Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 8s (2024)

Novel Picture of Topological Spaces in the Frame of Fuzzy Soft Sequential Sets

***K.Japhia Tino Mercy[1], K.Bageerathi[2]**

[1] Research Scholar (Registration Number: 22122022092006), PG and Research Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur, Affiliated to Manonmaniam Sundaranar University, Abishekapatti,Tirunelveli-627 012.

[2] Assistant Professor, PG and Research Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur.

E-mail Id: [1]jaffy306@gmail.com,[2]bageerathi14@gmail.com

Article History:

***Received**: 20-04-2024*

***Revised**: 10-06-2024*

***Accepted**: 24-06-2024*

Abstract:

The goal of this paper is to introduce fuzzy soft sequential topological spaces over the universal set via fuzzy soft sequential set by merging the concept of fuzzy soft set and sequential set. We provide a comprehensive study of their properties and explore a relationship between fuzzy soft topology and fuzzy soft sequential topology. We offers a new outlook on the theory of fuzzy soft sequential neighborhood of a fuzzy soft sequential set and we investigate the quasi-coincident relation.

Keywords: fuzzy soft set, fuzzy soft sequential set, fuzzy soft sequential topological spaces, fuzzy soft sequential neighborhood system, fuzzy soft sequential quasi-coincident

1. Introduction

The concept of fuzzy sets by defining them in terms of mappings from a set into the unit interval on the real line was proposed by **L.A.Zadeh** [18] in 1965, which is dealing with uncertainty and representing vague concepts. In 1968, **C. L. Chang** [4] initiated fuzzy topological spaces as an extension to classical topological spaces. In 1999, **Molodtsov** [11] introduced the concept of soft set theory, which provides a new mathematical theory for dealing with uncertainty. In 2003, **Maji et al** [10] defined and studied the theory of soft sets which is used to construct new soft sets from the given soft sets. In 2011, **Shabir and Naz** [14] introduced soft topological spaces which are defined over an initial universe with a fixed set of parameters. Further in the same year, **Hussain and Ahmad** [6] extended the soft topology and strengthen the foundations of the theory of soft topological spaces. Also **Cagman, Enginoglu and Citak** [3] investigated soft topology and its properties. In recent years, the researchers have contributed a lot towards fuzzification of soft set theory. In 2001, **Maji et al** [9] proposed the concept of fuzzy soft set which is a new mathematical approach to vagueness by involving the ideas of both fuzzy sets and soft sets. Fuzzy soft sets was further revised and improved by **Ahmad and Kharal** [1] in the year 2009. In 2011, **Tanay and Kandemir** [16] introduced the topological structure of fuzzy soft sets and studied some of its structural properties. Based on this theory many researchers like **Roy and Samanta** [12] in the year 2012**, Atmaca and Zorlutuna** [13] in the year 2013, **Mahanta and Das** [8] in the year 2018 studied and investigated many properties in fuzzy soft topology. In 2002, **Bose and Indrajit Lahiri [2]** introduced the concept of sequential topological spaces. He defined any sequence of subsets of a non void set is called a sequential set.

To go along this line of research, we introduce sequence of fuzzy soft set and named as fuzzy soft sequential set. In this paper, we extend the concept of topological space namely fuzzy soft sequential topological space and analyze their properties.

2. Preliminaries

Definition 2.1 [18] A **fuzzy set** X over a universal set U is a set defined by a function μ_x performing a mapping $\mu_x : U \rightarrow [0,1]$, here this μ_x is the membership function of X, and the value $\mu_x(u)$ will be the grade of membership of u $\in U$.

Definition 2.2 [11] Let U be an initial universe set and E be a set of parameters. A pair (F, E) is called a **soft set** over U if and only if F is a mapping of E into the set of all subsets of the set U.

Definition 2.3 [9] Let U be a common universe, E be a set of parameters and $A \subseteq E$. A pair (F, A) is called a **fuzzy soft set** over U, where F is a mapping given by F: $A \rightarrow \mu_x(U)$, where $\mu_x(U)$ denote the set of all fuzzy subsets of U.

