

Common Fixed Point Solutions for Pair of Mappings Via $G - (\alpha, \beta, \psi) -$ Contraction with Admissibility

D. Sattemma^{1*,2*}, Dr. S. Vijaya Lakshmi³

^{1*}Research Scholar, Department of mathematics, Osmania University, Hyderabad-500007, Telangana, India

^{2*}Assistant professor, Department of Mathematics, MALD Government Degree College, Gadwal-509125, Telangana, India

³Professor, Department of Mathematics, University College of Science, Osmania University, Hyderabad-500007, Telangana, India (vijayalakshmi.sandra@gmail.com)

*Corresponding Author: satya.dyavathi@gmail.com

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Abstract:

We explain the idea of contraction for a pair of mappings and propose fixed point theorems in the arrangement of G -metric spaces using contraction. Relevant examples are provided.

Keywords: BCMP, G -metric space, quasi-metric space, and b -metric.

1. Introduction

In 1922, S. Banach [1] presented the well-known Banach contraction mapping theorem (BCMP) for the complete metric space. Several generalizations of BCMP utilizing various contractive conditions in the framework of metric, G -metric space, quasi-metric space, and b -metric space have been published in the literature by a wide spectrum of mathematicians during the last several decades.

Khan [2] employed distance mapping to develop a new contractive condition in fixed point theory, hence expanding the BCP into novel configurations. Abodyeh et al. [21] just suggested a new idea, almost-perfect contraction, to create new contractive conditions in order to modify and expand some well-known fixed point theorems.

Samet et al. [22] proposed the idea of α -admissibility. Karapinar et al. [23] developed the concept of triangular admissibility. Abdeljawad [24] extended the concept of α -admissibility to a pair of functions. Chary et al. [33] developed the novel idea of rectangular α - G -admissible mapping in 2021, as well as rectangular α - G -admissible concerning another function β in G -metric space.

2. Preliminaries

Chary et al. [33] have been established the concepts of G - α -admissible mapping; G - α -admissible with respect to β ; G - α , β -Cauchy; α , β -complete G -metric space; α , β - G -continuous; G - α -admissible rectangular mapping; G - α -admissible rectangular with respect to β . These concepts are used in this work.

3. Main Results

First, we will define the following terms.

Definition 3.1. Let H, J be two self-mappings on Ω and $\alpha : \Omega \times \Omega \times \Omega \rightarrow R^+ \cup \{0\}$ be a function. The pair (H, J) is called G - α -admissible if $\vartheta, \nu, \rho \in \Omega$ and $\alpha(\vartheta, \nu, \rho) \geq 1$ imply $\alpha(H\vartheta, J\nu, J\rho) \geq 1$ and $\alpha(J\vartheta, H\nu, H\rho) \geq 1$.

Definition 3.2. Let H, J be two self-mappings on Ω and $\alpha, \beta : \Omega \times \Omega \times \Omega \rightarrow R^+ \cup \{0\}$ be functions. The pair (H, J) is called G - (α, β) -admissibility if $\vartheta, \nu, \rho \in \Omega$ and $\alpha(\vartheta, \nu, \rho) \geq \beta(\vartheta, \nu, \rho)$ imply $\alpha(H\vartheta, J\nu, J\rho) \geq \beta(H\vartheta, J\nu, J\rho)$ and $\alpha(J\vartheta, H\nu, H\rho) \geq \beta(J\vartheta, H\nu, H\rho)$.

Example 3.3. Define H, J from R to R by $H\vartheta = \vartheta^2$ and $J\vartheta = \begin{cases} -\vartheta^2 & \text{if } \vartheta < 0 \\ \vartheta^2 & \text{if } \vartheta \geq 0 \end{cases}$. Additionally, define $\alpha, \beta : \Omega \times \Omega \times \Omega \rightarrow R^+ \cup \{0\}$ and $\alpha(\vartheta, \nu, \rho) = e^{\vartheta+\nu+\rho}$ and $\beta(\vartheta, \nu, \rho) = e^\vartheta$ then (H, J) is a pair of G - (α, β) -admissibility.

