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Application of Fractional Calculus Operators to Fredholm type Integral Equations

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Abstract:
The purpose of this article is to study of fractional calculus operators of Riemann-Liouville and Weyl in solving the Fredholm type integral equations. The kernels of these equations are the product of incomplete -functions and generalized polynomials. Further, the Mellin transform technique is also used to find the solution of the Fredholm integral equations. A number of intriguing special cases have also been mentioned.

Keywords: Reimann-Liouville fractional integral, Weyl fractional integral, Incomplete -functions, Fredholm type integral equation, Fox's -function, Mellin transform, General class of polynomials, Generalized polynomials.

1. Introduction:

Fractional calculus is one of the branches of applied mathematics that is based on any real or complex order derivatives and integrals. Many advanced applications of fractional calculus are found in various areas of science and engineering. Several authors (refer [1]-[6]) have used fractional derivatives and integrals to solve problems like fluid dynamics, thermonuclear fusion, dynamical systems, control theory, quantum mechanics, image processing and many more. Integral equations usually occur in physics, mechanics and applied mathematics. Applications of Fredholm integral equations include the spectrum concentration problem, linear forward modelling, inverse problems, and signal processing theory. In recent years, several authors have studied integral equations of Fredholm type taking kernel as special functions and polynomials (refer [7]-[13]).

2. Objectives:

The following Fredholm type integral equations are to be solved in order to accomplish the goal of this work.

$$\int_0^\infty x^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{x}\right)^r,\dots,w_k\left(\frac{y}{x}\right)^r\right]\gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^\lambda\left|\begin{matrix}(u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_q,V_q)\end{matrix}\right.\right]$$
$$\times f(x)dx = g(y) \qquad (0<y<\infty), \qquad (1)$$

$$\int_0^\infty x^{-\delta} S_V^U\left[\mu\left(\frac{y}{x}\right)^\zeta\right] H_{P,Q}^{M,N}\left[T\left(\frac{y}{x}\right)^\tau\left|\begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right.\right]\gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^\lambda\left|\begin{matrix}(u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_q,V_q)\end{matrix}\right.\right]$$

$$\times f(x)dx = g(y) \qquad (0 < y < \infty), \tag{2}$$

$$\int_0^\infty x^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{x}\right)^r,\dots,w_k\left(\frac{y}{x}\right)^r\right]\Gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^\lambda \middle| \begin{matrix}(u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_q,V_q)\end{matrix}\right]$$

$$\times f(x)dx = g(y) \qquad (0 < y < \infty) \tag{3}$$

and

$$\int_0^\infty x^{-\delta} S_V^U\left[\mu\left(\frac{y}{x}\right)^\zeta\right] H_{P,Q}^{M,N}\left[T\left(\frac{y}{x}\right)^\tau \middle| \begin{matrix}(e_P,E_P)\\(f_Q,F_Q)\end{matrix}\right]\Gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^\lambda \middle| \begin{matrix}(u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_q,V_q)\end{matrix}\right]$$

$$\times f(x)dx = g(y) \qquad (0 < y < \infty). \tag{4}$$

The general class of polynomials is familiarized by Srivastava [14] as:

$$S_{V'}^{U'}[y] = \sum_{l=0}^{[V'/U']} \frac{(-V')_{U'l}}{l!} A_{V',l}\, y^l, \tag{5}$$

where V' is non-negative integer, $U' \in Z_+$ and the co-efficients $A_{V',l}(V', l \geq 0)$ are constants (real or complex) and are arbitrary in nature.

The generalized polynomial is given by Srivastava [15] in the following way:

$$S_{N_1,\dots,N_k}^{M_1,\dots,M_k}[y_1,\dots,y_k] = \sum_{\alpha_1=0}^{[N_1/M_1]},\dots,\sum_{\alpha_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\alpha_1}}{\alpha_1!},\dots,\frac{(-N_k)_{M_k\alpha_k}}{\alpha_k!} B[N_1,\alpha_1;\dots;N_k,\alpha_k]\, y_1^{\alpha_1},\dots,y_k^{\alpha_k}, \tag{6}$$

where $N_1,\dots,N_k$ are non-negative integers, $(M_1,\dots,M_k) \in Z_+$ and the co-efficient

$B[N_1,\alpha_1;\dots;N_k,\alpha_k]$ are constants (real or complex). All are arbitrary in nature.

Fox's H-function is represented in form of series as follows (refer [16]).

$$H_{P,Q}^{M,N}\left[z \middle| \begin{matrix}(u_P,U_P)\\(v_Q,V_Q)\end{matrix}\right] = \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G \phi'(L_G)\, z^{L_G}}{G!V_g}, \tag{7}$$

where $$\phi'(L_G) = \frac{\prod_{j=1, j\neq G}^{M} \Gamma(v_j - V_j L_G) \prod_{j=1}^{N} \Gamma(1-u_j + U_j L_G)}{\prod_{j=M+1}^{Q} \Gamma(1-v_j + V_j L_G) \prod_{j=N+1}^{P} \Gamma(u_j - U_j L_G)} \tag{8}$$

and $L_G = (v_g + G)/V_g$.

Special functions are significant in a wide range of scientific and engineering domains. The familiar H-function has spanned a greater range than the other special functions. However, in the fields of heat conduction and astrophysics, it has been revealed that certain issues cannot be solved using these special functions. During this period, the perception of incomplete Gamma functions existed, and academics researched several articles (refer [17]-[21]) connected to incomplete special functions alongside equivalent higher transcendental special functions.

Incomplete Gamma functions $\gamma(s,w)$ and $\Gamma(s,w)$ are defined as:

$$\gamma(s,w)=\int_0^w t^{s-1}e^{-t}dt \qquad (\Re(s)>0; w\geq 0) \tag{9}$$

and

$$\Gamma(s,w)=\int_w^\infty t^{s-1}e^{-t}dt \qquad (w\geq 0; \Re(s)>0\; when\; w=0) \tag{10}$$

respectively, satisfy the following decomposition formula

$$\gamma(s,w)+\Gamma(s,w)=\Gamma(s), \qquad (\Re(s)>0), \tag{11}$$

where $\Gamma(s)$ is the familiar Gamma function.