Definition 2.4 [16] Let U be a common universe, E be a set of parameters and $A, B \subseteq E$. Let (F, A) , (G, B) be an element of $FS(U, E)$, (Briefly, family of all fuzzy soft subsets) and $\tilde{\tau}$ be a subfamily of $FS(U, E)$. Then $\tilde{\tau}$ is called a **fuzzy soft topology** on U if the following conditions are satisfied:

(i) $\widetilde{\emptyset}$, $U \in \tilde{\tau}$
(ii) (F, A) , $(G, B) \in \tilde{\tau} \Rightarrow (F, A) \widetilde{\cap} (G, B) \in \tilde{\tau}$
(iii)$\{(F, A)_k | k \in K\} \subset \tilde{\tau} \Rightarrow \widetilde{\bigcup}_{k\in K}(F, A)_k \in \tilde{\tau}$.

The pair $(U, \tilde{\tau})$ is called a fuzzy soft topological space.

Definition 2.5 [13] Let (F, A) , $(G, B) \in FS(U, E)$. (F, A) is said to be **fuzzy soft quasi-coincident** with (G, B), denoted by $(F, A)\, q(G, B)$, if there exists an $e \in E, x \in X$ such that $F(e)(x) + G(e)(x) > 1$. If (F, A) is not soft quasi-coincident with (G, B), then we write $(F, A)\ \bar{q}(G, B)$.

Definition 2.6 [18] A **is contained in** B (or, equivalently, A is a subset of B, or A is smaller than or equal to B) if and only if $\mu_A \leqq \mu_B$. In symbols, $A \subset B \Leftrightarrow \mu_A \leqq \mu_B$.

Definition 2.7 [2] Any sequence of subsets of a non void set X is called a **sequential set** in X. That is, $A(s) = \{A_n\}_{n=1}^{\infty}$, where each A_n is a subset of X, is a sequential set in X. The subsets $A_n, n \in \mathbb{N}$ are called the components of $A(s)$.

Definition 2.8[7] A sequence of fuzzy soft sets is a mapping from $\mathbb{N}$ to the family of all fuzzy soft sets and is denoted by $\{(F, A)_n\}$ or $\{(F, A)_n; n = 1,2, \ldots\}$.That is, $\{(F, A)_n, n \in \mathbb{N}\}$ where $(F, A)_n$ for each $n \in \mathbb{N}$ represents components of fuzzy soft set in $\{(F, A)_n\}$ and $n \in \mathbb{N}$, the set of all natural numbers. A sequence of fuzzy soft sets is called **fuzzy soft sequential set**.

3. Main Results

Definition 3.1. A family $\tilde{\tau}$ of fuzzy soft sequential sets on U satisfying the properties:

(i) $\widetilde{\emptyset}, \widetilde{U} \in \tilde{\tau}$
(ii) $\{(F,A)_n\}, \{(G,B)_n\} \in \tilde{\tau} \Rightarrow \{(F,A)_n\} \widetilde{\cap} \{(G,B)_n\} \in \tilde{\tau}$ and
(iii) For any family $\{\{(F_\lambda,A)_n\}, \lambda \in \Lambda\} \in \tilde{\tau} \Rightarrow \widetilde{\bigcup}_{\lambda\in\Lambda} \{(F_\lambda,A)_n\} \in \tilde{\tau}$

is called a **fuzzy soft sequential topology** on U and the triplet $(U,\tilde{\tau},E)$ is called a **fuzzy soft sequential topological space** over U (briefly, $FSSTS$).

Definition 3.2. Let $(U,\tilde{\tau},E)$ be a fuzzy soft sequential topology on U. Then the members of $\tilde{\tau}$ are called **fuzzy soft sequential open sets** (FSS-open sets). The Complement of a fuzzy soft sequential open set is called **fuzzy soft sequential closed set** (FSS -closed set).

