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Stability Of Equilibrium States For A Stochastically Perturbed Pielou's Equation [

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Abstract:

It is widely recognized that the theory of stochastic difference equations is a mathematical area of great interest. Concrete systems and processes is one of its subareas, which is of some interest nowadays. Specifically, in fields such as systems biology, ecology, biochemistry, genetics, and physiology dynamics, numerous population models are described by studying linear and nonlinear difference equations and systems. These studies have been very productive and helpful to develop the basic theory of the qualitative behaviour of nonlinear rational and exponential difference equations and systems. In this paper, we investigate a type of Pielou's difference equation that is subject to additive stochastic perturbations modelled as a sequence of independent random variables (ξ_n) with zero mean and unit variance. These perturbations are proportional to the deviation of the system state (X_n) from its equilibrium points, resulting in the following stochastic difference equation.

$$X_{n+1} = \frac{\alpha X_n}{1 + X_{n-1}} + \sigma(X_n - \mathrm{X}^*)\xi_{n+1}$$

where parameters $a \, and$ σ have arbitrary values. Utilizing the general method of constructing Lyapunov functional for stochastic difference equations in discrete time, we establish necessary and sufficient conditions for the asymptotic mean square stability of the two equilibrium points (zero and positive). These conditions also serve as sufficient conditions for the stability in probability of the equilibrium points of the original nonlinear equation. To validate the derived results, numerical examples and simulations of equation solutions are presented, accompanied by numerous graphical illustrations of stability trajectories of solutions.

Keywords: Equilibrium points; difference equation; stochastic perturbations; stability in probability, Pielou's equation,.

1. Introduction

Stochastic discrete dynamical systems is a branch of mathematics that deals with analyzing and modelling systems influenced by randomness. The origins of stochastic difference equations or stochastic discrete dynamical systems trace back to investigations into phenomena characterized by random fluctuations, such as particle motion and stock market variability. Nonlinear stochastic difference equations represent a significant category within nonlinear difference equations. Unlike their linear counterparts, these equations are inherently complex and do not lend themselves easily to exact solutions or predictions, occasionally exhibiting chaotic behaviour under certain conditions. Recent interest in these systems stems from their relevance in mathematical models describing real-world scenarios across various fields such as probability theory, statistical problems, stochastic time series, combinatorial analysis, number theory, geometry, electrical networks, radiation quanta, genetics, economics, psychology, and sociology. Such equations also arise as discrete analogs and numerical solutions of stochastic differential and delay differential equations, modelling a wide array of phenomena.

Discrete time model is the most appropriate mathematical description of life histories of organism whose reproduction occurs only once a year during a very short season. These models are widely used in fisheries and many organisms. In 1965, Pielou's equation, as a discrete analogue of the delay logistic equation, was proposed by Pielou [8, 9].

$$X_{n+1} = \frac{a\,X_n}{1+\,X_{n-1}}, n = 0,1,2,\dots, \tag{1.1}$$

where a is a positive real number.

In order to find equilibrium points (fixed points) of the previous equation, we solve the following algebraic equation

$$X^* = \frac{aX^*}{1+\,X^*}. \tag{1.2}$$

Note that

$$X_1^* = 0 \tag{1.3}$$

And

$$X_2^* = a - 1 \tag{1.4}$$

Represent the solutions of equation (1.2) which simultaneously serve as the equilibrium points of the equation (1.1).

In the following section, we will consider that Equation (1.1) is subject to stochastic perturbations that are directly proportional to the deviation of the system state X_n from the equilibrium point X^*. Under this assumption, we establish some sufficient conditions for the stability in probability of the equilibrium points 1.2 and 1.3. The results are demonstrated through the trajectories of the analyzed equations.

The aim of this paper is to examine the stability in probability of the equilibrium state of Pielou's Equation type (1.1) affected by stochastic perturbations.

2. Stochastic perturbations

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $\mathcal{F}_n$ be a family of sigma-fields of $\mathcal{F}$ indexed by $k = \{0, 1, \dots\}$, E be the mathematical expectation, $\xi_n, n \in N$, be a sequence of $\mathcal{F}_n$ -adapted random variables such that $E(\xi_n) = 0$, $E(\xi_n^2) = 1$. Let a process X_n be a solution of the equation

$$X_{n+1} = \frac{aX_n}{1+ X_{n-1}} + \sigma (X_n - X^*)\xi_{n+1}, \qquad n = 0, 1, 2, \ldots, \tag{2.1}$$

with an $\mathcal{F}_0$-adapted initial condition

$$X_i = \lambda_i, \quad i \in k_0 = \{-1,0\}. \tag{2.2}$$

Here X^* is an equilibrium point (1.3) or (1.4) of the equation (1.1), and σ is an arbitrary constant. It should be noted that the equilibrium point X^* is also a solution of Equation (2.1).