Srivastava hosted the incomplete H -functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$, are described in Mellin-Barnes type of contour integrals as shown below:

$$\begin{aligned}\gamma_{p,q}^{m,n}(z)&=\gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(u_1,U_1,w),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]\\&=\gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(u_1,U_1,w),(u_2,U_2),\ldots,(u_p,U_p)\\(v_1,V_1),\ldots,(v_q,V_q)\end{matrix}\right.\right]\\&=\frac{1}{2\pi i}\int_L g^{'}(s,w)\,z^{-s}\,ds,\end{aligned} \tag{12}$$

where

$$g^{'}(s,w)=\frac{\gamma(1-u_1-U_1s,w)\prod_{j=1}^{m}\Gamma(v_j+V_js)\prod_{j=2}^{n}\Gamma(1-u_j-U_js)}{\prod_{j=m+1}^{q}\Gamma(1-v_j-V_js)\prod_{j=n+1}^{p}\Gamma(u_j+U_js)} \tag{13}$$

and

$$\begin{aligned}\Gamma_{p,q}^{m,n}(z)&=\Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(u_1,U_1,w),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]\\&=\Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(u_1,U_1,w),(u_2,U_2),\ldots,(u_p,U_p)\\(v_1,V_1),\ldots,(v_q,V_q)\end{matrix}\right.\right]\end{aligned} \tag{14}$$

$$= \frac{1}{2\pi i}\int_L G^{'}(s,w)\, z^{-s}\, ds,$$

where

$$G^{'}(s,w) = \frac{\Gamma(1-u_1-U_1 s, w)\prod_{j=1}^{m}\Gamma(v_j+V_j s)\prod_{j=2}^{n}\Gamma(1-u_j-U_j s)}{\prod_{j=m+1}^{q}\Gamma(1-v_j-V_j s)\prod_{j=n+1}^{p}\Gamma(u_j+U_j s)}. \tag{15}$$

Incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ in (12) and (14) exist for all $w \geq 0$ according to the same contour and set of circumstances as mentioned in [22].

Some special cases of incomplete H-function [23] are as follows:

(i) Specializing $w=0$ in (14), the incomplete H-function $\Gamma_{p,q}^{m,n}(z)$ moderates to the Fox's H-function.

$$\Gamma_{p,q}^{m,n}\left[z \middle| \begin{matrix} (u_1,U_1,0),(u_2,U_2),\ldots,(u_p,U_p) \\ (v_1,V_1),\ldots,(v_q,V_q) \end{matrix} \right] = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (u_1,U_1),\ldots,(u_p,U_p) \\ (v_1,V_1),\ldots,(v_q,V_q) \end{matrix} \right]. \tag{16}$$

The H-function of one variable [24] is defined in the following manner:

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (u_1,U_1),\ldots,(u_p,U_p) \\ (v_1,V_1),\ldots,(v_q,V_q) \end{matrix} \right] \tag{17}$$

$$= \frac{1}{2\pi i}\int_L \theta(s)\, z^{s}\, ds, \quad (i=\sqrt{-1}, z \in \mathbb{C} \setminus \{0\}),$$ where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(v_j-V_j s)\prod_{j=1}^{n}\Gamma(1-u_j+U_j s)}{\prod_{j=m+1}^{q}\Gamma(1-v_j+V_j s)\prod_{j=n+1}^{p}\Gamma(u_j-U_j s)}, \tag{18}$$

where m lies between $[1,q]$ and n lies between $[0,p]$ when $m,q \in \mathbb{N}$ and $n,p \in \mathbb{N}_0$.

Here $\mathbb{N}$ represents positive integers, $\mathbb{C}$ denote the set of complex numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

(ii) Substituting m by 1, n by p and changing q to $q+1$ in functions given in (12) and (14), the functions moderates to incomplete Fox-Wright functions (IFWFs) ${}_p\psi_q^{(\gamma)}$ and ${}_p\psi_q^{(\Gamma)}$ by taking suitable parameters:

$$\gamma_{p,q+1}^{1,p}\left[-z \middle| \begin{matrix} (1-u_1,U_1,w),(1-u_j,U_j)_{2,p} \\ (0,1),(1-v_j,V_j)_{1,q} \end{matrix} \right] = {}_p\psi_q^{(\gamma)}\left[z \middle| \begin{matrix} (u_1,U_1,w),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,q} \end{matrix} \right] \tag{19}$$

and

$$\Gamma^{1,p}_{p,q+1}\left[-z\left|\begin{matrix}(1-u_1,U_1,w),(1-u_j,U_j)_{2,p}\\(0,1),(1-v_j,V_j)_{1,q}\end{matrix}\right.\right]={}_p\psi_q^{(\Gamma)}\left[z\left|\begin{matrix}(u_1,U_1,w),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]. \quad (20)$$

(iii) Further, specializing $w=0$ in (20), ${}_p\psi_q^{(\Gamma)}$ (IFWF) moderates to Fox-Wright function (FWF) ${}_p\psi_q$:

$${}_p\psi_q^{(\Gamma)}\left[z\left|\begin{matrix}(u_1,U_1,0),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]={}_p\psi_q\left[z\left|\begin{matrix}(u_1,U_1),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_p,V_p)\end{matrix}\right.\right]. \quad (21)$$

(iv) Taking $U_j=V_j=1$ in (19) and (20), the incomplete Fox-Wright functions become incomplete generalized hypergeometric functions (IGHFs) ${}_p\gamma_q$ and ${}_p\Gamma_q$ [25]:

$${}_p\psi_q^{(\gamma)}\left[z\left|\begin{matrix}(u_1,1,w),(u_j,1)_{2,p}\\(v_j,1)_{1,q}\end{matrix}\right.\right]={}_p\gamma_q\left[z\left|\begin{matrix}(u_1,w),u_2,\dots,u_p\\v_1,\dots,v_q\end{matrix}\right.\right] \quad (22)$$

and

$${}_p\psi_q^{(\Gamma)}\left[z\left|\begin{matrix}(u_1,1,w),(u_j,1)_{2,p}\\(v_j,1)_{1,q}\end{matrix}\right.\right]={}_p\Gamma_q\left[z\left|\begin{matrix}(u_1,w),u_2,\dots,u_p\\v_1,\dots,v_q\end{matrix}\right.\right]. \quad (23)$$