Example 3.3. Let us consider the universe $U = \{x_1, x_2\}$ and $A = \{e_1, e_2\} \widetilde{\subseteq} E$ where $\{e_1, e_2\}$ be a collection of sets of parameters. Let us take a fuzzy soft sequential topological space $(U,\tilde{\tau},E)$ where $\tilde{\tau} = \left\{\widetilde{\emptyset}, \widetilde{U}, \{(F,A)_n\}\right\}$. The nth component of the fuzzy soft sequential set $\{(F,A)_n\}$ is defined as $(F,A)_n = \left\{ \left(e_1, \left\{ \frac{x_1}{(1/3n)}, \frac{x_2}{(n/n+2)} \right\}\right), \left(e_2, \left\{ \frac{x_1}{(2/5n)}, \frac{x_2}{(1/n)} \right\}\right) \right\}$. Then $\tilde{\tau}$ is a fuzzy soft sequential topology on U.

Definition 3.4. Let U be the universal set and E be a set of parameters and $\tilde{\tau} = \{\widetilde{\emptyset}, \widetilde{U}\}$. Then $\tilde{\tau}$ is called the **fuzzy soft sequential indiscrete topology** on U.

Definition 3.5. Let U be the universal set and E be a set of parameters and let $\tilde{\tau}$ be the collection of all fuzzy soft sequential sets which can be defined over U. Then $\tilde{\tau}$ is called the **fuzzy soft sequential discrete topology** on U.

Definition 3.6. $\{(F,A)_n\}$ is **contained in** $\{(G,B)_n\}$, symbolically $\{(F,A)_n\} \widetilde{\subseteq} \{(G,B)_n\}$, if and only if for each $e_i \in A, B$ and for all $x_j \in U$, $F_n(e_i)(x_j) \widetilde{\leq} G_n(e_i)(x_j)$ for all $n \in \mathbb{N}$.

Definition 3.7. $\{(F,A)_n\}$ is **weakly contained in** $\{(G,B)_n\}$, symbolically $\{(F,A)_n\} \widetilde{\subseteq}_w \{(G,B)_n\}$, if and only if for each $e_i \in A, B$ and for all $x_j \in U$, there exists $n \in \mathbb{N}$ such that $F_n(e_i)(x_j) \widetilde{\leq} G_n(e_i)(x_j)$.

Definition 3.8. Let $(U,\tilde{\tau}_1,E)$ and $(U,\tilde{\tau}_2,E)$ be two fuzzy soft sequential topological spaces. Then the following holds:

(i) If $\tilde{\tau}_1 \widetilde{\subseteq} \tilde{\tau}_2$, then $\tilde{\tau}_2$ is **fuzzy soft sequential finer than** $\tilde{\tau}_1$ or $\tilde{\tau}_1$ is **fuzzy soft sequential coarser than** $\tilde{\tau}_2$.
(ii) If either $\tilde{\tau}_1 \widetilde{\subseteq} \tilde{\tau}_2$ or $\tilde{\tau}_2 \widetilde{\subseteq} \tilde{\tau}_1$, then $\tilde{\tau}_1$ is **comparable** with $\tilde{\tau}_2$.

Theorem 3.9 Let $(U,\tilde{\tau}_1,E)$ and $(U,\tilde{\tau}_2,E)$ be two fuzzy soft sequential topological spaces over U, then $(U,\tilde{\tau}_1 \widetilde{\cap} \tilde{\tau}_2,E)$ is a fuzzy soft sequential topological space over$U$.

Proof.

(i) $\widetilde{\emptyset}, \widetilde{U} \in \tilde{\tau}_1 \widetilde{\cap} \tilde{\tau}_2$.
(ii) Let the two fuzzy soft sequential sets $\{(F,A)_n\}$, $\{(G,B)_n\} \in \tilde{\tau}_1 \widetilde{\cap} \tilde{\tau}_2$. Then $\{(F,A)_n\}$, $\{(G,B)_n\} \in \tilde{\tau}_1$ and $\{(F,A)_n\}$, $\{(G,B)_n\} \in \tilde{\tau}_2$. Therefore $\{(F,A)_n\} \widetilde{\cap} \{(G,B)_n\} \in \tilde{\tau}_1$ and $\{(F,A)_n\} \widetilde{\cap} \{(G,B)_n\} \in \tilde{\tau}_2$. Hence $\{(F,A)_n\} \widetilde{\cap} \{(G,B)_n\} \in \tilde{\tau}_1 \widetilde{\cap} \tilde{\tau}_2$.

