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Solution of a Boundary Value Problem Involving General Class of Polynomial, Struve's Function and PSI function of One Variable

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Abstract

The authors of this paper have established some integrals that involves the PSI-function of one variable and Struve's function with the general class of polynomials. Additionally, solved a boundary value problem involving the steady state temperature distribution of a rectangular plate using PSI-function, Struve's function and general class of polynomials. This can be considered another novel technique to solve the boundary value problem. Finally, some special cases were incorporated.

Keywords: PSI-function, General class of polynomial, Struve's function and a Boundary value problem.

1. Introduction

When tackling complex problems in physics and engineering, one essential concept often arises is the steady state solution. In many cases, these steady state solutions are derived from boundary value problems (BVPs), which involve solving differential equations with specific conditions at the boundaries. Since the steady-state heat equation is important in many domains, various techniques exist to obtain the solution analytically and numerically.

There has been a lot of literature on the solutions of steady state heat equations. Many special functions such as Hypergeometric function [2], G-function [1], H-function [3, 10] and I-functions [8] involved in the solutions of different boundary value problems [4, 6].

In this paper, we use PSI-function (Pragathi - Satyanarayana's I-function) [5] to solve a boundary value problem. This function was proved to be the generalization of I – functions defined by Saxena [8] and Arjun K Rathie [7]. The integral representation of PSI-function in terms of Mellin-Barnes type integration is as follows:

$$\Psi^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}\left(a_j,\alpha_j;A_j\right)_{1,n};\left(a_{ji},\alpha_{ji};A_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j;B_j\right)_{1,m};\left(b_{ji},\beta_{ji};B_{ji}\right)_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \phi(s)\,z^s ds \qquad (1.1)$$

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Where $$\phi(s)=\frac{\prod_{j=1}^{m}\Gamma^{B_j}\left(b_j-\beta_j s\right)\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\alpha_j s\right)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma^{B_{ji}}\left(1-b_{ji}+\beta_{ji}s\right)\prod_{j=n+1}^{p_i}\Gamma^{A_{ji}}\left(a_{ji}-\alpha_{ji}s\right)\right]} \qquad (1.2)$$

Useful functions and Integral Formulae:

1. The general class of polynomials [11] is defined as follows:

$$S_n^m(x)=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^k \qquad (1.3)$$

where n = 0,1, 2,…; m is arbitrary positive integer and the coefficients $A_{n,k}$ (n, $k \geq 0$) are arbitrary constants.

2. The Struve's function [9] is defined as follows:

$$H_{v,y,u}^{\lambda,k}(z)=\sum_{t=0}^{\infty}\frac{(-1)^t(z/2)^{v+2t+1}}{\Gamma(kt+y)\Gamma(v+\lambda t+u)} \qquad (1.4)$$

where Re (k)>0, Re (λ)>0, Re (y)>0 and Re (v + u) >0.

3. The orthogonal property for cosine functions [2] is :

$$\int_{-c}^{c}\cos\left(\frac{n\pi x}{c}\right)\cos\left(\frac{m\pi x}{c}\right)dx=\begin{cases}0 & for\ \ n\neq m\\ c & for\ \ n=m\end{cases} \qquad (1.5)$$

4. The integral formula due to Kumar [2] is :

$$\int_{0}^{a/2}\left(\cos\frac{\pi x}{a}\right)^n\left(\cos\frac{2\pi x}{a}\right)^m dx=\frac{a\Gamma(n+1)}{2^{n+1}\Gamma\left(\frac{n}{2}+m+1\right)\Gamma\left(\frac{n}{2}-m+1\right)} \qquad (1.6)$$

Boundary Value Problem:

We consider a boundary value problem for a rectangular plate as shown in figure 1.1 with boundary conditions.

