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# Menger Space and Some Contraction Mappings

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**Abstract:**

Menger space is a probabilistic metric space introduced by K. Menger [15] in 1942 as one of the generalizations of metric space. The two different types of contraction mappings in probabilistic metric spaces are the creation of V.M. Sehgal [20-21], and T.L. Hicks [14]. Since then many researchers have introduced generalizations of these contraction mappings and established fixed point theorems in Menger space.

In this paper, we discuss Menger space and contraction mapping in Menger space. Also, presents the interrelationship between some contraction mapping with examples.

**Keywords:** Probabilistic metric space, Menger space, Sehgal, and Hicks contraction.

## 1. Introduction

In the early 19th century mathematicians were studying various spaces, mainly spaces of functions where they had various notions of convergence. For each space, its concepts of convergence were introduced and studied. To overcome this problem, in 1906 French mathematician M. Frechet [11] gave the axiomatic notions of metric space, this name metric space was given by F. Hausdorff in 1914. These metric spaces provide a deterministic framework for measuring distances between two points. Representing or giving the deterministic approach for the distance between two points in space by a single number is over-idealization. In such cases where we can't predict the distance by a single number, give a probable answer. So, looking at the distance concept as a statistical/ probabilistic rather than a determinate one is appropriate. The Austrian mathematician Karl Menger in 1942 had given the idea of statistical metric space later called probabilistic metric space to overcome uncertainties in cases of the distance between points in spaces. The idea is to replace the distance function with a probability distance function.

And $F(u,v)(t) = P(d(u,v) < t), t > 0$. That is the value of the distribution function in between $u$ and $v$ is equal to the probability that the distance between $u$ and $v$ is less than $t$. So, Menger space is a space in which the concept of distance is considered to be probabilistic, rather than deterministic and the theory of Menger spaces is of fundamental importance in probabilistic functional analysis. Schweizer and Sklar [22] have investigated several of these structures. Particularly, a lot of work has been done on the existence of fixed points of mappings in such spaces. For more details on these spaces see references [2-4], [8-10], [23]- [28].

Fundamental works in probabilistic metric spaces for mathematics researchers were done by V. M. Sehgal [20], who introduced a natural probabilistic version of Banach contraction [1]. Using this contraction, V. M. Sehgal and A.T. Bharucha Reid [21] established the first fixed point theorem in

Menger space in 1972. Another probabilistic contraction mapping was introduced by T. L. Hicks in 1983[14]. Then after many researchers introduced its extension and generalized forms see references [5-7], [[17-19].

## 2. Preliminaries notes:

**Definition 2.1 [5]:** A function F: $\mathbb{R} \rightarrow \mathbb{R}^{+}$ is said to be a ***distribution function*** if a function is non-decreasing function, left continuous with inf $\{F(x) : x \in \mathbb{R}\} = 0$,

and $\sup\{F(x) : x \in \mathbb{R}\} = 1$.

We denote L for the set of all distribution functions, and H stands for the heavy side function, which is defined as:

$$H(x) = \begin{cases} 0, & if\ x \leq 0 \\ 1, & if\ x > 0\,. \end{cases}$$

For reference, we also record the definition of metric space.

**Definition 2.2 [8]:** A metric space is an ordered pair $(X, d)$, where $X$ is an abstract set and $d$ is a mapping of $X \times X \rightarrow R$, satisfying the following axioms:

**M$_1$:** $d(p,q) = 0$ if and only if $p = q$ ( Identity);

**M$_2$:** $d(p,q) \geq 0$ ( Positivity);

**M$_3$:** $d(p,q) = d(q,p)$ ( Symmetry);

**M$_4$:** $d(p,q) \leq d(p,r) + d(r,q)$ ( Triangle inequality).

**Definition 2.2 [7]:** Let $F: X \times X \rightarrow L$ (set of all distribution functions) be a distribution function i.e., $F$ associates a distribution function $F(p,q)$ with every pair $(p,q)$ of points in a non-empty set $X$. Then, a pair $(X,F)$ is said to be a ***Probabilistic metric space*** (abbreviated as ***Pm-space***) if the distribution function $F(p,q)$, also denoted by $F_{p,q}$, satisfies the following conditions:

**(I)** $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$,

**(II)** $F_{p,q}(0) = 0$ for every $p, q \in X$,

**(III)** $F_{p,q}(x) = F_{q,p}(x)$ for every $p, q \in X$, and

**(IV)** $F_{p,q}(x+y) = 1$ if and only if $F_{p,r}(x) = 1$ and $F_{r,q}(y) = 1$.

Here, $F_{p,q}(x)$ represents the value of $F_{p,q}\ at\ x \in \mathbb{R}$.