(iii) Let $\{\{(F_\lambda, A)_n\}, \lambda \in \Lambda\}$ be a family of fuzzy soft sequential sets in $\tilde{\tau}_1 \tilde{\cap} \tilde{\tau}_2$. Then $\{(F_\lambda, A)_n\} \in \tilde{\tau}_1$ and $\{(F_\lambda, A)_n\} \in \tilde{\tau}_2 \; \forall \lambda \in \Lambda$. Therefore $\tilde{\cup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}_1$ and $\tilde{\cup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}_2$. Hence $\tilde{\cup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}_1 \tilde{\cap} \tilde{\tau}_2$.

Remark 3.10. Let $(U, \tilde{\tau}_1, E)$ and $(U, \tilde{\tau}_2, E)$ be two fuzzy soft sequential topological spaces over U, then $(U, \tilde{\tau}_1 \tilde{\cup} \tilde{\tau}_2, E)$ need not be a fuzzy soft sequential topological space over U.

Proposition 3.11. If $\tilde{\tau}$ is a fuzzy soft topology on U, then $\tilde{\tau}^{\mathbb{N}}$ forms a fuzzy soft sequential topology on U.

Proof. Let $\tilde{\tau}$ be a fuzzy soft topology on U.

(i) $\tilde{\emptyset}, \tilde{U} \in \tilde{\tau}^{\mathbb{N}}$, Since $\tilde{\emptyset}, U \in \tilde{\tau}$.

(ii) Let $\{(F, A)_n\}, \{(G, B)_n\} \in \tilde{\tau}^{\mathbb{N}}$. Then $(F, A)_n, (G, B)_n \in \tilde{\tau}$ for all $n \in \mathbb{N}$. Thus $(F, A)_n \tilde{\cap} (G, B)_n \in \tilde{\tau}$ for all $n \in \mathbb{N}$. Thus $\{(F, A)_n\} \tilde{\cap} \{(G, B)_n\} \in \tilde{\tau}^{\mathbb{N}}$.

(iii) Let $\{\{(F_\lambda, A)_n\} \in \tilde{\tau}, \lambda \in \Lambda\}$ be a family of fuzzy soft sequential sets in $\tilde{\tau}^{\mathbb{N}}$. Then for each $n \in \mathbb{N}$, $(F, A)_\lambda \in \tilde{\tau} \; for \; all \; \lambda \in \Lambda$. Thus $\tilde{\cup}_{\lambda \in \Lambda} (F, A)_\lambda \in \tilde{\tau}$.
Hence $\tilde{\cup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}^{\mathbb{N}}$.

Proposition 3.12. Any fuzzy soft sequential topological space induces fuzzy soft topological space.

Proof. Let $\tilde{\tau}$ be a fuzzy soft sequential topology, $\tilde{\tau}_1$ be a fuzzy soft topology and $n \in \mathbb{N}$, the set of all natural numbers. Then

(i) $\tilde{\emptyset}, U \in \tilde{\tau}_1$, since $\tilde{\emptyset}, \tilde{U} \in \tilde{\tau}$.

(ii) Let $(F, A), (G, B) \in \tilde{\tau}_1$. Then there exists fuzzy soft sequential sets $\{(F, A)_n\}, \{(G, B)_n\} \in \tilde{\tau}$ such that $(F, A)_n = (F, A)$ and $(G, B)_n = (G, B)$. Since $\{(F, A)_n\} \tilde{\cap} \{(G, B)_n\} \in \tilde{\tau}$ implies $(F, A) \tilde{\cap} (G, B) \in \tilde{\tau}_1$.

(iii) Let $\{(F, A)_\lambda \,/\, \lambda \in \Lambda\}$ be a family of fuzzy soft open sets. Then for each $\lambda \in \Lambda$, there exists a fuzzy soft sequential set $\{\{(F_\lambda, A)_n\} \,/\, \lambda \in \Lambda\} \in \tilde{\tau}$ having n^{th} component $(F, A)_\lambda$. Since $\tilde{\cup}_{\lambda \in \Lambda} \{(F_\lambda, A)_n\} \in \tilde{\tau}$, $\tilde{\cup}_{\lambda \in \Lambda} (F, A)_\lambda \in \tilde{\tau}_1$.