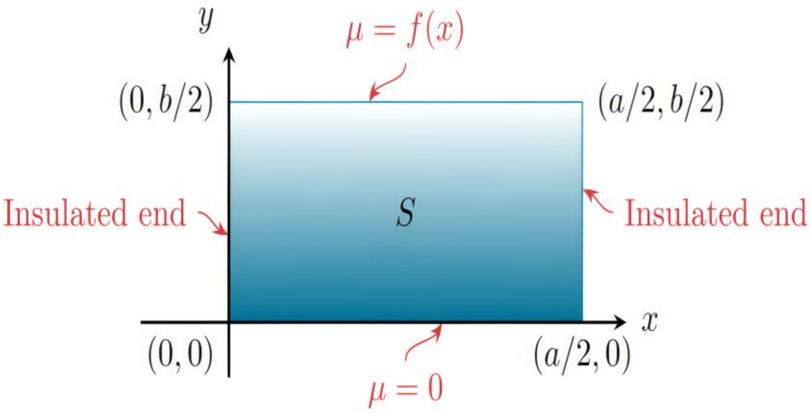

Figure 1.1 Rectangular plate with insulated ends

In this article, we are going to find the steady state temperature $\mu\,(x, y)$ for the rectangular plate whose edges x = 0 and x =a/2 are insulated.

Here the Laplace's equation is defined as $\frac{\partial^2\mu}{\partial x^2}+\frac{\partial^2\mu}{\partial y^2}=0,\ 0<x<\frac{a}{2}, 0<y<\frac{b}{2}$ (1.7)

Considering the boundary conditions: $\left.\frac{\partial\mu}{\partial x}\right|_{x=0}=\left.\frac{\partial\mu}{\partial x}\right|_{x=\frac{a}{2}}=0\ ,\ \ 0<y<\frac{b}{2}\ ,$ (1.8)

$$\mu(x,0)=0$$

and $$\mu\left(x,\frac{b}{2}\right)=f(x)=\left(\cos\frac{\pi x}{a}\right)^n S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right],0<x<\frac{a}{2} \tag{1.9}$$

for section 3,

and $$\mu\left(x,\frac{b}{2}\right)=f(x)=\left(\cos\frac{\pi x}{a}\right)^n H_{v,l,u}^{\lambda,k}\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right],0<x<\frac{a}{2} \tag{1.10}$$

for section 4.

In the next section, it is to derive integral formulae that are useful in solving the given boundary value problem.

2. Main integral

Theorem 1. Prove that

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^n\cos\left(\frac{2\pi px}{a}\right)S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx=\frac{a}{2^{n+1}}\sum_{k=0}^{[l/t]}\frac{(-l)_{tk}}{k!}A_{l,k}\left(\frac{h}{4^\rho}\right)^k *$$

$$\Psi^{m,.n+1}_{p_i+1,q_i+2;r}\left[\frac{z}{4^\sigma}\left|\begin{matrix}(-n-2\rho k,2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i}\\(b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho k-p,\sigma;1\right),\left(-\frac{n}{2}-\rho k+p,\sigma;1\right)\end{matrix}\right.\right] \tag{2.1}$$

Proof: Substituitng corresponding formulae mentioned in (1.1) and (1.3) in the left hand side of the theorem 1, we will get

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^n\cos\left(\frac{2\pi px}{a}\right)S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx$$

$$=\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^n\cos\left(\frac{2\pi px}{a}\right)\sum_{k=0}^{[l/t]}\frac{(-l)_{tk}}{k!}A_{l,k}h^k\left(\cos\frac{\pi x}{a}\right)^{2\rho k}\frac{1}{2\pi i}\int_L\phi(s)z^s\left(\cos\frac{\pi x}{a}\right)^{2\sigma s}ds\,dx$$

Simplifying,

$$= \sum_{k=0}^{[l/t]} \frac{(-l)_{tk}}{k!} A_{l,k} h^k \frac{1}{2\pi i} \int_L \phi(s) \left[\int_0^{a/2} \left(\cos\frac{\pi x}{a} \right)^{n+2\rho k+2\sigma s} \cos\left(\frac{2\pi p x}{a} \right) dx \right] z^s ds$$