Proof. Let $\vartheta, \nu, \rho \in \Omega$ such that $\alpha(\vartheta, \nu, \rho) \geq \beta(\vartheta, \nu, \rho)$ then $e^{\vartheta+\nu+\rho} \geq e^\vartheta$. So $\vartheta + \nu + \rho \geq \vartheta$ and hence ν, ρ are non-negative real numbers. Therefore

$$\alpha(H\vartheta, J\nu, J\rho) = \alpha(\vartheta^2, \nu^2, \rho^2) = e^{\vartheta^2+\nu^2+\rho^2} \geq e^{\vartheta^2} = \beta(H\vartheta, J\nu, J\rho)$$

imply that

$$\alpha(H\vartheta, J\nu, J\rho) \geq \beta(H\vartheta, J\nu, J\rho).$$

Now, if $\vartheta \geq 0$, then $\alpha(J\vartheta, H\nu, H\rho) = \alpha(\vartheta^2, \nu^2, \rho^2) = e^{\vartheta^2+\nu^2+\rho^2} \geq e^{\vartheta^2} = \beta(J\vartheta, H\nu, H\rho)$.

While if $\vartheta < 0$, then $\alpha(J\vartheta, H\nu, H\rho) = \alpha(-\vartheta^2, \nu^2, \rho^2) = e^{-\vartheta^2+\nu^2+\rho^2} \geq e^{-\vartheta^2} = \beta(J\vartheta, H\nu, H\rho)$

imply

$$\alpha(J\vartheta, H\nu, H\rho) \geq \beta(J\vartheta, H\nu, H\rho).$$

Definition 3.4. Let G be a metric on Ω . Let H, J be two self-mappings on Ω , ψ be a perfect self-mapping on $R^+ \cup \{0\}$, $\alpha, \beta : \Omega \times \Omega \times \Omega \rightarrow R^+ \cup \{0\}$ be functions. We say that the pair (H, J) is an G - (α, β, ψ) -contraction if there exists $k \in [0, 1)$ such that $\vartheta, \nu, \rho \in \Omega$ and $\alpha(\vartheta, \nu, \rho) \geq \beta(\vartheta, \nu, \rho)$ imply

$$\begin{aligned} \psi(G(H\vartheta, J\nu, J\rho)) \leq & \max\{k\psi(G(\vartheta, \nu, \rho)), k\psi(G(\vartheta, H\vartheta, H\vartheta)), k\psi(G(\nu, J\nu, J\nu)), \\ & k\psi(G(\rho, J\rho, J\rho)), k\psi(G(\nu, H\rho, H\rho)), k\psi(G(\rho, H\vartheta, H\vartheta)), \\ & k\psi(G(\vartheta, J\vartheta, J\vartheta)), \frac{1}{3}k\psi(G(\vartheta, J\nu, J\nu)), \frac{1}{3}k\psi(G(\nu, J\rho, J\rho)), \\ & \frac{1}{3}k\psi(G(\rho, J\vartheta, J\vartheta))\} \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned} \psi(G(J\vartheta, H\nu, H\rho)) \leq & \max\{k\psi(G(\vartheta, \nu, \rho)), k\psi(G(\vartheta, J\vartheta, J\vartheta)), k\psi(G(\nu, H\nu, H\nu)), \\ & k\psi(G(\rho, J\vartheta, J\vartheta)), k\psi(G(\nu, H\rho, H\rho)), k\psi(G(\rho, H\vartheta, H\vartheta)), \\ & k\psi(G(\rho, J\vartheta, J\vartheta)), \frac{1}{3}k\psi(G(\vartheta, H\nu, H\nu)), \frac{1}{3}k\psi(G(\nu, H\rho, H\rho)), \\ & \frac{1}{3}k\psi(G(\rho, J\vartheta, J\vartheta))\} \dots\dots\dots(2) \end{aligned}$$