Definition 3.13. $(U, \tilde{\tau})$ in the Proposition 3.12, is called the **n^{th} component fuzzy soft topological space** of the fuzzy soft sequential topological space $(U, \tilde{\tau}, E)$.

Proposition 3.14. Let $\{(F, A)_n\}$ be a fuzzy soft sequential open (closed) set in fuzzy soft sequential topological space $(U, \tilde{\tau}, E)$. Then for each $n \in \mathbb{N}$, $(F, A)_n$ is fuzzy soft open (closed) set in $(U, \tilde{\tau})$.

Proof. Let $\{(F, A)_n\}$ be a fuzzy soft sequential open set in fuzzy soft sequential topological space$(U, \tilde{\tau}, E)$ and $(F, A)_n$ be a fuzzy soft open set in $(U, \tilde{\tau})$. From the Proposition 3.12, fuzzy soft topologies are induced by fuzzy soft sequential topologies and any fuzzy soft topology $\tilde{\tau}$ on $(U, \tilde{\tau})$ can be considered as a component of fuzzy soft sequential topology $\tilde{\tau}$ on U. Hence we get, for every $n \in \mathbb{N}$, $(F, A)_n$ is fuzzy soft open set in $(U, \tilde{\tau})$.

The following example shows that, the converse of the above proposition need not be true.

Example 3.15. Example 3.3 shows that, $\{(F, A)_n\}$ is a fuzzy soft sequential open set in $(U, \tilde{\tau}, E)$ implies $(F, A)_1$, $(F, A)_2$,….. $(F, A)_n$,….. is fuzzy soft open set in $(U, \tilde{\tau})$.

Now, let us take, for each $n \in \mathbb{N}$, $(F,A)_n$ is fuzzy soft open set in $(U,\tilde{\tau})$ implies $\{(F,A)_n\}$ is not a fuzzy soft sequential open set in $(U,\tilde{\tau},E)$.

Theorem 3.16. Arbitrary intersection of fuzzy soft sequential closed sets is fuzzy soft sequential closed set.

Proof. Let $\{\{(F_\lambda,A)_n\} \,/\, \lambda \in \Lambda\}$ be an arbitrary collection of fuzzy soft sequential closed sets. We take $\{(F,A)_n\} = \tilde{\bigcap}_{\lambda\in\Lambda}\{(F_\lambda,A)_n\}$. Then $U - \{(F,A)_n\} = U - \tilde{\bigcap}_{\lambda\in\Lambda}\{(F_\lambda,A)_n\} = \tilde{\bigcup}_{\lambda\in\Lambda}(U - \{(F_\lambda,A)_n\})$ (by De Morgan's law).Since $\{(F_\lambda,A)_n\}$ is fuzzy soft sequential closed set in U for all $\lambda \in \Lambda$, $U - \{(F_\lambda,A)_n\}$ is fuzzy soft sequential open set in U. Since arbitrary union of fuzzy soft sequential open set is fuzzy soft sequential open, $U - \{(F,A)_n\}$ is fuzzy soft sequential open set. Hence $\{(F,A)_n\}$ is fuzzy soft sequential closed set.

Theorem 3.17. Finite union of fuzzy soft sequential closed sets are fuzzy soft sequential closed.

Proof. Let $\{(F_i,A)_n\}_{i=1}^{k}$ be a finite collection of fuzzy soft sequential closed sets. Let $\{(F,A)_n\} = \tilde{\bigcup}_{i=1}^{k}\{(F_i,A)_n\}$. Then $U - \{(F,A)_n\} = U - \tilde{\bigcup}_{i=1}^{k}\{(F_i,A)_n\} = \tilde{\bigcap}_{i=1}^{k}(U - \{(F_i,A)_n\}$ (by De Morgan's law). Since $\{(F_i,A)_n\}$ is fuzzy soft sequential closed set in $U, U - \{(F_i,A)_n\}$ fuzzy soft sequential open set in U. Since finite intersection of fuzzy soft sequential open set is fuzzy soft sequential open set, $U - \{(F,A)_n\}$ is fuzzy soft sequential open set. Hence $\{(F,A)_n\}$ is fuzzy soft sequential closed set.