Putting in the equation, (2.1) $X_n = X^* + Y_n$ we obtain

$$X^* + Y_{n+1} = \frac{a(X^*+Y_n)}{1+ X^*+Y_{n-1}} + \sigma (Y_n)\xi_{n+1}$$
$$= \frac{aX^*+ a\,Y_n}{1+ X^*+Y_{n-1}} + \sigma (Y_n)\xi_{n+1}$$

which gives the equation

$$Y_{n+1} = \frac{aX^*+ a\,Y_n}{1+ X^*+Y_{n-1}} - X^* + \sigma (Y_n)\xi_{n+1}. \tag{2.3}$$

It is evident that the stability of the zero solution of Equation (2.3) corresponds to the stability of the solution X^* of Equation (2.1).

Note that the equation (2.3) is a nonlinear equation with an order of non- linearity higher than one. From[29], it is defined that, in this regard, if the zero solution of the linear approximation of (2.3) is asymptotically mean square stable, then this condition is sufficient for the stability in probability of the zero solution of the nonlinear equation (2.3).

Lemma 1: the linear approximation of (2.3) has the form

$$Z_{n+1} = \frac{a}{1+X^*} Z_n - \frac{aX^*}{(1+ X^*)^2} Z_{n-1} + \sigma(Z_n)\xi_{n+1}. \tag{2.4}$$

Proof: Using the equality

$$\frac{1}{\mathrm{p+y}} = \frac{1}{\mathrm{p}} - \frac{\mathrm{y}}{\mathrm{p}^2} + o(y), \qquad \text{where p} \neq 0 \text{ and } \lim_{y\to 0}\frac{o(y)}{y} = 0.$$

We have

$$\frac{aX^* + a\,Y_n}{1+ X^* + Y_{n-1}} = X^* - \frac{aX^*}{(1+ X^*)^2} Y_{n-1} + \frac{a}{1+X^*} Z_n + o(y) \tag{2.5}$$

Substituting (2.5) in (2.3), we get (2.4)

Now putting $X^* = 0$ in (2.3) and (2.4) give us the following equations

$$Y_{n+1} = \frac{a\,Y_n}{1+Y_{n-1}} + \sigma (Y_n)\xi_{n+1} \tag{2.6}$$

And

$$Z_{n+1} = a\, Z_n + \sigma\,(Z_n)\xi_{n+1}. \qquad (2.7)$$

In a similar way for the equilibrium point(1.5), we can obtain

$$Y_{n+1} = 1 + a\,\left[\frac{a-1+\,Y_n}{a+Y_{n-1}} - 1\right] + \sigma\,(Y_n)\xi_{n+1}. \qquad (2.8)$$

And

$$Z_{n+1} = Z_n - \left(\frac{a-1}{a}\right) Z_{n-1} + \sigma\,(Z_n)\xi_{n+1}. \qquad (2.9)$$

Next, we present the essential definitions and lemmas necessary for deriving our main results.

Definition 2.1: The zero solution of(2.6), $(2.8)$is called stable in probability if for any $\varepsilon > 0$, $\epsilon_1 > 0$ there exists a $\beta > 0$ such that

$$P\left\{\sup_{i\in k}|Y_i| \; > \epsilon\right\} < \epsilon_1$$

for any initial function λ which is less than β with probability 1, i.e.,

$$P\left\{\sup_{i\in k_0}|\lambda_i| > \beta\right\} = 1.$$

Definition 2.2: The zero solution of (2.7), (2.9) is called mean square stable if for any $\epsilon > 0$ there exists a $\beta > 0$ such that $E|Z_i|^2 < \epsilon$, $i \in k$, for any initial function λ_i such that $\sup_{i\in S_0} E\,|\lambda_i| < \beta$. The zero solution of (2.7), (2.9) is called asymptotically mean square stable if it is mean square stable and for each initial function λ_i such that $\sup_{i\in S_0} E\,|\lambda_i|^2 < \infty$ the solution of (2.7), (2.9) satisfies the condition $\lim_{i\to\infty} E|Z_i|^2 = 0$.