Example.3.5. Define $G : [0, \frac{1}{4}] \times [0, \frac{1}{4}] \times [0, \frac{1}{4}] \rightarrow R^+ \cup \{0\}$ by $G(\vartheta, \nu, \rho) = |\vartheta - \nu| + |\nu - \rho| + |\rho - \vartheta|$ and $H, J : [0, \frac{1}{4}] \rightarrow [0, \frac{1}{4}]$ by $H\vartheta = \vartheta^2$ and $J\vartheta = \vartheta^4$. Also define ψ on $R^+ \cup \{0\} \rightarrow R^+ \cup \{0\}$. By $\psi(\vartheta) = \frac{\vartheta}{1+\vartheta}$ and $\alpha, \beta : [0, \frac{1}{4}] \times [0, \frac{1}{4}] \times [0, \frac{1}{4}] \rightarrow R^+ \cup \{0\}$ by $\alpha(\vartheta, \nu, \rho) = e^\vartheta$ and $\beta(\vartheta, \nu, \rho) = e^{\vartheta+\nu+\rho}$ then (H, J) is an $G - (\alpha, \beta, \psi)$ - contraction.

Proof. Given $\vartheta, \nu, \rho \in [0, \frac{1}{4}]$ is such that $\alpha(\vartheta, \nu, \rho) \geq \beta(\vartheta, \nu, \rho)$ then $e^\vartheta \geq e^{\vartheta+\nu+\rho}$. Therefore, we conclude that $\nu = \rho = 0$. Since $\vartheta \leq \frac{1}{4}$, we have

$$\begin{aligned} \psi(G(H\vartheta, J\nu, J\rho)) &= \psi(G(\vartheta^2, 0, 0)) = \psi(2\vartheta^2) = \frac{2\vartheta^2}{1+2\vartheta^2} \leq \frac{1}{3} \frac{2\vartheta}{1+2\vartheta} = \frac{1}{3} \psi(G(\vartheta, \nu, \rho)) \quad \text{and} \\ \psi(G(J\vartheta, H\nu, H\rho)) &= \psi(G(\vartheta^4, 0, 0)) = \psi(2\vartheta^4) = \frac{2\vartheta^4}{1+2\vartheta^4} \leq \frac{1}{3} \frac{2\vartheta}{1+2\vartheta} = \frac{1}{3} \psi(G(\vartheta, \nu, \rho)). \end{aligned}$$

So pair (H, J) is an $G - (\alpha, \beta, \psi)$ - contraction.

The main outcome of this study is

Theorem. 3.6. On the set Ω , let $\alpha, \beta : \Omega \times \Omega \times \Omega \rightarrow R^+ \cup \{0\}$ be two functions and H, J be two self-mappings on Ω . Assume there exists a metric G on Ω such that the following hypothesis hold:

Let $\alpha, \beta : \Omega \times \Omega \times \Omega \rightarrow R^+ \cup \{0\}$ be two functions on Ω , with H and J being two self-mappings. Assume there is a metric G on which the following hypothesis holds:

1. (G, Ω) is an $\alpha, \beta - G -$ Complete metric space.
2. H and J are $\alpha, \beta - G -$ Continuous.
3. (H, J) is an $G - (\alpha, \beta, \psi)$ - Contraction.
4. (H, J) is a pair of $(\alpha, \beta) - G -$ admissibility.