**Example 2.1:** Let $(X,d)$ be metric space where $X = [0,2]$with usual metric $d(x,y) = |x-y|$ and distribution function $F$ defined as:

$$F_{x,y}(t) = \begin{cases} e^{\frac{d(x,y)}{t}}, & if\ t > 0, \\ 0, & if\ t = 0. \end{cases} \quad \text{for all } x, y \in X.$$

Then, $(X,F)$ be PM space.

**Definition 2.3 [13]:** A function $T : [0,1] \times [0,1] \rightarrow [0,1]$ is referred to as a ***Triangular norm***

(shortly T-norm) if it satisfies the following conditions:

**T$_1$:** $T\ (0,0) = 0,$

**T$_2$:** $T\ (a,1) = a$ for all $a \in [0,1]$,

**T$_3$:** $T\ (a,b) = t\ (b,a)$ for all $a,b \in [0,1]$,

**T$_4$:** $if\ a \leq c, b \leq d$ then $T\ (a,b) \leq T(c,d)$ and

**T$_5$:** $T\ (t\ (a,b),c) = T\left(a,t\ (b,c)\right)$, where $a,b,c,d \in [0,1]$.

**Example 2.2** of t-norms

$T(a,b) = \max\{(a+b)-1{,}0\}$ and $T(a,b) = \min\{a,b\}$

The four basic standard t-norms are:

(i) The minimum t-norm, $T_M$, is defined by $T_M(x,y) = \min\{x,y\}$,

(ii) The product t-norm, $T_p$, is defined by $T_p(x,y) = x,y$,

(iii) The Lukasiewicz t-norm, $T_L$, is defined by $T_L(x,y) = \max\{x+y-1{,}0\}$,

(iv) The weakest t-norm, the drastic product, $T_D$, is defined by

$$T_D(x,y) = \begin{cases} \min(x,y) \ \ if & \max(x,y) = 1, and \\ 0 & otherwise \end{cases}$$

Concerning the point-wise ordering, we have the following inequalities

$T_D < T_L < T_P < T_M$.

**Definition 2.4: [13]** A Menger space is a triplet $(X,F,T)$, where $X$ is a non-empty set, $F$ is a function defined on $X \times X$ to the set of distribution functions and $T$ is a $t$-norm such that the following are satisfied:

**(I)** $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$,

**(II)** $F_{p,q}(0) = 0$ for every $p,q \in X$,

**(III)** $F_{p,q}(x) = F_{q,p}(x)$ for every $p,q \in X$, and

**(IV)** $F_{p,q}(t+s) \geq T(F_{p,r}(t), F_{r,q}(s))$, for every $t,s > 0$ & $p,q,r \in X$.

**Definition 2.5:** [6] Let $(X,F,T)$ be a **Menger Space** and $T$ be a continuous t-norm

**(i)** A sequence $\{x_n\}$ in $X$ is said to **converge** to a point $x$ in $X$ (written $x_n \to x$) if for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x}(\epsilon) > 1-\lambda$ for all $n > N_{\epsilon,\lambda}$.

**(ii)** A sequence $\{x_n\}$ in $X$ is said to be a **Cauchy** if, for every $\epsilon > 0$ and $\lambda \in (0,1)$, there exists positive integer $N_{\epsilon,\lambda}$ such that $F_{x_n,x_m}(\epsilon) > 1-\lambda$ for all $n,m > N_{\epsilon,\lambda}$.

**(iii)** A Menger space $(X,F,T)$ is said to be complete if every Cauchy sequence is convergent in $X$.

**(iv)** A mapping $f: X \to X$ is said to be **continuous** in Menger space $(X, F, T)$ at point $x \in X$ if for each $\lambda \in (0,1)$ there exists a real number $\delta \in (0,1)$ satisfying the condition:
$F_{x,y}(t) \geq 1 - \delta \ implies \ F_{fx,fy}(t) \geq 1 - \lambda$ for each $t > 0$ and $x, y \in X$.

**Definition 2.6: [8]** **Fixed point** of a self-mapping function $f: X \to X$ is an element $x \in X$ such that $f(x) = x$.

**Example 2.3:** $\boldsymbol{f(x) = x^2}$ have two fixed points 0 &1 but $f\ (x) = \ x + 2$ have no fixed point.

**Definition 2.7: [8] Common fixed point of self-mapping functions** $f, g: X \to X$ is an element $x \in X$

such that $f(x) = x = g(x)$.

**Example 2.4: Let** $f, g: X \to X$ be functions such that $f(x) = x^2$ and $g(x) = tanx$, then 0 is a common

fixed point of $f$ and $g$.

**Definition 2.8: [1]** Let $(X, d)$ be a metric space. Then, a mapping $f: X \to X$ is said to be a **contraction mapping** if there exists a fixed constant $\alpha \in [0,1)$ such that $d\big(f(x), f(y)\big) \leq \alpha\, d(x, y), \forall\, x, y \in X$.

