applications, 9(1):284–292, 2021. http://math-frac.org/Journals/EJMAA/