(v) Specializing $w=0$ in (23), ${}_p\Gamma_q$ (IGHF) moderates to the known function generalized hypergeometric (GHF) ${}_pF_q$:

$${}_p\Gamma_q\left[z\left|\begin{matrix}(u_1,0),u_2,\dots,u_p\\v_1,\dots,v_q\end{matrix}\right.\right]={}_pF_q\left[z\left|\begin{matrix}u_1,\dots,u_p\\v_1,\dots,v_q\end{matrix}\right.\right]. \quad (24)$$

Let $\aleph$ be a set of all functions $\mathscr{f}$, defined on $R^+=[0,\infty)$ and satisfies the followings:

(i) $\mathscr{f}\in C^\infty(R^+)$,

(ii) $\lim_{y\to\infty}[y^\gamma \mathscr{f}^r(y)]=0$ for all non-negative integers γ and r,

(iii) $\mathscr{f}(y)=0(1)$ when $y\to 0$,

for correspondence to the space of good functions defined on the whole real line $(-\infty,\infty)$.

μ^{th} order Reimann-Liouville fractional integral is given as follows:

$$D^{-\mu}\left\{\mathscr{f}(y)\right\}={}_0D_y^{-\mu}\left\{\mathscr{f}(y)\right\}=\frac{1}{\Gamma(\mu)}\int_0^y(y-\omega)^{\mu-1}\mathscr{f}(\omega)\,d\omega,\quad (\mathrm{Re}(\mu)>0:\mathscr{f}\in\aleph) \quad (25)$$

and h^{th} order Weyl fractional integral is described in the following manner:

$$W^{-h}\left\{\mathscr{f}(y)\right\}={}_yD_\infty^{-h}\left\{\mathscr{f}(y)\right\}=\frac{1}{\Gamma(h)}\int_y^\infty(\xi-y)^{h-1}\mathscr{f}(\xi)\,d\xi,\quad (\mathrm{Re}(h)>0:\mathscr{f}\in\aleph). \quad (26)$$

3. Main Results:

The Fredholm integral equation with kernel, the product of Generalized Polynomials and the incomplete H-functions.

Lemma 1: Consider

(i) $m,n,p,q \in \{\square^{+} \cup 0\}$: non-negative integers thus $1 \le m \le q;\ 0 \le n \le p.$

(ii) $\Re(\delta-\sigma) > 0, \Re\left[\sigma + \lambda\left(\frac{v_j}{V_j}\right)\right] > 0 \ \ (j=1,\dots,m).$

(iii) $w \ge 0, \lambda > 0$ and $\delta \in \square$.

(iv) $|\arg(z)| < \frac{1}{2}\pi\aleph,$

where $\aleph = \sum_{j=1}^{n} U_j + \sum_{j=1}^{m} V_j - \sum_{j=n+1}^{p} U_j - \sum_{j=m+1}^{q} V_j > 0,$

Then,

$$W^{\sigma-\delta}\left[x^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{x}\right)^{r},\dots,w_k\left(\frac{y}{x}\right)^{r}\right]\gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]\right]$$

$$= x^{-\sigma}\sum_{\delta_1=0}^{[N_1/M_1]},\dots,\sum_{\delta_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\dots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!}B[N_1,\delta_1;\dots;N_k,\delta_k]\,w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\dots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times\gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),\left(1-\sigma-\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,m},\left(1-\delta-\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{m+1,q}\end{matrix}\right.\right]. \tag{27}$$

Proof: We use the formulation of Weyl fractional integral provided by (26), to prove the lemma 1. Equations (6) and (12) are used to precisely describe the incomplete H-function in form of Mellin-Barnes contour integral and generalized polynomial in series form. In order to get the desired result, change the order of summation and integrals under the aforementioned circumstances, then solve the integral using substitution method, and interpret the resultant contour integral by means of incomplete H-function.

Theorem 1: Consider

(i) $m,n,p,q \in \{\square^{+} \cup 0\}$: non-negative integers thus $1 \le m \le q;\ 0 \le n \le p.$

(ii) $\Re\left[\sigma + \lambda\left(\frac{u_j - 1}{U_j}\right)\right] < 0 \ \ (j=1,\dots,n),\ \Re\left[\sigma + \lambda\left(\frac{v_j}{V_j}\right)\right] > 0 \ \ (j=1,\dots,m).$

(iii) $w \ge 0, \lambda > 0$ and $\delta \in \square$.

Consequently, the relationship shown below is valid:

$$\int_0^{\infty} x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]}, \dots, \sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1 !}, \dots, \frac{(-N_k)_{M_k\delta_k}}{\delta_k !} B[N_1,\delta_1;\dots;N_k,\delta_k] w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1}, \dots, w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$
$$\times \gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-\sigma-\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-\delta-\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix}\right] f(x)\, dx$$
$$= \int_0^{\infty} x^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{x}\right)^{r}, \dots, w_k\left(\frac{y}{x}\right)^{r}\right] \gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p) \\ (v_1,V_1),\dots,(v_q,V_q) \end{matrix}\right]$$
$$\times D^{\sigma-\delta}\left\{ f(x)\right\} dx, \tag{28}$$

for $f \in \aleph$ and $y > 0$.