3. Stability of equilibruim points

By applying equation (2.6) and the inequalities derived from [22], we obtain

$$|a| < 1,$$

$$\sigma^2 < 1 - a^2.$$

the necessary and sufficient conditions for the asymptotic mean square stability of the zero solution (1.4) of equation (2.7). These conditions are also sufficient for the stability in probability of the zero solution of equation (2.6), and consequently, for the stability in probability of the zero equilibrium point of equation (2.1).

Similarly, we derive the necessary and sufficient conditions for the asymptotic mean square stability of the zero solution of equation (2.9). These conditions also serve as sufficient conditions for the stability in probability of the zero solution of equation (2.8), and consequently, for the equilibrium point (1.5) of equation (2.1).

$$\left|\frac{1-a}{a}\right| < 1,$$

$$\sigma^2 < \frac{\left[1+\left(\frac{1-a}{a}\right)\right]\left[\left(1-\left(\frac{1-a}{a}\right)\right)^2-1\right]}{1-\frac{1-a}{a}}.$$

Note that the stability conditions are independent of the parameter σ.

4. Numerical Examples

In this section, we give some numerical examples for validating the outcomes of previous sections. All plots in this section are drawn using Matlab.

Example 1:

We let $X_0 = 0.3,\ a = 2.4\ ,\ \sigma = 0.3\ , X^* = 1.4.$ So, we deal with the equation

$$X_{n+1} = \frac{2.4X_n}{1+X_{n-1}} + 0.3(X_n - 1.4)\xi_{n+1}. \tag{4.1}$$

We find that

$$\frac{1-a}{a} = \frac{1-2.4}{2.4} < 0, \qquad \left|\frac{1-a}{a}\right| = \left|\frac{1-2.4}{2.4}\right| < 1\,, \qquad \frac{\left(\frac{2a-1}{a}\right)^2 - 1}{2a-1} > \sigma^2 = 0.3^2.$$

So, the positive equilibrium of (4.1) is stable in probability. (See Figure 1).

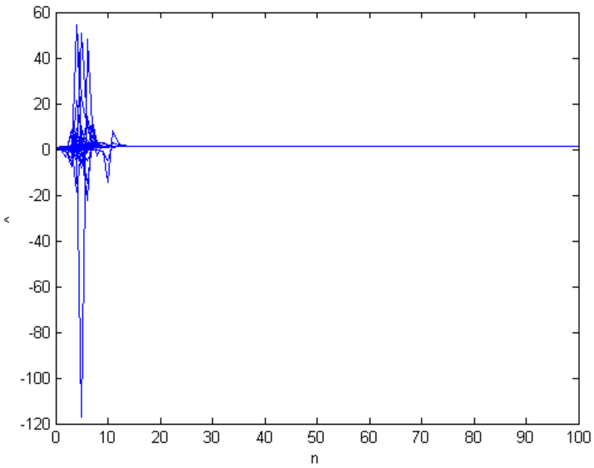

Figure 1 : 300 trajectories of the equation (4.1).

In Figure 1, 300 trajectories of the solutions of equation (4.1). One can see that all trajectories converge in probability to the stable positive equilibrium, so all trajectories converge to this equilibrium, (by using MATLAB).

Example 2 :

We let $X_0 = 0.3,\ a = 0.4\ ,\ \ \sigma = 0.75,\ X^* = 0$. So, we deal with the equation

$$X_{n+1} = \frac{0.4X_n}{1+X_{n-1}} + 0.75X_n\xi_{n+1}. \tag{4.2}$$

We find that

$$|a| = |0.4| < 1, \qquad 1 - a^2 = 1 - (0.4)^2 = 0.84 > \sigma^2 = 0.75^2.$$

So, the zero equilibrium of (4.2) is stable in probability. (See Figure 2).

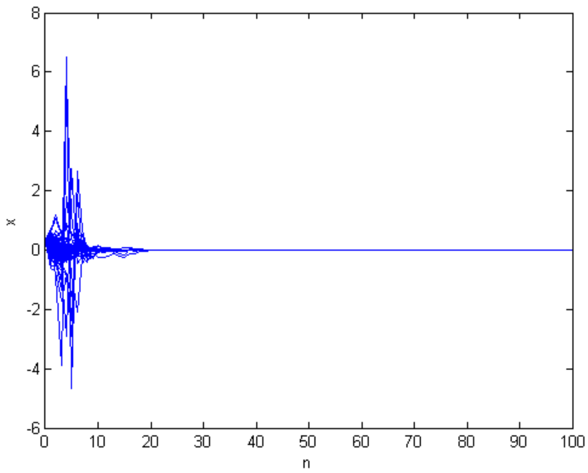

Figure 2 : 300 trajectories of the equation (4.2).