5. If $\mathcal{G}, \nu, \rho \in \Omega$ satisfy the condition $\alpha(\mathcal{G}, \nu, \nu) \geq \beta(\mathcal{G}, \nu, \nu)$ and $\alpha(\nu, \rho, \omega) \geq \beta(\nu, \rho, \omega)$ then $\alpha(\mathcal{G}, \rho, \omega) \geq \beta(\mathcal{G}, \rho, \omega)$

6. There exists $\mathcal{G}_0 \in \Omega$ Such that $\alpha(H\mathcal{G}_0, JH\mathcal{G}_0, JH\mathcal{G}_0) \geq \beta(H\mathcal{G}_0, JH\mathcal{G}_0, JH\mathcal{G}_0)$ and $\alpha(JH\mathcal{G}_0, H\mathcal{G}_0, H\mathcal{G}_0) \geq \beta(JH\mathcal{G}_0, H\mathcal{G}_0, H\mathcal{G}_0)$ Then both mappings H and J have common fixed point.

Proof: In considering premise (6), we begin with $\mathcal{G}_0 \in \Omega$ in such a way that

$$\alpha(H\mathcal{G}_0, JH\mathcal{G}_0, JH\mathcal{G}_0) \geq \beta(H\mathcal{G}_0, JH\mathcal{G}_0, JH\mathcal{G}_0) \text{ and } \alpha(JH\mathcal{G}_0, H\mathcal{G}_0, H\mathcal{G}_0) \geq \beta(JH\mathcal{G}_0, H\mathcal{G}_0, H\mathcal{G}_0).$$

Now, Let $\mathcal{G}_1 = H\mathcal{G}_0, \mathcal{G}_2 = J\mathcal{G}_1$ Then $\alpha(\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_1) \geq \beta(\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_1)$ and $\alpha(\mathcal{G}_1, \mathcal{G}_0, \mathcal{G}_0) \geq \beta(\mathcal{G}_1, \mathcal{G}_0, \mathcal{G}_0)$.

In the view of hypothesis (4), we have

$$\alpha(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_2) = \alpha(H\mathcal{G}_0, J\mathcal{G}_1, J\mathcal{G}_1) \geq \beta(H\mathcal{G}_0, J\mathcal{G}_1, J\mathcal{G}_1) = \beta(\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_2) \text{ and}$$

$$\alpha(\mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1) = \alpha(J\mathcal{G}_1, H\mathcal{G}_0, H\mathcal{G}_0) \geq \beta(J\mathcal{G}_1, H\mathcal{G}_0, H\mathcal{G}_0) = \beta(\mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_1)$$

Again we put $\mathcal{G}_3 = H\mathcal{G}_2$, Then, hypothesis (4) implies that

$$\alpha(\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_3) = \alpha(J\mathcal{G}_1, H\mathcal{G}_2, H\mathcal{G}_2) \geq \beta(J\mathcal{G}_1, H\mathcal{G}_2, H\mathcal{G}_2) = \beta(\mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_3) \text{ and}$$

$$\alpha(\mathcal{G}_3, \mathcal{G}_2, \mathcal{G}_2) = \alpha(H\mathcal{G}_2, J\mathcal{G}_1, J\mathcal{G}_1) \geq \beta(H\mathcal{G}_2, J\mathcal{G}_1, J\mathcal{G}_1) = \beta(\mathcal{G}_3, \mathcal{G}_2, \mathcal{G}_2)$$

putting $\mathcal{G}_4 = J\mathcal{G}_3$ and referring to hypothesis (4), we conclude

$$\alpha(\mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_4) = \alpha(H\mathcal{G}_2, J\mathcal{G}_3, J\mathcal{G}_3) \geq \beta(H\mathcal{G}_2, J\mathcal{G}_3, J\mathcal{G}_3) = \beta(\mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_4) \text{ and}$$

$$\alpha(\mathcal{G}_4, \mathcal{G}_3, \mathcal{G}_3) = \alpha(J\mathcal{G}_3, H\mathcal{G}_2, H\mathcal{G}_2) \geq \beta(J\mathcal{G}_3, H\mathcal{G}_2, H\mathcal{G}_2) = \beta(\mathcal{G}_4, \mathcal{G}_3, \mathcal{G}_3)$$