Remark 3.18. The set of all fuzzy soft sequential closed sets in fuzzy soft sequential topological space $(U,\tilde{\tau},E)$, form a fuzzy soft sequential topology on U.

4. Fuzzy soft sequential neighborhood system and Properties of fuzzy soft sequential quasi-coincident of a fuzzy soft sequential set

Definition 4.1. Let $(U,\tilde{\tau},E)$ be a fuzzy soft sequential topological space over U and $\{(F,A)_n\}$ be a fuzzy soft sequential set over U. A fuzzy soft sequential set $\{(F,A)_n\}$ is called a **fuzzy soft sequential neighborhood** (briefly, FSS-nbhd) of a fuzzy soft sequential set $\{(G,B)_n\}$ if and only if there exists a fuzzy soft sequential open set $\{(H,C)_n\}$ such that $\{(G,B)_n\} \tilde{\subseteq} \{(H,C)_n\} \tilde{\subseteq} \{(F,A)_n\}$.

A fuzzy soft sequential neighborhood $\{(F,A)_n\}$ is said to be an **open** if and only if $\{(F,A)_n\}$ is fuzzy soft sequential open set.

The collection of all fuzzy soft sequential neighborhood of $\{(F,A)_n\}$ is called the **fuzzy soft sequential neighborhood system** of $\{(F,A)_n\}$ up to topology $\tilde{\tau}$ and is denoted by $\mathcal{N}_{\{(F,A)_n\}}$.

Theorem 4.2. A fuzzy soft sequential set $\{(F,A)_n\}$ over U is a fuzzy soft sequential open set if and only if $\{(F,A)_n\}$ is a fuzzy soft sequential neighborhood of each fuzzy soft sequential set $\{(G,B)_n\}$ contained in $\{(F,A)_n\}$.

Proof.

Necessary Part. Let $\{(F,A)_n\}$ be a fuzzy soft sequential open set and $\{(G,B)_n\}$ be any fuzzy soft sequential set contained in $\{(F,A)_n\}$. Since $\{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\} \tilde{\subseteq} \{(F,A)_n\}$, $\{(F,A)_n\}$ is a fuzzy soft sequential neighborhood of each $\{(G,B)_n\}$ contained in $\{(F,A)_n\}$.

Sufficient Part. Let $\{(F,A)_n\}$ be a fuzzy soft sequential neighborhood for every fuzzy soft sequential set contained it. Since $\{(F,A)_n\} \tilde{\subseteq} \{(F,A)_n\}$, there exists a fuzzy soft sequential open set $\{(G,B)_n\}$ such that $\{(F,A)_n\} \tilde{\subseteq} \{(G,B)_n\} \tilde{\subseteq} \{(F,A)_n\}$ .Hence $\{(F,A)_n\} = \{(G,B)_n\}$ and $\{(F,A)_n\}$ is fuzzy soft sequential open set.

Theorem 4.3 If $\mathcal{N}_{\{(F,A)_n\}}$ is the fuzzy soft sequential neighborhood system of fuzzy soft sequential set $\{(F,A)_n\}$, then

(i) Finite intersection of members of $\mathcal{N}_{\{(F,A)_n\}}$ belongs to $\mathcal{N}_{\{(F,A)_n\}}$.

(ii) Each fuzzy soft sequential set which contains a member of $\mathcal{N}_{\{(F,A)_n\}}$ belongs to $\mathcal{N}_{\{(F,A)_n\}}$.

Proof.

(i) If $\{(G,B)_n\}, \{(H,C)_n\} \in \mathcal{N}_{\{(F,A)_n\}}$. Then there exists $\{(G', B')_n\}, \{(H', C')_n\} \in \tilde{\tau}$ such that $\{(F,A)_n\} \tilde{\subseteq} \{(G', B')_n\} \tilde{\subseteq} \{(G,B)_n\}$ and $\{(F,A)_n\} \tilde{\subseteq} \{(H', C')_n\} \tilde{\subseteq} \{(H,C)_n\}$. Since $\{(G', B')_n\} \tilde{\cap} \{(H', C')_n\} \in \tilde{\tau}$, $\{(F,A)_n\} \tilde{\subseteq} \{(G', B')_n\} \tilde{\cap} \{(H', C')_n\} \tilde{\subseteq} \{(G,B)_n\} \tilde{\cap} \{(H,C)_n\}$. Hence $\{(G,B)_n\} \tilde{\cap} \{(H,C)_n\} \in \mathcal{N}_{\{(F,A)_n\}}$.