Proof: If ∇ denote the left-hand side of the equation (28),

$$\nabla = \int_0^{\infty} x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]}, \dots, \sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1 !}, \dots, \frac{(-N_k)_{M_k\delta_k}}{\delta_k !} B[N_1,\delta_1;\dots;N_k,\delta_k] w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1}, \dots, w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$
$$\times \gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-\sigma-\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-\delta-\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix}\right] f(x)\, dx.$$

Utilizing the result of lemma 1 and equation (26), we have

$$\nabla = \int_0^{\infty} \frac{f(x)}{\Gamma(\delta-\sigma)}\left\{ \int_x^{\infty} (\eta-x)^{\delta-\sigma-1}\eta^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{\eta}\right)^{r}, \dots, w_k\left(\frac{y}{\eta}\right)^{r}\right] \gamma_{p,q}^{m,n}\left[z\left(\frac{y}{\eta}\right)^{\lambda}\right] d\eta \right\} dx. \tag{29}$$

We get the following equation by altering the order of integration and applying equation (25).

$$\nabla = \int_{\eta=0}^{\infty} \eta^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{\eta}\right)^{r}, \dots, w_k\left(\frac{y}{\eta}\right)^{r}\right] \gamma_{p,q}^{m,n}\left[z\left(\frac{y}{\eta}\right)^{\lambda}\right] D^{-(\delta-\sigma)}\left\{ f(\eta)\right\} d\eta. \tag{30}$$

On changing η to x in the equation (30), we attain the required proof of the theorem 1.

Lemma 2: Consider

(i) $m,n,p,q \in \left\{\mathbb{Z}^{+} \cup 0\right\}$ thus $1 \le m \le q\,; 0 \le n \le p.$, where $\mathbb{Z}^{+}$ is positive integers.

(ii) $\Re(\delta-\sigma) > 0, \Re\left[\sigma + \lambda\left(\frac{v_j}{V_j}\right)\right] > 0 \ \ (j = 1,\dots,m).$

(iii) $w \ge 0, \lambda > 0$ and $\delta \in \mathbb{C}$.

(iv) $\left|\arg(z)\right| < \frac{1}{2}\pi \aleph$,

where $\aleph = \sum_{j=1}^{n} U_j + \sum_{j=1}^{m} V_j - \sum_{j=n+1}^{p} U_j - \sum_{j=m+1}^{q} V_j > 0.$

Then,

$$W^{\sigma-\delta}\left[x^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{x}\right)^r,\dots,w_k\left(\frac{y}{x}\right)^r\right]\Gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]\right]$$

$$= x^{-\sigma}\sum_{\delta_1=0}^{[N_1/M_1]},\dots,\sum_{\delta_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\dots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!}B[N_1,\delta_1;\dots;N_k,\delta_k]\,w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\dots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \Gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),\left(1-\sigma-\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,m},\left(1-\delta-\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{m+1,q}\end{matrix}\right.\right]. \quad (31)$$

Proof: Similar lines of the proof of lemma1 and equations (6), (14), (26) are used to reach the result of Lemma 2.

Theorem 2: For

(i) $m,n,p,q \in \{\mathbb{N}^+ \cup 0\}$ thus $1 \le m \le q;\ 0 \le n \le p.$

(ii) $\Re\left[\sigma + \lambda\left(\frac{u_j - 1}{U_j}\right)\right] < 0 \quad (j = 1,\dots,n),\ \Re\left[\sigma + \lambda\left(\frac{v_j}{V_j}\right)\right] > 0 \quad (j = 1,\dots,m).$

(iii) $w \ge 0, \lambda > 0$ and $\delta \in \mathbb{C}$.

Consequently, the relationship shown below is valid:

$$\int_0^{\infty} x^{-\sigma}\sum_{\delta_1=0}^{[N_1/M_1]},\dots,\sum_{\delta_k=0}^{[N_k/M_k]}\frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\dots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!}B[N_1,\delta_1;\dots;N_k,\delta_k]\,w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\dots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \Gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),\left(1-\sigma-\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,m},\left(1-\delta-\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{m+1,q}\end{matrix}\right.\right]\mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_{N_1,\dots,N_k}^{M_1,\dots,M_k}\left[w_1\left(\frac{y}{x}\right)^r,\dots,w_k\left(\frac{y}{x}\right)^r\right]\Gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p)\\(v_1,V_1),\dots,(v_q,V_q)\end{matrix}\right.\right]$$

$$\times D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\}dx, \quad (32)$$

when $\mathscr{f} \in \aleph$ and $y > 0.$

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Proof: Lemma 2 is used to achieve this outcome. The proof resembles the proof of Theorem 1.

The Fredholm integral equations with kernel, the product of the general polynomial, one variable H-function and the incomplete H-functions:

Lemma 3: Consider

(i) $m,n,p,q \in \{\mathbb{N}^{+} \cup 0\}$ thus $0 \le m \le q, 0 \le n \le p; U_j > 0\,(j=1,...,p),$
$V_j > 0(j=1,...,q); u_j\,(j=1,...,p)$ and $v_j\,(j=1,...,q)$ are complex parameters.

(ii) $\Re(\delta) > \Re(\sigma); \Re\left[\sigma + K\zeta + \tau L_G + \lambda\left(\frac{v_j}{V_j}\right)\right] > 0$; where $(j=1,...,m); \lambda > 0$

(iii) $|\arg(z)| < \frac{1}{2}\pi\aleph,$

where $\aleph = \sum_{j=1}^{n}|u_j U_j| + \sum_{j=1}^{m}|v_j V_j| - \sum_{j=n+1}^{p}|u_j U_j| - \sum_{j=m+1}^{q}|v_j V_j| > 0,$

Then

$$W^{\sigma-\delta}\left\{x^{-\delta} S_V^U\left[\mu\left(\frac{y}{x}\right)^{\zeta}\right] H_{P,Q}^{M,N}\left[T\left(\frac{y}{x}\right)^{\tau}\left|\begin{matrix}(e_P, E_P)\\(f_Q, F_Q)\end{matrix}\right.\right] \gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,q}\end{matrix}\right.\right]\right\}$$

$$= x^{-\sigma}\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\,\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}\left(\frac{y}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\,\gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q}\end{matrix}\right.\right]. \qquad (33)$$

Proof: We utilize the Weyl fractional integral given by (26) to prove the lemma 3. Express the general polynomial, the Fox's H-function in series and the incomplete H-function in contour integral form with the help of equations (5), (7) and (12). To achieve the desired outcome, swap the order of summation and integrals under the aforementioned requirements, then evaluate the integral using substitution method, and resultant contour type integral is represented as incomplete H-function.