Applying (1.6), we get

$$= \sum_{k=0}^{[l/t]} \frac{(-l)_{tk}}{k!} A_{l,k} h^k \frac{1}{2\pi i} \int_L \phi(s) \frac{a\Gamma(n+2\rho k+2\sigma s+1)}{2^{n+2\rho k+2\sigma s+1}\Gamma\left(\frac{n+2\rho k+2\sigma s}{2}+p+1\right).\Gamma\left(\frac{n+2\rho k+2\sigma s}{2}-p+1\right)} z^s ds$$

Using the notation of (1.1) and (1.2), we can write the above integration as -

$$= \frac{a}{2^{n+1}} \sum_{k=0}^{[l/t]} \frac{(-l)_{tk}}{k!} A_{l,k} \left(\frac{h}{4^\rho} \right)^k *$$

$$\Psi^{m,.n+1}_{p_i+1,q_i+2;r} \left[\frac{z}{4^\sigma} \middle| \begin{matrix} (-n-2\rho k, 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,n}, (a_{ji}, \alpha_{ji}; A_{ji})_{n+1,p_i} \\ (b_j, \beta_j; B_j)_{1,m}, (b_{ji}, \beta_{ji}; B_{ji})_{m+1,q_i}, \left(-\frac{n}{2}-\rho k-p, \sigma; 1\right), \left(-\frac{n}{2}-\rho k+p, \sigma; 1\right) \end{matrix} \right]$$

Theorem 2. Prove that

$$\int_0^{a/2} \left(\cos\frac{\pi x}{a} \right)^n \cos\left(\frac{2\pi p x}{a} \right) H^{\lambda,k}_{v,l,u}\left[h\left(\cos\frac{\pi x}{a} \right)^{2\rho} \right] \Psi\left[z\left(\cos\frac{\pi x}{a} \right)^{2\sigma} \right] dx = \frac{a}{2^{n+1}} \sum_{t=0}^{\infty} (-1)^t \left(\frac{h}{2^{2\rho}} \right)^{y(t)} *$$

$$\Psi^{m,.n+1}_{p_i+1,q_i+4;r} \left[\frac{z}{4^\sigma} \middle| \begin{matrix} (-n-2\rho y(t), 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,n}, (a_{ji}, \alpha_{ji}; A_{ji})_{n+1,p_i} \\ (b_j, \beta_j; B_j)_{1,m}, (b_{ji}, \beta_{ji}; B_{ji})_{m+1,q_i}, \left(-\frac{n}{2}-\rho y(t)-p, \sigma; 1\right), \\ \left(-\frac{n}{2}-\rho y(t)+p, \sigma; 1\right), (1-kt-l, 0; 1), (1-v-\lambda t-u, 0; 1) \end{matrix} \right] \tag{2.2}$$

Proof: Substituitng corresponding formulae mentioned in (1.1) and (1.4) in the left hand side, we will get

$$\int_0^{a/2} \left(\cos\frac{\pi x}{a} \right)^n \cos\left(\frac{2\pi p x}{a} \right) H^{\lambda,k}_{v,l,u}\left[h\left(\cos\frac{\pi x}{a} \right)^{2\rho} \right] \Psi\left[z\left(\cos\frac{\pi x}{a} \right)^{2\sigma} \right] dx$$

$$=\sum_{t=0}^{\infty}\frac{(-1)^t(h/2)^{v+2t+1}}{\Gamma(kt+l)\Gamma(v+\lambda t+u)}\frac{1}{2\pi i}\int_L\phi(s)\left[\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{n+2\rho(v+2t+1)+2\sigma s}\cos\left(\frac{2\pi px}{a}\right)dx\right]z^s ds$$