Proceeding in a similar fashion, we create a sequence $\{\mathcal{G}_n\} \in \Omega$ with $\mathcal{G}_{2n+1} = H\mathcal{G}_{2n}$ and $\mathcal{G}_{2n+2} = H\mathcal{G}_{2n+1}$

So that $\alpha(\mathcal{G}_n, \mathcal{G}_{n+1}, \mathcal{G}_{n+1}) \geq \beta(\mathcal{G}_n, \mathcal{G}_{n+1}, \mathcal{G}_{n+1}) \forall n \in N$ and $\alpha(\mathcal{G}_{n+1}, \mathcal{G}_n, \mathcal{G}_n) \geq \beta(\mathcal{G}_{n+1}, \mathcal{G}_n, \mathcal{G}_n) \forall n \in N$.

Based on premise (5), we observe that $\alpha(\mathcal{G}_n, \mathcal{G}_m, \mathcal{G}_m) \geq \beta(\mathcal{G}_n, \mathcal{G}_m, \mathcal{G}_m) \forall n, m \in N$. If there exists $p \in N$ so that $\mathcal{G}_{2p} = \mathcal{G}_{2p+1}$ then $\mathcal{G}_{2p} = H\mathcal{G}_{2p+1}$ and hence H has fixed point. Based on premise (1), we have

$$\begin{aligned} \psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})) &= \\ \psi(G(H\mathcal{G}_{2p}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})) &\leq \max\{k\psi(G(\mathcal{G}_{2p}, \mathcal{G}_{2p+1}, \mathcal{G}_{2p+1})), k\psi(G(\mathcal{G}_{2p}, H\mathcal{G}_{2p}, H\mathcal{G}_{2p})), \\ &k\psi(G(\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), k\psi(G(\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), \\ &k\psi(G(\mathcal{G}_{2p+1}, H\mathcal{G}_{2p+1}, H\mathcal{G}_{2p+1})), k\psi(G(\mathcal{G}_{2p+1}, H\mathcal{G}_{2p}, H\mathcal{G}_{2p})), \\ &k\psi(G(\mathcal{G}_{2p}, J\mathcal{G}_{2p}, J\mathcal{G}_{2p})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), \\ &\frac{1}{3}k\psi(G(\mathcal{G}_{2p}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+1}, J\mathcal{G}_{2p}, J\mathcal{G}_{2p}))\} \end{aligned}$$

$$\begin{aligned} &\leq \max\{k\psi(G(\mathcal{G}_{2p}, \mathcal{G}_{2p}, \mathcal{G}_{2p})), k\psi(G(\mathcal{G}_{2p}, H\mathcal{G}_{2p}, H\mathcal{G}_{2p})), \\ &\quad k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})), k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})), \\ &\quad k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p}, \mathcal{G}_{2p})), k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p}, \mathcal{G}_{2p})), \\ &\quad k\psi(G(\mathcal{G}_{2p}, \mathcal{G}_{2p+1}, \mathcal{G}_{2p+1})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})), \\ &\quad \frac{1}{3}k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+1}, \mathcal{G}_{2p+1}))\} \\ &\leq \max\{k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, J\mathcal{G}_{2p+2}))\} \\ &\leq k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})) \end{aligned}$$

The last inequality is correct only if $\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})) = 0$. The properties ψ and G imply that $\mathcal{G}_{2p+1} = \mathcal{G}_{2p+2}$. Hence $\mathcal{G}_{2p} = H\mathcal{G}_{2p} = J\mathcal{G}_{2p}$. Thus, H and J have a common fixed point. If there is a natural number p with $\mathcal{G}_{2p+1} = \mathcal{G}_{2p+2}$, then $\mathcal{G}_{2p+1} = J\mathcal{G}_{2p+1}$ and hence J has a fixed point. Based on premise (2), we observe that