Theorem 3: With the usual states (i), (ii) and (iii) mentioned in lemma 3, the following relation holds:

$$\int_0^{\infty} x^{-\sigma}\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\,\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}\left(\frac{y}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\, \gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix} \right] \mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_V^U\left[\mu\left(\frac{y}{x}\right)^{\zeta}\right] H_{P,Q}^{M,N}\left[T\left(\frac{y}{x}\right)^{\tau} \middle| \begin{matrix} (e_P,E_P) \\ (f_Q,F_Q) \end{matrix}\right] \gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),(u_2,U_2),\dots,(u_p,U_p) \\ (v_1,V_1),\dots,(v_q,V_q) \end{matrix}\right]$$

$$\times D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx, \tag{34}$$

Provided that $\mathscr{f} \in \aleph$ and $y > 0$.

Proof: Lemma 3 and equation (26) along with following the steps outlined in the demonstration of theorem 1 are used to establish theorem 3.

Lemma 4: Let

(i) $m,n,p,q \in \left\{\square^{+} \cup 0\right\}$ thus $0 \le m \le q, 0 \le n \le p; U_j > 0\,(j=1,\dots,p),$

$V_j > 0(j=1,\dots,q); u_j\,(j=1,\dots,p)$ and $v_j\,(j=1,\dots,q)$ are complex parameters.

(ii) $\Re(\delta) > \Re(\sigma); \Re\left[\sigma + K\zeta + \tau L_G + \lambda\left(\frac{v_j}{V_j}\right)\right] > 0;$ where $(j=1,\dots,m); \lambda > 0.$

(iii) $\left|\arg(z)\right| < \frac{1}{2}\pi\aleph,$

where $\aleph = \sum_{j=1}^{n}\left|u_j U_j\right| + \sum_{j=1}^{m}\left|v_j V_j\right| - \sum_{j=n+1}^{p}\left|u_j U_j\right| - \sum_{j=m+1}^{q}\left|v_j V_j\right| > 0,$

Then

$$W^{\sigma-\delta}\left\{x^{-\delta} S_V^U\left[\mu\left(\frac{y}{x}\right)^{\zeta}\right] H_{P,Q}^{M,N}\left[T\left(\frac{y}{x}\right)^{\tau} \middle| \begin{matrix} (e_P,E_P) \\ (f_Q,F_Q) \end{matrix}\right] \Gamma_{p,q}^{m,n}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,q} \end{matrix}\right]\right\}$$

$$= x^{-\sigma} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K}\, \mu^K \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G \phi'(L_G)}{G!\,V_g} T^{L_G} \left(\frac{y}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\, \Gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix}\right]. \tag{35}$$

Proof: The proof is comparable to lemma 3's proof. Equations (5), (7) and (14) can be used to reach this conclusion.

Theorem 4: With the usual conditions (i), (ii) and (iii) of lemma 4, the following relation holds:

$$\int_0^\infty x^{-\sigma} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K}\ \mu^K \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G\ \phi^{'}(L_G)}{G!\,V_g} T^{L_G} \left(\frac{y}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\ \Gamma^{m,n+1}_{p+1,q+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix} \right] \mathscr{f}(x)\,dx$$

$$= \int_0^\infty x^{-\delta} S_V^U \left[\mu\left(\frac{y}{x}\right)^{\zeta} \right] H^{M,N}_{P,Q}\left[T\left(\frac{y}{x}\right)^{\tau} \middle| \begin{matrix} (e_P,E_P) \\ (f_Q,F_Q) \end{matrix} \right] \Gamma^{m,n}_{p,q}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),(u_2,U_2),...,(u_p,U_p) \\ (v_1,V_1),...,(v_q,V_q) \end{matrix} \right]$$

$$\times D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx, \tag{36}$$

provided that $\mathscr{f} \in \aleph$ and $y > 0.$

Proof: This result can be obtained using lemma 4 and the proof resembles from the proof given for the theorem 3.

Mellin transform method for the outcome of Fredholm integral equation which contains the product of general polynomial, one variable H-function in series and incomplete H-function as kernel:

Theorem 5: When $\mathscr{f}, g \in \aleph,\ D^{\delta-\sigma}\{\mathscr{f}(y)\}$ exists $\lambda > 0, y > 0, \left|\arg(z)\right| < \frac{1}{2}\pi\aleph, \aleph > 0$ ($\aleph$ is same as mentioned in lemma 3), $\Re(\delta) > \Re(\sigma) > 0,$ then the following integral equation

$$g(y) = \int_0^\infty x^{-\delta} S_V^U \left[\mu\left(\frac{y}{x}\right)^{\zeta} \right] H^{M,N}_{P,Q}\left[T\left(\frac{y}{x}\right)^{\tau} \middle| \begin{matrix} (e_P,E_P) \\ (f_Q,F_Q) \end{matrix} \right] \gamma^{m,n}_{p,q}\left[z\left(\frac{y}{x}\right)^{\lambda} \right] \mathscr{f}(x)\,dx \tag{37}$$

has solution given by

$$\mathscr{f}(y) = \frac{\lambda}{2\pi i} y^{\delta-1} \lim_{\gamma\to\infty} \int_{c-i\gamma}^{c+i\gamma} y^{-s} \left\{ \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K}\ \mu^K \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G\ \phi^{'}(L_G)}{G!\,V_g} T^{L_G} \right.$$

$$\left. \times\ z^{-\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right)} g^{'}\left(\frac{-s-K\zeta-\tau L_G}{\lambda}\right) \right\}^{-1} \phi(s)\,ds, \tag{38}$$

provided that $\max_{1\le j\le n}\left\{ \Re\left(\frac{u_j-1}{U_j}\right) \right\} < -\Re\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right) < \min\left\{ \Re\left(\frac{v_j}{V_j}\right) \right\}.$