Applying (1.6), we get

$$=\sum_{t=0}^{\infty}\frac{(-1)^t(h/2)^{v+2t+1}}{\Gamma(kt+l)\Gamma(v+\lambda t+u)}\quad\frac{1}{2\pi i}\int_L\phi(s)$$

$$\frac{a\Gamma(n+2\rho(v+2t+1)+2\sigma s+1)}{2^{n+2\rho(v+2t+1)+2\sigma s+1}\Gamma\left(\frac{n+2\rho(v+2t+1)+2\sigma s+1}{2}+p+1\right).\Gamma\left(\frac{n+2\rho(v+2t+1)+2\sigma s+1}{2}-p+1\right)}z^s ds$$

Let us assume y(t) = v + 2t + 1. Then

$$=\sum_{t=0}^{\infty}\frac{(-1)^t(h/2)^{y(t)}}{2^{2r}\,y(t)}\frac{a}{2^{n+1}}*$$

$$\frac{1}{2\pi i}\int_L\phi(s)\frac{a\Gamma(n+2\rho y(t)+2\sigma s+1)}{\Gamma\left(\frac{n+2\rho y(t)+2\sigma s+1}{2}+p+1\right).\Gamma\left(\frac{n+2\rho y(t)+2\sigma s+1}{2}-p+1\right)}z^s\,2^{2\sigma s}ds$$

Using the notation of (1.1) and (1.2) , we can write the above integral in PSI-function representation as -

$$=\frac{a}{2^{n+1}}\sum_{t=0}^{\infty}(-1)^t\left(\frac{h}{2^{2\rho}}\right)^{y(t)}*$$

$$\Psi^{m,.n+1}_{p_i+1,q_i+4;r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho y(t),2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i}\\(b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho y(t)-p,\sigma;1\right),\end{matrix}\right.\right.$$

$$\left.\left(-\frac{n}{2}-\rho y(t)+p,\sigma;1\right),(1-kt-l,0;1),(1-v-\lambda t-u,0;1)\right]$$

3. Solution of steady state temperature in rectangular plate involving General Class of Polynomial and PSI-function of one variable:

We wish to find the steady-state temperature $\mu(x, y)$ in a rectangular plate described in Section.1 whose vertical edges x=0, x=a/2 are insulated.

Using General Class of polynomial [11] and PSI function [5], one of the boundary condition can be defined as

$$\mu\left(x,\frac{b}{2}\right)= f(x)=\left(\cos\frac{\pi x}{a}\right)^n S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right], 0<x<\frac{a}{2} \qquad (3.1)$$

From the general solution of (1.7) given by Zill et al,[12], the superposition principle is -

$$\mu(x,y)=A_o y+\sum_{r=1}^{\infty}A_r .Sinh\left(\frac{2r\pi y}{a}\right)\cos\left(\frac{2r\pi x}{a}\right),\ 0<x<\frac{a}{2}, 0<y<\frac{b}{2} \qquad (3.2)$$

Substituting y= b/2 in (3.2) gives

$$\mu\left(x,\frac{b}{2}\right)=A_o\frac{b}{2}+\sum_{r=1}^{\infty}A_r .Sinh\left(\frac{r\pi b}{a}\right)\cos\left(\frac{2r\pi x}{a}\right),\ 0<x<\frac{a}{2}, 0<y<\frac{b}{2} \qquad (3.3)$$

Which is a half-range expansion in cosine series. We integrate (3.3) both sides w.r.t. x between the limits 0 and a/2.

$$\Rightarrow \quad \int_0^{a/2}\mu\left(x,\frac{b}{2}\right)dx = A_0\frac{b}{2}\int_0^{a/2}dx+\sum_{r=0}^{\infty}A_r Sinh\left(\frac{r\pi b}{a}\right)\int_0^{a/2}Cos\left(\frac{2r\pi x}{a}\right)dx$$

$$\Rightarrow \quad \int_0^{a/2}\mu\left(x,\frac{b}{2}\right)dx = A_0\frac{b}{2}.\frac{a}{2}$$

From (3.1), we have

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^n S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx = A_0.\frac{b}{2}.\frac{a}{2}$$

$$\Rightarrow \quad A_o=\frac{4}{ab}\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^n S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx$$