**Example 2.5:** Let function $f: [0,2] \to [0,2]$ be defined by

$$f(x) = \begin{matrix} 0 \text{ for } x \in [0,1] \\ 1 \quad \text{for } x \in (1,2] \end{matrix}$$

Then, $f^2$ is a contraction but $f$ is not a contraction. (why?)

## 3. Contraction Mapping in Menger Space:

In 1966, V. M. Sehgal [20] first defined probabilistic contraction in his PhD dissertation at Wayne State University as:

**Definition 3.1:** Let $(X, F)$ be a probabilistic metric space. A mapping $T: X \to X$ is a **probabilistic contraction or Sehgal contraction** if there exists $k \in (0,1)$ such that $F_{Tp,Tq}(kt) \geq F_{p,q}(t)$

for all $p, q \in X$, and $t > 0$.

The interpretation of Sehgal Contraction is as follows: The probability that the distance between the image points $Tp$ and $Tq$ is less than $kt$ is at least equal to the probability that the distance between $p, q$ that is less than $t$.

In 1983, T.L. Hicks [14] defined another contraction mapping in probabilistic metric space as:

**Definition 3.2:** A mapping $T: X \to X$ in probabilistic metric space $(X, F)$ is said to be Hicks contraction or **C-contraction** if there exists $k \in (0,1)$ such that for every $p, q \in X$, and every $t > 0$:

$$F_{p,q}(t) > 1 - t \Rightarrow F_{Tp,\ Tq}(kt) > 1 - kt.$$

In 2005, D. Mihet [16] introduced a weaker form of Hicks contraction and defined it as:

**Definition 3.3:** A mapping $T: X \to X$ is said to be **weak - Hicks contraction (w-H contraction)** if there exists $k \in (0,1)$ such that, for all $p, q \in S$.

$$(w-H): t \in (0,1),\ F_{p,q}(t) > 1-t \Rightarrow F_{Tp,\ Tq}(kt) > 1-kt.$$

Example: Let $X = [0,\infty)$, and $F_{p,q}(t) = \frac{\min(p,q)}{\max(p,q)}, \forall\, p,q \in X,\ p \neq q.$ Then, $(X,F,T)$ be a complete Menger space under triangular norm $T = T_P > T_L$. It can be seen that the mapping $g: X \to X$, $g(x) = \{ \begin{smallmatrix} 0, if\ x=0 \\ 1, if\ x>0 \end{smallmatrix}$

is a $w-H$ contraction for $k \in (0,1)$.

## 3.1 Generalized form of probabilistic contraction:

A probabilistic (m, k) contraction is a generalization of Sehgal contraction, where $m \geq 1$ and $k \in (0,1)$ and is defined as:

**Definition 3.1.1:** [13] If $(X,F)$ is a PM - space, $m \geq 1$ and $k \in (0,1)$, a function $f: X \to X$ is called probabilistic **(m, k)-contraction** if for any $p,q \in X$ there is an $i$ with $1 < i < m$ such that for every $t > 0$,

$$F_{f^i p, f^i q}(k^i t) \geq F_{p,q}(t).$$

If $m = 1$ and $k \in (0,1)$ then a probabilistic $(1,k)$-Sehgal contraction, $f$ is a probabilistic

Sehgal contraction.

**As a generalization of C-contraction, we have**

**Definition 3.1.2:** [13] If $(X,\varphi)$ is a PM - space, $m \geq 1$ and $k \in (0,1)$, a function $f: X \to X$ is called a **(m, k)-C-contraction** if for any $p,q \in X$ there is an $i$ with $l < i < m$ such that for every $t > 0$.

$$F_{p,q}(t) > 1-t \Rightarrow F_{f^i p, f^i q}(k^i t) > 1-k^i t.$$

If $m = 1$ and $k \in (0,1)$ then a probabilistic $(1,k)$-C-contraction $f$ is a probabilistic C-contraction.

**g-contraction mapping** is another generalization of Hick's C-contraction in probabilistic metric space which is defined as:

**Definition 3.1.3: [16]** Let $f,g$ be two mappings defined on a Menger space $(X,F,T)$ with values into itself, and let us suppose that $g$ is bijective. The mapping $f$ is called a **probabilistic g-contraction** with a constant $k \in (0,1)$ if

$t > 0$ and $F_{g(x),g(y)}(t) > 1-t$ impies $F_{f(x),f(y)}(kt) > 1-kt$.

## 4. Results and Conclusions:

### 4.1 Interrelationship:

#### 4.1.1 Every metric space is a probabilistic metric space:

Every metric space can be shown as a probabilistic metric space if we set

$F_{p,q}(x) = H(x - d(p,q))$ for every pair of points $(p,q)$ in the metric space.