$$\begin{aligned} &\psi(G(\mathcal{G}_{2p+2}, \mathcal{G}_{2p+3}, \mathcal{G}_{2p+3})) = \\ &\psi(G(J\mathcal{G}_{2p+1}, H\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+2})) \leq \max\{k\psi(G(\mathcal{G}_{2p+1}, \mathcal{G}_{2p+2}, \mathcal{G}_{2p+2})), k\psi(G(\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+1}, H\mathcal{G}_{2p+1})), \\ &\quad k\psi(G(\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), k\psi(G(\mathcal{G}_{2p+2}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), \\ &\quad k\psi(G(\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+2})), k\psi(G(\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+1}, H\mathcal{G}_{2p+1})), \\ &\quad k\psi(G(\mathcal{G}_{2p+2}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+1}, H\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+2})), \\ &\quad \frac{1}{3}k\psi(G(\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+2}, H\mathcal{G}_{2p+2})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+2}, J\mathcal{G}_{2p+1}, J\mathcal{G}_{2p+1}))\} \\ &\leq \max\{k\psi(G(\mathcal{G}_{2p+2}, \mathcal{G}_{2p+3}, \mathcal{G}_{2p+3})), \frac{1}{3}k\psi(G(\mathcal{G}_{2p+2}, \mathcal{G}_{2p+3}, J\mathcal{G}_{2p+3}))\} \\ &\leq k\psi(G(\mathcal{G}_{2p+2}, \mathcal{G}_{2p+3}, \mathcal{G}_{2p+3})) \end{aligned}$$

The last inequality holds only if $\psi(G(\mathcal{G}_{2p+2}, \mathcal{G}_{2p+3}, \mathcal{G}_{2p+3})) = 0$. The properties of ψ and G imply that $\mathcal{G}_{2p+2} = \mathcal{G}_{2p+3}$. Hence $\mathcal{G}_{2p+1} = H\mathcal{G}_{2p+1} = J\mathcal{G}_{2p+1}$. Thus we conclude that \mathcal{G}_{2p+1} is a common fixed point of H and J . As a result, we determine that H and J have a single fixed point i.e \mathcal{G}_{2p+1} .

Now, presume that's the case $\mathcal{G}_i \neq \mathcal{G}_{i+1} \forall i \in N$. For $i \in N \cup \{0\}$, we get

$$\begin{aligned} \psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})) &= \\ \psi(G(H\mathfrak{g}_{2i}, J\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1})) &\leq \max\{k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i}, H\mathfrak{g}_{2i}, H\mathfrak{g}_{2i})), \\ &k\psi(G(\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1})), \\ &k\psi(G(\mathfrak{g}_{2i+1}, H\mathfrak{g}_{2i+1}, H\mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i+1}, H\mathfrak{g}_{2i}, H\mathfrak{g}_{2i})), \\ &k\psi(G(\mathfrak{g}_{2i}, J\mathfrak{g}_{2i}, J\mathfrak{g}_{2i})), \frac{1}{3}k\psi(G(\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1})), \\ &\frac{1}{3}k\psi(G(\mathfrak{g}_{2i}, J\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i+1})), \frac{1}{3}k\psi(G(\mathfrak{g}_{2i+1}, J\mathfrak{g}_{2i}, J\mathfrak{g}_{2i}))\} \\ &\leq \max\{k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), \\ &k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), \\ &k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), \\ &k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), \frac{1}{3}k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), \\ &\frac{1}{3}k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), \frac{1}{3}k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1}))\} \\ &\leq \max\{k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), \\ &\frac{1}{3}k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, J\mathfrak{g}_{2i+2})), \frac{1}{3}k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})), \} \\ &\leq \max\{k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2}))\} \end{aligned}$$

Thus if $\max\{k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2}))\} = k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2}))$ then $\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})) \leq k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2}))$. Since $k < 1$, condition (1) on ψ implies that $\mathfrak{g}_{2i+1} = \mathfrak{g}_{2i+2}$, a contradiction, therefore

$$\max\{k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})), k\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2}))\} = k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1}))$$