Applying Theorem 1, and taking p = 0 in (2.1), we arrive to

$$A_0=\frac{1}{b.2^{n-1}}\sum_{k=0}^{[l/t]}\frac{(-l)_{tk}}{k!}A_{l,k}\left(\frac{h}{4^{\rho}}\right)^k \Psi^{m,.n+1}_{p_i+1,q_i+2;r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho k,2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i}\\ (b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho k,\sigma;1\right),\left(-\frac{n}{2}-\rho k,\sigma;1\right)\end{matrix}\right.\right]$$

(3.4)

To find A_r, multiplying (3.3) both sides with $Cos\left(\frac{2r\pi x}{a}\right)$ and integrating w.r.t. x between the limits 0 and a/2, we get

$$\int_0^{a/2}\mu\left(x,\frac{b}{2}\right)\cos\left(\frac{2\pi rx}{a}\right)dx = A_0\frac{b}{2}\int_0^{a/2}\cos\left(\frac{2\pi px}{a}\right)dx+\sum_{r=0}^{\infty}A_r\sinh\frac{r\pi b}{a}\int_0^{a/2}\left(\cos\frac{2\pi rx}{a}\right)\left(\cos\frac{2\pi bx}{a}\right)dx$$

From (1.5),

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^n\cos\left(\frac{2\pi rx}{a}\right)S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx \;=\; A_r\frac{a}{4}Sinh\left(\frac{r\pi b}{a}\right)$$

$$\Rightarrow A_r=\frac{4}{a.Sinh\left(\frac{r\pi b}{a}\right)}\int_0^{a/2}\left(\cos\left(\frac{\pi x}{a}\right)\right)^n\cos\left(\frac{2\pi rx}{a}\right)S_l^t\left[h\left(Cos\,\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx$$

Applying Theorem.1 and assuming p = r in (2.1), we arrive to

$$A_r=\frac{1}{\sinh\left(\frac{r\pi b}{a}\right)2^{n-1}}\sum_{k=0}^{[l/t]}\frac{(-l)_{tk}}{k!}A_{l,k}\left(\frac{h}{4^{\rho}}\right)^k.$$

$$\Psi_{p_i+1,q_i+2;r}^{m,.n+1}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho k,2\sigma;1),\left(a_j,\alpha_j;A_j\right)_{1,n},\left(a_{ji},\alpha_{ji};A_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j;B_j\right)_{1,m},\left(b_{ji},\beta_{ji};B_{ji}\right)_{m+1,q_i},\left(-\frac{n}{2}-\rho k-r,\sigma;1\right),\left(-\frac{n}{2}-\rho k+r,\sigma;1\right)\end{matrix}\right.\right] \tag{3.5}$$

Hence the general solution from (3.4) and (3.5) is given by

$$\mu(x,y)=\sum_{k=0}^{[l/t]}\frac{(-l)_k}{k!}.\left(\frac{h}{4^{\rho}}\right)^k\left[\frac{y}{b.2^{n-1}}\psi(\theta_1)+\sum_{r=1}^{\infty}\frac{\sinh\left(\frac{2r\pi y}{a}\right)\cos\left(\frac{2r\pi x}{a}\right)}{2^{n-1}\sinh\left(\frac{r\pi b}{a}\right)}\psi(\theta_2)\right] \tag{3.6}$$

Where

$$\psi(\theta_1)=\Psi_{p_i+1,q_i+2;r}^{m,.n+1}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho k,2\sigma;1),\left(a_j,\alpha_j;A_j\right)_{1,n},\left(a_{ji},\alpha_{ji};A_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j;B_j\right)_{1,m},\left(b_{ji},\beta_{ji};B_{ji}\right)_{m+1,q_i},\left(-\frac{n}{2}-\rho k,\sigma;1\right),\left(-\frac{n}{2}-\rho k,\sigma;1\right)\end{matrix}\right.\right] \tag{3.7}$$