Proof: Changing $\mathscr{f}$ by $D^{\delta-\sigma}\{\mathscr{f}\}$ in (37), we have

$$g(y) = \int_0^\infty x^{-\sigma} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K}\ \mu^K \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G\ \phi^{'}(L_G)}{G!\,V_g} T^{L_G} \left(\frac{y}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\ \gamma^{m,n+1}_{p+1,q+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix} \right] D^{\delta-\sigma}\{\mathscr{f}(x)\}\,dx.$$

Now, multiply both sides the above equation by y^{s-1} then integrate it w.r.t. y from 0 to ∞, we reach

$$\phi(s)=\int_0^\infty y^{s-1}g(y)\,dy$$

$$=\int_0^\infty x^{-\sigma}\,D^{\delta-\sigma}\left\{f(x)\right\}\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\;\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}\left(\frac{1}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\left\{\int_0^\infty y^{s+K\zeta+\tau L_G-1}\gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda}\left|\begin{matrix}(u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p}\\(v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q}\end{matrix}\right.\right]dy\right\}dx\,.$$

Utilizing the result of Mellin transform of incomplete H -function, the above result moderates to

$$\phi(s)=\frac{1}{\lambda}\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\;\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}z^{-\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right)}$$

$$g'\left(\frac{-s-K\zeta-\tau L_G}{\lambda}\right)\frac{\Gamma(\sigma-s)}{\Gamma(\delta-s)}\int_0^\infty x^{s-\sigma}\,D^{\delta-\sigma}\left\{f(x)\right\}dx.$$

Inverting the above equation by applying theorem of Mellin inversion, we have

$$D^{\delta-\sigma}\{f(x)\}=\frac{\lambda}{2\pi i}\lim_{\gamma\to\infty}\int_{c-i\gamma}^{c+i\gamma}\left\{\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\;\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}\right.$$

$$\left.\times\,z^{-\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right)}g'\left(\frac{-s-K\zeta-\tau L_G}{\lambda}\right)\right\}^{-1}\frac{\Gamma(\delta-s)}{\Gamma(\sigma-s)}x^{\sigma-s-1}\phi(s)\,ds.$$

Operating both sides of the above equation by $D^{\sigma-\delta}$, which gives

$$f(x)=\frac{\lambda}{2\pi i}D^{\sigma-\delta}\left[\lim_{\gamma\to\infty}\int_{c-i\gamma}^{c+i\gamma}\left\{\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\;\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}\right.\right.$$

$$\left.\left.\times\,z^{-\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right)}g'\left(\frac{-s-K\zeta-\tau L_G}{\lambda}\right)\right\}^{-1}\frac{\Gamma(\delta-s)}{\Gamma(\sigma-s)}x^{\sigma-s-1}\phi(s)\,ds\right],$$

this finally yields

$$f(y)=\frac{\lambda}{2\pi i}y^{\delta-1}\lim_{\gamma\to\infty}\int_{c-i\gamma}^{c+i\gamma}y^{-s}\left\{\sum_{K=0}^{[V/U]}\frac{(-V)_{UK}}{K!}A_{V,K}\;\mu^K\sum_{G=0}^{\infty}\sum_{g=1}^{M}\frac{(-1)^G\,\phi'(L_G)}{G!\,V_g}T^{L_G}\right.$$

$$\left.\times\,z^{-\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right)}g'\left(\frac{-s-K\zeta-\tau L_G}{\lambda}\right)\right\}^{-1}\phi(s)\,ds,$$

which is required solution.

Theorem 6: If $\mathscr{f}, g \in \aleph,\ D^{\delta-\sigma}\{\mathscr{f}(y)\}$ exists $\lambda > 0, y > 0, |\arg(z)| < \frac{1}{2}\pi\aleph, \aleph > 0$ ($\aleph$ is same as mentioned in lemma 3), $\Re(\delta) > \Re(\sigma) > 0,$ then the following integral equation

$$g(y) = \int_0^\infty x^{-\sigma} \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K}\ \mu^K \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G\ \phi'(L_G)}{G!\,V_g} T^{L_G} \left(\frac{y}{x}\right)^{K\zeta+\tau L_G}$$

$$\times\, \Gamma_{p+1,q+1}^{m,n+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(1-K\zeta-\tau L_G-\sigma,\lambda\right),(u_j,U_j)_{2,p} \\ (v_j,V_j)_{1,m},\left(1-K\zeta-\tau L_G-\delta,\lambda\right),(v_j,V_j)_{m+1,q} \end{matrix} \right] D^{\delta-\sigma}\left\{\mathscr{f}(x)\right\} dx \qquad (39)$$

has a solution given by

$$\mathscr{f}(y) = \frac{\lambda}{2\pi i} y^{\delta-1} \lim_{\gamma\to\infty} \int_{c-i\gamma}^{c+i\gamma} y^{-s} \left\{ \sum_{K=0}^{[V/U]} \frac{(-V)_{UK}}{K!} A_{V,K}\ \mu^K \sum_{G=0}^{\infty}\sum_{g=1}^{M} \frac{(-1)^G\ \phi'(L_G)}{G!\,V_g} \right.$$

$$\left. \times T^{L_G}\ z^{-\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right)} G'\left(\frac{-s-K\zeta-\tau L_G}{\lambda}\right)\right\}^{-1} \phi(s)\, ds, \qquad (40)$$

provided that $\max\limits_{1\le j\le n}\left\{\Re\left(\frac{u_j-1}{U_j}\right)\right\} < -\Re\left(\frac{s+K\zeta+\tau L_G}{\lambda}\right) < \min\left\{\Re\left(\frac{v_j}{V_j}\right)\right\}.$

Proof: Result of this theorem is obtained on the similar lines of proof presented for theorem.