Hence

$$\psi(G(\mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+2}, \mathfrak{g}_{2i+2})) \leq k\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})) \dots\dots\dots (3)$$

Using arguments similar to above, we may show that

$$\psi(G(\mathfrak{g}_{2i}, \mathfrak{g}_{2i+1}, \mathfrak{g}_{2i+1})) \leq k\psi(G(\mathfrak{g}_{2i-1}, \mathfrak{g}_{2i}, \mathfrak{g}_{2i})) \dots\dots\dots (4)$$

Combining eqn (3) and (4) together, we reach

$$\psi(G(\mathfrak{g}_i, \mathfrak{g}_{i+1}, \mathfrak{g}_{i+1})) \leq k\psi(G(\mathfrak{g}_{i-1}, \mathfrak{g}_i, \mathfrak{g}_i)) \dots\dots\dots (5)$$

By recurring eqn (5) n-times, we deduce

$$\begin{aligned} \psi(G(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1})) &\leq k\psi(G(\vartheta_{i-1}, \vartheta_i, \vartheta_i)) \\ &\leq k^2\psi(G(\vartheta_{i-2}, \vartheta_{i-1}, \vartheta_{i-1})) \\ &\vdots \\ &\leq k^n\psi(G(\vartheta_0, \vartheta_1, \vartheta_1)) \end{aligned} \dots\dots\dots(6)$$

On allowing $n \rightarrow +\infty$ in eqn (6), we get

$$\lim_{n \rightarrow +\infty} \psi(G(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1})) = 0 \dots\dots\dots (7)$$

Criterion (2) for the role ψ indicates that

$$\lim_{n \rightarrow +\infty} G(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1}) = 0 \dots\dots\dots (8)$$

We seek confirmation that $\{\vartheta_i\}$ is a G-Cauchy sequence in Ω .

Take $i, j \in N$ with $j > i$. We categorize the documentation into four separate instances.

Case 1: i is odd integer and j is even integer. Therefore, there exists $l \in N$ and an odd integer ‘ h ’ such that $i = 2l + 1$ and $j = i + h = 2l + 1 + h$. Since $\alpha(\vartheta_i, \vartheta_j, \vartheta_j) \geq \beta(\vartheta_i, \vartheta_j, \vartheta_j)$ we have n

$$\begin{aligned} \psi(G(\vartheta_i, \vartheta_j, \vartheta_j)) &= \psi(G(\vartheta_{2l+1}, \vartheta_{2l+1+h}, \vartheta_{2l+1+h})) = \psi(G(H\vartheta_{2l}, J\vartheta_{2l+h}, J\vartheta_{2l+h})) \\ &\leq \max\{k\psi(G(\vartheta_{2l}, \vartheta_{2l+h}, \vartheta_{2l+h})), k\psi(G(\vartheta_{2l}, H\vartheta_{2l}, H\vartheta_{2l})), \\ &\quad k\psi(G(\vartheta_{2l+h}, J\vartheta_{2l+h}, J\vartheta_{2l+h})), k\psi(G(\vartheta_{2l+h}, J\vartheta_{2l+h}, J\vartheta_{2l+h})), \\ &\quad k\psi(G(\vartheta_{2l+h}, H\vartheta_{2l+h}, H\vartheta_{2l+h})), k\psi(G(\vartheta_{2l+h}, H\vartheta_{2l}, H\vartheta_{2l})), \\ &\quad k\psi(G(\vartheta_{2l}, J\vartheta_{2l}, J\vartheta_{2l})), \frac{1}{3}k\psi(G(\vartheta_{2l}, J\vartheta_{2l+h}, J\vartheta_{2l+h})), \\ &\quad \frac{1}{3}k\psi(G(\vartheta_{2l+h}, J\vartheta_{2l+h}, J\vartheta_{2l+h})), \frac{1}{3}k\psi(G(\vartheta_{2l+h}, J\vartheta_{2l+h}, J\vartheta_{2l+h}))\} \\ &\leq \max\{k\psi(G(\sum_{s=2l}^{2l+h-1} \vartheta_s, \vartheta_{s+1}, \vartheta_{s+1})), k\psi(G(\vartheta_{2l}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\ &\quad k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), \\ &\quad k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\ &\quad k\psi(G(\vartheta_{2l}, \vartheta_{2l+1}, \vartheta_{2l+1})), \frac{1}{3}k\psi(G(\vartheta_{2l}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), \\ &\quad \frac{1}{3}k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), \frac{1}{3}k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1}))\} \end{aligned}$$