(ii) Let $\{(G,B)_n\} \in \mathcal{N}_{\{(F,A)_n\}}$ and $\{(H,C)_n\}$ be a fuzzy soft sequential set which contains $\{(G,B)_n\}$. Since $\{(G,B)_n\} \in \mathcal{N}_{\{(F,A)_n\}}$, there exists a fuzzy soft sequential open set $\{(G', B')_n\}$ such that $\{(F,A)_n\} \tilde{\subseteq} \{(G', B')_n\} \tilde{\subseteq} \{(G,B)_n\}$. Since $\{(G,B)_n\} \tilde{\subseteq} \{(H,C)_n\}$, $\{(F,A)_n\} \tilde{\subseteq} \{(G', B')_n\} \tilde{\subseteq} \{(H,C)_n\}$. Hence $\{(H,C)_n\} \in \mathcal{N}_{\{(F,A)_n\}}$.

Definition 4.4. Fuzzy soft sequential sets $\{(F,A)_n\}$ and $\{(G,B)_n\}$ are said to be fuzzy soft sequential quasi-coincident, denoted by $\{(F,A)_n\}q\{(G,B)_n\}$, if and only if there exists $e_i \in A, B$ and $x_j \in U$ such that $F_{\mathrm{n}}(e_i)(x_j) + G_{\mathrm{n}}(e_i)(x_j) \gtrsim 1$ $(or) F_{\mathrm{n}}(e_i)(x_j) \gtrsim G_{\mathrm{n}}^{\mathrm{c}}(e_i)(x_j)$ for all $n \in \mathbb{N}$.

If there exists $e_i \in A, B$ and $x_j \in U$ such that $F_{\mathrm{n}}(e_i)(x_j) + G_{\mathrm{n}}(e_i)(x_j) \lesseqqgtr 1$ for all $n \in \mathbb{N}$, then $\{(F,A)_n\}$ and $\{(G,B)_n\}$ are not fuzzy soft sequential quasi-coincident and it is denoted by $\{(F,A)_n\}\bar{q}\{(G,B)_n\}$.

Definition 4.5. Fuzzy soft sequential sets $\{(F,A)_n\}$ and $\{(G,B)_n\}$ are said to be fuzzy soft sequential weakly quasi-coincident, denoted by $\{(F,A)_n\}q_w\{(G,B)_n\}$, if and only if there exists $e_i \in A, B$ and $x_j \in U$ such that $F_{\mathrm{n}}(e_i)(x_j) + G_{\mathrm{n}}(e_i)(x_j) \gtrsim 1$ $(or) F_{\mathrm{n}}(e_i)(x_j) \gtrsim G_{\mathrm{n}}^{\mathrm{c}}(e_i)(x_j)$ for some $n \in \mathbb{N}$.

If $\{(F,A)_n\}$ and $\{(G,B)_n\}$ are not fuzzy soft sequential weakly quasi-coincident, then it can be written as $\{(F,A)_n\}\overline{q_w}\{(G,B)_n\}$.

Proposition 4.6 The fuzzy soft sequential sets $\{(F,A)_n\}$ and $\{(G,B)_n\}$ are fuzzy soft sequential quasi-coincident, if and only if each pair of non-zero fuzzy soft sets $(F,A)_n$ and $(G,B)_n$ are also so.

Proof. Let the fuzzy soft sequential sets $\{(F,A)_n\}$ and $\{(G,B)_n\}$ are fuzzy soft sequential quasi-coincident $\Leftrightarrow$ there exists $e_i \in A$, B and $x_j \in U$ such that

$$F_n(e_i)(x_j) + G_n(e_i)(x_j) \gtrsim 1 \ for\ all\ n \in \mathbb{N}$$

$\Leftrightarrow$ each pair of non-zero fuzzy soft sets $(F, A)_n$ and $(G, B)_n$ are also fuzzy soft quasi-coincident.