In Figure 2, 300 trajectories of the solutions of equation(4.2). One can see that all trajectories converge in probability to the stable zero equilibrium, so all trajectories converge to this equilibrium, (by using MATLAB).

Example 3 :

We let $X_0 = 0.3,\ a = 0.4\ ,\ \ \sigma = 2.1,\ X^* = 0$. *So*, we deal with the equation

$$X_{n+1} = \frac{0.4X_n}{1 + X_{n-1}} + 2.1X_n\xi_{n+1}. \tag{4.3}$$

Thus, the zero equilibrium of (4.3) is unstable. Since,

$$\sigma^2 = 2.1^2 > (1 - 0.4^2).$$

(See Figure 3).

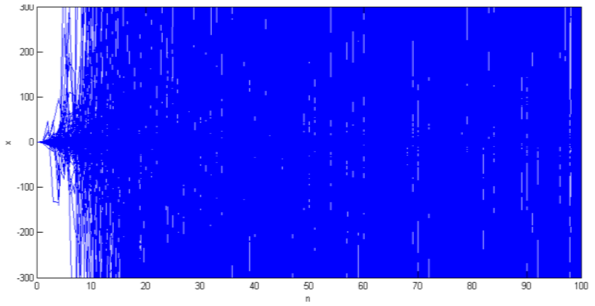

Figure 3: 300 trajectories of the equation (4.3).

In Figure 3, 300 trajectories of the solutions of equation(4.3). One can see that all trajectories go out of the zero equilibrium, (by using MATLAB).

Example 4 :

We let We let$X_0 = 0.3,\ a = 2.4\ ,\ \sigma = 1.4, X^* = 1.4.$ So, we deal with the equation

$$X_{n+1} = \frac{2.4X_n}{1 + X_{n-1}} + 1.4(X_n - 1.4)\xi_{n+1}. \qquad (4.4)$$

Thus, the positive equilibrium of (4.4) is unstable. Since,

$$\frac{\left(\frac{2a-1}{a}\right)^2 - 1}{2a - 1} < \sigma^2 = 1.4^2$$

(See Figure 4).

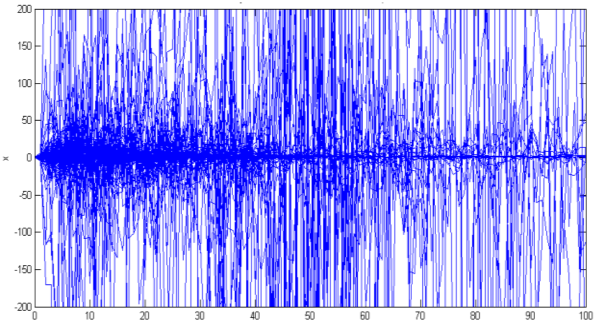

Figure 4: 300 trajectories of the equation(4.4).

In Figure 4, 300 trajectories of the solutions of equation (4.4). one can see that all trajectories go out of the positive equilibrium, (by using MATLAB).

5. Conclusion

This paper devoted to investigation of the stability in probability of the two equilibrium points (zero and positive) of Pielou's difference equation that is subject to additive stochastic perturbations

$$X_{n+1} = \frac{\alpha X_n}{1 + X_{n-1}} + \sigma(X_n - X^*)\xi_{n+1} \tag{5.1}$$

Where parameters α and σ have arbitrary values, X^* is an equilibrium point of (5.1), $\xi_n, n \in N$, be a sequence of $\mathcal{F}_n$ -adapted random variables such that $E(\xi_n) = 0, E(\xi_n^2) = 1$ and the initials X_0, X_{-1} are non-negative real numbers. It is shown that for equilibrium points of the considered equation it is possible to get conditions for stability in probability in an analytical form and can be obtained numerically via MATLAB. The obtained results are illustrated by examples and figures with stable and unstable solutions. The authors continues this work and hopes to involve all other interested researchers and applications in it.

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