and

$$\psi(\theta_2)=\Psi_{p_i+1,q_i+2;r}^{m,.n+1}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho k,2\sigma;1),\left(a_j,\alpha_j;A_j\right)_{1,n},\left(a_{ji},\alpha_{ji};A_{ji}\right)_{n+1,p_i}\\ \left(b_j,\beta_j;B_j\right)_{1,m},\left(b_{ji},\beta_{ji};B_{ji}\right)_{m+1,q_i},\left(-\frac{n}{2}-\rho k-r,\sigma;1\right),\left(-\frac{n}{2}-\rho k+r,\sigma;1\right)\end{matrix}\right.\right] \tag{3.8}$$

4. Solution of steady state temperature in rectangular plate involving Struve's function and Pragathi-Satyanarayana's I-function of one variable:

we wish to find the steady-state temperature $\mu(x, y)$ in a rectangular plate described in Section 1 whose vertical edges x=0, x=a/2 are insulated.

Using Struve's function [9] and PSI function [5], one of the boundary condition can be defined as

$$\mu\left(x,\frac{b}{2}\right)= f(x)=\left(\cos\frac{\pi x}{a}\right)^{n} H_{v,l,u}^{\lambda,k}\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right],0<x<\frac{a}{2} \qquad (4.1)$$

From the general solution of (1.7) given by Zill et al,[12], the superposition principle is –

$$\mu(x,y)=A_o y+\sum_{r=1}^{\infty}A_r.Sinh\left(\frac{2r\pi y}{a}\right)\cos\left(\frac{2r\pi x}{a}\right),\ 0<x<\frac{a}{2},0<y<\frac{b}{2} \qquad (4.2)$$

Substituting y= b/2 in (4.2) gives

$$\mu\left(x,\frac{b}{2}\right)=A_o\frac{b}{2}+\sum_{r=1}^{\infty}A_r.Sinh\left(\frac{r\pi b}{a}\right)\cos\left(\frac{2r\pi x}{a}\right),\ 0<x<\frac{a}{2},0<y<\frac{b}{2} \qquad (4.3)$$

which is a half-range expansion in cosine series. We integrate (4.3) both sides w.r.t. x between the limits 0 and a/2.

$$\Rightarrow \quad \int_0^{a/2}\mu\left(x,\frac{b}{2}\right)dx=A_0\frac{b}{2}\int_0^{a/2}dx+\sum_{r=0}^{\infty}A_r Sinh\left(\frac{r\pi b}{a}\right)\int_0^{a/2}Cos\left(\frac{2r\pi x}{a}\right)dx$$

$$\Rightarrow \quad \int_0^{a/2}\mu\left(x,\frac{b}{2}\right)dx=A_0\frac{b}{2}.\frac{a}{2}$$

From (4.1), we have

$$\int_0^{a/2}\left(\cos\frac{\pi x}{a}\right)^{n} H_{v,l,u}^{\lambda,k}\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]dx=A_0.\frac{b}{2}.\frac{a}{2}$$

Applying Theorem.2 and taking p = 0 in (2.2), we arrive to

$$A_o=\frac{4}{ab}\int_0^{a/2}\mu\left(x,\frac{b}{2}\right)dx=\frac{1}{b.2^{n-1}}\sum_{t=0}^{\infty}(-1)^t\left(\frac{h}{2^{2\rho+1}}\right)^{y(t)} *$$

$$\Psi_{p_i+1,q_i+4;r}^{m,,n+1}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho y(t),2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i}\\(b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho y(t),\sigma;1\right),(1-kt-l,0;1),(1-v-\lambda t-u,0;1)\end{matrix}\right.\right], \qquad (4.4)$$

To find A_r, multiplying (4.2) both sides with $Cos\left(\frac{2r\pi x}{a}\right)$ and integrating w.r.t. x between the limits 0 and a/2, we get,

$$\int_0^{a/2} \mu\left(x,\frac{b}{2}\right) Cos\left(\frac{2\pi r x}{a}\right) dx = A_r \frac{a}{4}\sinh\left(\frac{r\pi b}{a}\right)$$