4. **Special cases:**

[a] Considering (19) and (20), we get the following results from theorem 1 and 2.

$$\int_0^\infty x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]},\ldots,\sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\ldots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!} B[N_1,\delta_1;\ldots;N_k,\delta_k]\, w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\ldots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \frac{\Gamma(\delta+\sum_{i=1}^{k} r\delta_i)}{\Gamma(\sigma+\sum_{i=1}^{k} r\delta_i)}\ {}_{p+1}\psi_{q+1}^{(\gamma)}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(\sigma+\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p} \\ \left(\delta+\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{1,q} \end{matrix} \right] \mathscr{f}(x)\, dx$$

$$= \int_0^\infty x^{-\delta} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\left[w_1\left(\frac{y}{x}\right)^r,\ldots,w_k\left(\frac{y}{x}\right)^r\right] {}_p\psi_q^{(\gamma)}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),(u_2,U_2),\ldots,(u_p,U_p) \\ (v_1,V_1),\ldots,(v_q,V_q) \end{matrix} \right]$$

$$\times D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx \qquad (41)$$

and

$$\int_0^\infty x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]},\ldots,\sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\ldots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!} B[N_1,\delta_1;\ldots;N_k,\delta_k]\, w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\ldots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \frac{\Gamma(\delta+\sum_{i=1}^{k} r\delta_i)}{\Gamma(\sigma+\sum_{i=1}^{k} r\delta_i)}\, {}_{p+1}\psi_{q+1}^{(\Gamma)}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),\left(\sigma+\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{2,p} \\ \left(\delta+\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{1,q} \end{matrix} \right] \mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\left[w_1\left(\frac{y}{x}\right)^{r},\ldots,w_k\left(\frac{y}{x}\right)^{r}\right] {}_{p}\psi_{q}^{(\Gamma)}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1,w),(u_2,U_2),\ldots,(u_p,U_p) \\ (v_1,V_1),\ldots,(v_q,V_q) \end{matrix} \right]$$

$$\times D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx. \qquad (42)$$

[b] Setting $w=0$ in (42) and utilizing the result of (21), we attain the required result.

$$\int_0^{\infty} x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]},\ldots,\sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\ldots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!} B[N_1,\delta_1;\ldots;N_k,\delta_k] w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\ldots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \frac{\Gamma(\delta+\sum_{i=1}^{k} r\delta_i)}{\Gamma(\sigma+\sum_{i=1}^{k} r\delta_i)}\, {}_{p+1}\psi_{q+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} \left(\sigma+\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_j,U_j)_{1,p} \\ \left(\delta+\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_j,V_j)_{1,q} \end{matrix} \right] \mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\left[w_1\left(\frac{y}{x}\right)^{r},\ldots,w_k\left(\frac{y}{x}\right)^{r}\right] {}_{p}\psi_{q}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,U_1),\ldots,(u_p,U_p) \\ (v_1,V_1),\ldots,(v_q,V_q) \end{matrix} \right] \times D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx.$$

(43)

[c] Put $U_j = V_j = 1$ in (41) and (42), we accomplish the following result from (22) and (23).

$$\int_0^{\infty} x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]},\ldots,\sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\ldots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!} B[N_1,\delta_1;\ldots;N_k,\delta_k] w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\ldots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \frac{\Gamma(\delta+\sum_{i=1}^{k} r\delta_i)}{\Gamma(\sigma+\sum_{i=1}^{k} r\delta_i)}\, {}_{p+1}\psi_{q+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \middle| \begin{matrix} (u_1,1,w),\left(\sigma+\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_2,1),\ldots,(u_p,1) \\ \left(\delta+\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_1,1),\ldots,(v_q,1) \end{matrix} \right] \mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_{N_1,\ldots,N_k}^{M_1,\ldots,M_k}\left[w_1\left(\frac{y}{x}\right)^{r},\ldots,w_k\left(\frac{y}{x}\right)^{r}\right] {}_{p}\gamma_{q}\left[\begin{matrix} (u_1,w),u_2,\ldots,u_p; \\ v_1,\ldots,v_q \qquad ; \end{matrix}\; z\left(\frac{y}{x}\right)^{\lambda} \right] D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx \qquad (44)$$

and

$$\int_0^{\infty} x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]},\ldots,\sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1!},\ldots,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!} B[N_1,\delta_1;\ldots;N_k,\delta_k] w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},\ldots,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \frac{\Gamma(\delta+\sum_{i=1}^{k} r\delta_i)}{\Gamma(\sigma+\sum_{i=1}^{k} r\delta_i)} \, {}_{p+1}\psi_{q+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \left| \begin{matrix} (u_1,1,w),\left(\sigma+\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_2,1),...,(u_p,1) \\ \left(\delta+\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_1,1),...,(v_q,1) \end{matrix} \right. \right] \mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_{N_1,...,N_k}^{M_1,...,M_k}\left[w_1\left(\frac{y}{x}\right)^{r},...,w_k\left(\frac{y}{x}\right)^{r} \right] {}_p\Gamma_q\left[\begin{matrix} (u_1,w),u_2,...,u_p; \\ v_1,...,v_q \quad ; \end{matrix} z\left(\frac{y}{x}\right)^{\lambda} \right] D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx. \tag{45}$$

[d] Setting $w=0$ in (45) and utilizing the result of (24), we obtain the following result.

$$\int_0^{\infty} x^{-\sigma} \sum_{\delta_1=0}^{[N_1/M_1]},...,\sum_{\delta_k=0}^{[N_k/M_k]} \frac{(-N_1)_{M_1\delta_1}}{\delta_1!},...,\frac{(-N_k)_{M_k\delta_k}}{\delta_k!} B[N_1,\delta_1;...;N_k,\delta_k]\, w_1^{\delta_1}\left(\frac{y}{x}\right)^{r\delta_1},...,w_k^{\delta_k}\left(\frac{y}{x}\right)^{r\delta_k}$$

$$\times \frac{\Gamma(\delta+\sum_{i=1}^{k} r\delta_i)}{\Gamma(\sigma+\sum_{i=1}^{k} r\delta_i)} \, {}_{p+1}\psi_{q+1}\left[z\left(\frac{y}{x}\right)^{\lambda} \left| \begin{matrix} \left(\sigma+\sum_{i=1}^{k} r\delta_i,\lambda\right),(u_1,1),...,(u_p,1) \\ \left(\delta+\sum_{i=1}^{k} r\delta_i,\lambda\right),(v_1,1),...,(v_q,1) \end{matrix} \right. \right] \mathscr{f}(x)\,dx$$

$$= \int_0^{\infty} x^{-\delta} S_{N_1,...,N_k}^{M_1,...,M_k}\left[w_1\left(\frac{y}{x}\right)^{r},...,w_k\left(\frac{y}{x}\right)^{r} \right] {}_pF_q\left[\begin{matrix} u_1,...,u_p; \\ v_1,...,v_q; \end{matrix} z\left(\frac{y}{x}\right)^{\lambda} \right] D^{\sigma-\delta}\left\{\mathscr{f}(x)\right\} dx. \tag{46}$$

Similarly, we can find special cases for theorem 3 and 4.