Proposition 4.7. Let $\{(F, A)_n\}$ and $\{(G, B)_n\}$ be a fuzzy soft sequential sets over U. Then the following properties are true.

(i) $\{(F, A)_n\} \tilde{\sqsubseteq} \{(G, B)_n\}$ if and only if $\{(F, A)_n\} \bar{q}\{(G, B)_n\}^c$

(ii) $\{(F, A)_n\} \tilde{\sqsubseteq}_w \{(G, B)_n\}$ if and only if $\{(F, A)_n\} q_w\{(G, B)_n\}^c$

(iii) $\{(F, A)_n\} \tilde{\not\subset} \{(G, B)_n\}^c$ if and only if $\{(F, A)_n\} q\{(G, B)_n\}$

(iv) $\{(F, A)_n\} q \{(G, B)_n\}$ implies $\{(F, A)_n\} \tilde{\sqcap} \{(G, B)_n\} \neq \tilde{\emptyset}$

(v) $\{(F, A)_n\} \bar{q} \{(F, A)_n\}^c$

Proof.

(i) $\{(F, A)_n\} \tilde{\sqsubseteq} \{(G, B)_n\}$ $\Leftrightarrow$ for each $e_i \in A, B$ and $x_j \in U$ such that $F_n(e_i)(x_j) \lesseqqgtr G_n(e_i)(x_j)$ for all $n \in \mathbb{N}$.

$\Leftrightarrow$ for each $e_i \in A, B$ and $x_j \in U$ such that $F_n(e_i)(x_j) + 1 - G_n(e_i)(x_j) \lesseqqgtr 1$ for all $n \in \mathbb{N}$.

$\Leftrightarrow$ for each $e_i \in A, B$ and $x_j \in U$ such that $\{(F, A)_n\} + \{(G, B)_n\}^c \lesseqqgtr 1$ for all $n \in \mathbb{N}$.

Hence $\{(F, A)_n\} \bar{q}\{(G, B)_n\}^c$.

(ii) The proof is similar to (i)

(iii) The proof is similar to (i)

(iv) Let $\{(F, A)_n\} q \{(G, B)_n\}$. Then there exists $e_i \in A, B$ and $x_j \in U$ such that $F_n(e_i)(x_j) + G_n(e_i)(x_j) \gtrsim 1 \; for\ all\ n \in \mathbb{N}$ implies $\{(F, A)_n\} \neq \tilde{\emptyset}$ and $\{(G, B)_n\} \neq \tilde{\emptyset}$ for all $n \in \mathbb{N}$. Hence $\{(F, A)_n\} \tilde{\sqcap} \{(G, B)_n\} \neq \tilde{\emptyset}$.

(v) Suppose $\{(F, A)_n\} q \{(F, A)_n\}^c$. Then there exists $e_i \in A$ and $x_j \in U$ such that $F_n(e_i)(x_j) + F_n^c(e_i)(x_j) \gtrsim 1 \; for\ all\ n \in \mathbb{N}$ implies there exists $e_i \in A$ and $x_j \in U$ such that $F_n(e_i)(x_j) + 1 - F_n(e_i)(x_j) \gtrsim 1 \; for\ all\ n \in \mathbb{N}$. Therefore, there exists $e_i \in A$ and $x_j \in U$ such that $F_n(e_i)(x_j) \gtrsim F_n(e_i)(x_j) \; for\ all\ n \in \mathbb{N}$, which is a contradiction. Hence $\{(F, A)_n\} \bar{q} \{(F, A)_n\}^c$.

5. Conclusion

In this paper, we have introduced fuzzy soft sequential topological spaces via fuzzy soft sequential set. We studied their properties and a relationship between fuzzy soft topology and fuzzy soft sequential topology are explored. We investigated fuzzy soft sequential neighborhood of a fuzzy soft sequential set, fuzzy soft sequential quasi-coincident along with their properties.

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