Using (4.1) and simplifying, we will get

$$A_r = \frac{1}{2^{n-1}\sinh\left(\frac{r\pi b}{a}\right)} \sum_{t=0}^{\infty} (-1)^t \left(\frac{h}{2^{2\rho+1}}\right)^{y(t)} . \Psi^{m,.n+1}_{p_i+1,q_i+4;r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho y(t),2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i} \\ (b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho y(t)-r,\sigma;1\right), \left(-\frac{n}{2}-\rho y(t)+r,\sigma;1\right),(1-kt-l,0;1),(1-v-\lambda t-u,0;1)\end{matrix}\right.\right] \quad (4.5)$$

Hence the general solution from (4.4) and (4.5) is given by

$$\mu(x,y) = \sum_{k=0}^{\infty} (-1)^t . \left(\frac{h}{2^{2\rho+1}}\right)^{y(t)} \left[\frac{y}{b.2^{n-1}}\psi(\theta_1) + \sum_{r=1}^{\infty} \frac{\sinh\left(\frac{2r\pi y}{a}\right)\cos\left(\frac{2r\pi x}{a}\right)}{2^{n-1}\sinh\left(\frac{r\pi b}{a}\right)}\psi(\theta_2)\right] \quad (4.6)$$

Where

$$\psi(\theta_1) = \Psi^{m,.n+1}_{p_i+1,q_i+4;r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho y(t),2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i} \\ (b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho y(t),\sigma;1\right),(1-kt-l,0;1),(1-v-\lambda t-u,0;1)\end{matrix}\right.\right] \quad (4.7)$$

$$\psi(\theta_2) = \Psi^{m,.n+1}_{p_i+1,q_i+4;r}\left[\frac{z}{4^{\sigma}}\left|\begin{matrix}(-n-2\rho y(t),2\sigma;1),(a_j,\alpha_j;A_j)_{1,n},(a_{ji},\alpha_{ji};A_{ji})_{n+1,p_i} \\ (b_j,\beta_j;B_j)_{1,m},(b_{ji},\beta_{ji};B_{ji})_{m+1,q_i},\left(-\frac{n}{2}-\rho y(t)-r,\sigma;1\right), \left(-\frac{n}{2}-\rho y(t)+r,\sigma;1\right),(1-kt-l,0;1),(1-v-\lambda t-u,0;1)\end{matrix}\right.\right] \quad (4.8)$$

5. Expansion Formulae

(i) Substituting y = b/2 in the solution (3,6), we arrive to

$$\mu\left(x,\frac{b}{2}\right) = f(x) = \left(\cos\frac{\pi x}{a}\right)^n S_l^t\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right]\Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]$$

$$= \sum_{k=0}^{[l/t]} \frac{(-l)_{tk}}{k!} \left(\frac{h}{4^{\rho}} \right)^k \left[\frac{1}{2^n} \psi(\theta_1) + \sum_{r=1}^{\infty} \frac{\cos\left(\frac{2\pi r x}{a}\right)}{2^{n-1}} \psi(\theta_2) \right] \quad (5.1)$$

Where $\psi(\theta_1)$ and $\psi(\theta_2)$ are assumed as in (3.7) and (3.8).

(ii) Substituting y = b/2 in the solution (4.6), we arrive to

$$\mu\left(x, \frac{b}{2}\right) = f(x) = \left(\cos\frac{\pi x}{a}\right)^n H_{v,l,u}^{\lambda,k}\left[h\left(\cos\frac{\pi x}{a}\right)^{2\rho}\right] \Psi\left[z\left(\cos\frac{\pi x}{a}\right)^{2\sigma}\right]$$

$$= \sum_{k=0}^{\infty} (-1)^t \left(\frac{h}{2^{2\rho+1}} \right)^{y(t)} \left[\frac{1}{2^n} \psi(\theta_1) + \sum_{r=1}^{\infty} \frac{\cos\left(\frac{2\pi r x}{a}\right)}{2^{n-1}} \psi(\theta_2) \right] \quad (5.2)$$

Where $\psi(\theta_1)$ and $\psi(\theta_2)$ are assumed as in (4.7) and (4.8).