5. Conclusion:

Using the fractional integral operators solved the integral equations of the Fredholm type, whose kernel involves the product of incomplete H-functions, H-function of Fox, general polynomial, and generalized polynomials. Furthermore, using the Mellin transform technique, the Fredholm integral equation including the combination of H-function given by Fox, incomplete H-functions and generalized polynomial in the kernel is also resolved. Our findings can be reduced to a huge numeral of integral equations with different special functions, which is mentioned as a particular case, by appropriately giving the value of the various parameters to the incomplete H-functions.

References:

[1] S.D. Purohit, Solutions of fractional partial differential equations of quantum mechanics, Adva. Appl. Math. Mech. 5, 639–651, 2013.

[2] D. Kumar, F.Y. Ayant and J. Choi, Application of product of the multivariable A-function and the multivariable Srivastava polynomials, East Asian Math. J., 34(3), 295–303, 2018.

[3] J. B. Sharma, K. K. Sharma, S. D. Purohit, A. Antagana, Hybrid Watermarking Algorithm using Finite Radon and Fractional Fourier Transform, Fundamenta Informaticae, 151(4), 523–543, 2017.

[4] D. L. Suthar, R. K. Parmar, S. D. Purohit, Fractional calculus with complex order and generalized hypergeometric functions, Nonlinear Sci. Lett. A, 8(2), 156–161, 2017.

[5] R. K. Parmar, Some generating relations for generalized extended hypergeometric functions involving generalized fractional derivative operator, J. Concr. Appl. Math., 12, 217–228, 2014.

[6] N. Ghiya and S. Manjula, Generalized fractional integrals involving product of multivariable H-function and a generalized M-series, GIS Science Journal, 8(7), 336-346, 2021.

[7] V. B. L. Chaurasia and D. Kumar, On the solutions of integral equations of Fredholm type with special functions, Tamsui Oxf. J. of Math. Sci., 28(1), 49-61, 2012.

[8] V.B.L. Chaurasia and Yudhveer Singh, New Generalization of Integral Equations of Fredholm Type using Aleph-function, International Journal of Modern Mathematical Sciences, 9(3), 208-220, 2014.

[9] V. B. L. Chaurasia and V. Gill, Extension of integral equations of Fredholm type involving special function, Thai Journal of Mathematics, 15(2), 491-501, 2017.

[10] F. Ayant and D. Kumar, Fredholm type integral equation with special functions, Acta Univ. Sapientiae, Mathematica, 10(1), 5-17, 2018.

[11] M. K. Bansal and Devendra Kumar, On the integral operators pertaining to a family of incomplete I-functions, AIMS Mathematics, 5(2), 1247–1259, 2020.

[12] M. K. Bansal, Devendra Kumar, J. Singh and K. S. Nisar, On the Solutions of a Class of Integral Equations Pertaining to Incomplete H-Function and Incomplete $\overline{H}$-Function, MDPI Mathematics, 8(819), 1-14, 2020.

[13] D. Kumar and F. Ayant, Fractional Calculus Operators Pertaining to Multivariable Aleph-function, J. of Bol. Soc. Paran. Mat., 40, 1-10, 2022.

[14] H. M. Srivastava, A contour integral involving Foxs H-function, Indian J. Math., 14, 1-6, 1972.

[15] H. M. Srivastava, The Weyl fractional integral of a general class of polynomials, Boll. Un. Mat. Ital. (Series B), 2(6), 219-228, 1983.

[16] B. L. J. Braaksma, Asymptotic expansions and analytic continuations for a class of Barnes-integrals, Composito Math., 15, 339-341, 1963.

[17] M. K. Bansal, D. Kumar, J. Singh and K. S. Nisar, On the solutions of a class of integral equations pertaining to incomplete H-function and incomplete $\overline{H}$-function, Math., 8(819), 1-14, 2020.

[18] M. K. Bansal, D. Kumar, K. S. Nisar and J. Singh, Application of incomplete H-functions in determination of Lambert's law, J. Interdis. Math., 22(7), 1205-1212, 2020.

[19] R. K. Parmar and R. K. Saxena, The incomplete generalized τ-hypergeometric and second τ-Appell functions, J. Korean Math. Soc., 53(2), 363-379, 2016.

[20] R. K. Parmar and R. K. Saxena, *Incomplete extended Hurwitz-Lerch Zeta functions and associated properties*, Commun. Korean Math. Soc., 32(2), 287-304, 2017.

[21] S. Meena, S. Bhatter, K. Jangid and S. D. Purohit, *Certain integral transforms concerning the product of family of polynomials and generalized incomplete functions*, Moroccan J. of Pure and Appl. Anal., 6(2), 243-254, 2020.

[22] H. M. Srivastava, R. K. Saxena and R. K. Parmar, *Some families of the incomplete H- function and the incomplete $\overline{H}$-functions and associated integral transforms and operators of fractional calculus with applications,* Russ. J. Math. Phys., 25(1), 116-138, 2018.

[23] M. K. Bansal, Junesang Choi, *A Note on Pathway Fractional Integral Formulas Associated with the Incomplete H-Functions*, Int. J. Appl. Comput. Math., 5(133), 1-10, 2019.

[24] H. M. Srivastava, K. C. Gupta and S. P. Goyal, *The H-functions of one and two variables with applications*, South Asian publishers, New Delhi, 1982.

[25] H. M. Srivastava, M. A. Chaudhry, R. P. Agarwal, *The incomplete Pochhammer symbols and their applications to hypergeometric and related functions*, Integral Transforms Spec. Funct. 23, 659- 683, 2012.