This can be illustrated as follows:

$$F_{p,q}(x) = H(x - d(p,q))$$
$$= H(x - 0) iff\ p = q$$

$= H(x) = 1, \quad x > 0$ as $H(x)$is a distribution function. This shows that $F_{p,q}(x) = 1$ for every $x > 0$ if and only if $p = q$.

For proving $F_{p,q}(0) = 0$, consider

$$F_{p,q}(0) = H\big(0 - d(p,q)\big) = 0\ as - d(p.q) < 0.$$

For proving, $F_{p,q}(x) = F_{q,p}(x)$, consider

$$F_{p,q}(x) = H(x - d(p,q))$$
$$= H(x - d(q,p)) \text{ as } d(p,q) = d(q,p)$$
$$= F_{q,p}(x)$$

Finally, we consider

$$F_{p,r}(x) = H(x - d(p,r))$$
$$= H(x) = 1, \Rightarrow d(p,r) = 0$$
$$\Rightarrow \mathrm{p} = \mathrm{r}.$$
$$F_{r,q}(y) = H(y - d(r,q))$$
$$= H(y) = 1, \Rightarrow \mathrm{d(r,q)} = 0$$
$$\Rightarrow \mathrm{r} = \mathrm{q}.$$

Therefore,

$$F_{p,q}(x + y) = H(x + y - d(p,q))$$
$$= H(x + y), \text{ for } d(p,q)) = 0, \text{ for every p} = \text{q}.$$
$$= 1, \mathrm{as\ x} > 0, \mathrm{y} > 0 \ \Rightarrow \mathrm{x + y} > 0.$$

Thus, $F_{p,q}(x + y) = 1$ if and only if $F_{p,r}(x) = 1$ and $F_{r,q}(y) = 1$.

**Theorem 4.1.1:** Let $(X, d)$ be a complete metric space and $f: X \to X$ be a mapping satisfying the following condition: there exists a constant $k \in (0,1)$, such that $d\big(f(x), f(y)\big) \leq \alpha\, d(x,y), \forall\, x, y \in X$. Then, $f$ has a fixed point $z$ in $X$, and for any $z_o \in X, f_{z_o}^{n} \to z$.

Proof: Defining mapping $F: X \to X$ by $F_{p,q}(x) = H\big(x - d(p,q)\big), x > 0 \in \mathbb{R}, \forall\, p, q \in X$.

We know that $(X, F, min)$ is a complete Menger space. Since, for each $x > 0$, we have

$$F_{fp,fq}(kx) = H\big(kx - d(fp, fq)\big)$$
$$\geq H\big(kx - kd(p,q)\big)$$
$$\geq H(x - d(p,q)$$
$$\geq F_{p,q}(x)$$

This implies that $f: X \to X$ is contraction mapping in $X$. Hence, by theorem, every contraction mapping in complete Menger space has a fixed point in $X$. [], we have a fixed point in $X$.

### 4.1.2: Contraction condition and their relationship:

- **Sehgal contraction:** $F_{Tp,Tq}(kt) \geq F_{p,q}(t)$;
- **Generalized (m, k)- Sehgal contraction:** $F_{f^i p,f^i q}\left(k^i t\right) \geq F_{p,q}(t)$;
- **Hicks C-contraction:** $F_{p,q}(t) > 1 - t \Rightarrow F_{Tp,\ Tq}(kt) > 1 - kt$;
- **w-H contraction:** $F_{p,q}(t) > 1 - t \Rightarrow F_{Tp,\ Tq}(kt) > 1 - kt$;
- **g-contraction:** $F_{g(x),g(y)}(t) > 1 - t$ impies $F_{f(x),f(y)}(kt) > 1 - kt$;
- **(m, k)-C-contraction:** $F_{p,q}(t) > 1 - t \Rightarrow F_{f^i p,f^i q}\left(k^i t\right) > 1 - k^i t$.

### 4.1.3: Interrelationship:

- (m, k) contraction ⟹ C-contraction ⟹ Sehgal contraction ⟹ Banach contraction
- g-contraction ⟹ C-contraction
- C-contraction ⟹ w-H contraction.

**Conclusions:** Menger solves the uncertainty cases of the distances between two points in spaces by introducing the probabilistic distance function in metric space. The structure of the probabilistic metric space allows probabilistic generalizations of the contraction mapping principle in some inequivalent ways. Probabilistic contraction mappings extend the study and research in probabilistic metric space which helps not only in mathematical cases but also in the geometric study of quantum mechanics.

Lastly, this paper helps mathematicians and researchers in the study of contraction mapping, its generalization, and its interrelationship in probabilistic metric space.

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