6. Special Cases

- Taking $A_{ji} = A_j = B_{ji} = B_j = 1$ in (4.6), we get *f(x)* in terms of Struve's function and I function of one variable given by Saxena [8] and we obtain modified (4.7) and (4.8) as -

$$\psi(\theta_1) = I_{p_i+1,q_i+4;r}^{m,n+1}\left[\frac{z}{4^{\sigma}} \middle| \begin{matrix} (-n-2\rho y(t), 2\sigma; 1), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i}, \left(-\frac{n}{2} - \rho y(t), \sigma; 1\right), (1-kt-l, 0; 1), (1-v-\lambda t-u, 0; 1) \end{matrix} \right]$$ and

$$\psi(\theta_2) = I_{p_i+1,q_i+4;r}^{m,n+1}\left[\frac{z}{4^{\sigma}} \middle| \begin{matrix} (-n-2\rho y(t), 2\sigma; 1), (a_j, \alpha_j)_{1,n}, (a_{ji}, \alpha_{ji})_{n+1,p_i} \\ (b_j, \beta_j)_{1,m}, (b_{ji}, \beta_{ji})_{m+1,q_i}, \left(-\frac{n}{2} - \rho y(t) - r, \sigma; 1\right), \left(-\frac{n}{2} - \rho y(t) + r, \sigma; 1\right), (1-kt-l, 0; 1), (1-v-\lambda t-u, 0; 1) \end{matrix} \right]$$

- Taking r = 1, $a_{ji} = a_j, b_{ji} = b_j, A_{ji} = A_j, B_{ji} = B_j$ in (4.6), we get *f(x)* in terms of Struve's function and I-function of one variable given by Arjun K Rathie [6], we obtain modified (4.7) and (4.8) as

$$\psi(\theta_1) = \Psi_{p_i+1,q_i+4}^{m,n+1}\left[\frac{z}{4^{\sigma}} \middle| \begin{matrix} (-n-2\rho y(t), 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,p_i} \\ (b_j, \beta_j; B_j)_{1,q_i}, \left(-\frac{n}{2} - \rho y(t), \sigma; 1\right), (1-kt-l, 0; 1), (1-v-\lambda t-u, 0; 1) \end{matrix} \right]$$

and

$$\psi(\theta_2) = \Psi_{p_i+1,q_i+4}^{m,n+1}\left[\frac{z}{4^{\sigma}} \middle| \begin{matrix} (-n-2\rho y(t), 2\sigma; 1), (a_j, \alpha_j; A_j)_{1,p_i} \\ (b_j, \beta_j; B_j)_{1,q_i}, \left(-\frac{n}{2} - \rho y(t) - 1, \sigma; 1\right), \left(-\frac{n}{2} - \rho y(t) + 1, \sigma; 1\right), (1-kt-l, 0; 1), (1-v-\lambda t-u, 0; 1) \end{matrix} \right]$$

Similarly we can impose above said conditions to (3.6) and we can arrive the solution in terms of general class of polynomials and I-function of one variable by Saxena [8] and Arjun K Rathie [7] respectively.

7. Conclusion

In this paper, we presented a novel technique to obtain the solution of a boundary value problem using PSI-function. Initially, we have introduced two theorems that pertain to the multiplication of general class of polynomials and Struve's function in conjunction with the PSI-function. Subsequently, by using the outcomes of the established theorems, the solutions to the boundary value problems are derived in relation to the PSI-function of a single variable. In this study by specializing several parameters as well as variables lead to wide variety of useful special functions (or product of such special functions) expressible in terms of I-function defined by Saxena [8], defined by Rathie [7]. One can further specialize the results to H-function [3, 10], Meijer's G-function [1], E-function [10] and hypergeometric function [2].

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