$$\begin{aligned}
 &\leq \max\{k\psi(\sum_{s=2l}^{2l+h-1} G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1})), k\psi(G(\vartheta_{2l}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\
 &\quad k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\
 &\quad \frac{1}{3}k\psi(G(\vartheta_{2l}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})) \\
 &\leq \max\{k\psi(\sum_{s=2l}^{2l+h-1} (G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1}))), k\psi(G(\vartheta_{2l}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\
 &\quad k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), k\psi(\sum_{s=2l+1}^{2l+h-1} G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1})), \\
 &\quad \frac{1}{3}k\psi(\sum_{s=2l}^{2l+h} G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1}))\} \\
 &\leq \max\{k \sum_{s=2l}^{2l+h-1} \psi(G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1})), k\psi(G(\vartheta_{2l}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\
 &\quad k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1})), k \sum_{s=2l+1}^{2l+h-1} \psi(G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1})), \\
 &\quad \frac{1}{3}k \sum_{s=2l}^{2l+h} \psi(G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1}))\} \\
 &\leq \max\{k(\sum_{s=2l}^{+\infty} \psi(G(\vartheta_s, \vartheta_{s+1}, \vartheta_{s+1}))), k\psi(G(\vartheta_{2l}, \vartheta_{2l+1}, \vartheta_{2l+1})), \\
 &\quad k\psi(G(\vartheta_{2l+h}, \vartheta_{2l+h+1}, \vartheta_{2l+h+1}))\} \\
 &\leq \max\{\frac{k^{2l+1}}{1-k} \psi(G(\vartheta_0, \vartheta_1, \vartheta_1)), k\psi(G(\vartheta_{2l}, H\vartheta_{2l}, H\vartheta_{2l})), \\
 &\quad k\psi(G(\vartheta_{2l+h}, H\vartheta_{2l+h}, H\vartheta_{2l+h}))\}
 \end{aligned}$$

By allowing $i, j \rightarrow +\infty$ in the previous solutions and applying argument (7), we obtain

$$\lim_{i, j \rightarrow +\infty} \psi(G(\vartheta_i, \vartheta_j, \vartheta_j)) = 0$$

The properties of ψ imply that

$$\lim_{i, j \rightarrow +\infty} G(\vartheta_i, \vartheta_j, \vartheta_j) = 0 \quad \dots\dots\dots (9)$$

Case 2: i and j are both even integers.

Applying the rectangular inequality of the metric ‘G’, we have

$$G(\vartheta_i, \vartheta_j, \vartheta_j) \leq G(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1}) + G(\vartheta_{i+1}, \vartheta_j, \vartheta_j)$$

Letting $i \rightarrow +\infty$ and in view of eqn (8) and (9), we get $\lim_{i, j \rightarrow +\infty} G(\vartheta_i, \vartheta_j, \vartheta_j) = 0$.

Case3: i is even integer and j is an odd integer.

Applying the rectangular inequality of the metric G , we have

$$G(\mathcal{G}_i, \mathcal{G}_j, \mathcal{G}_j) \leq G(\mathcal{G}_i, \mathcal{G}_{i+1}, \mathcal{G}_{i+1}) + G(\mathcal{G}_{i+1}, \mathcal{G}_{j-1}, \mathcal{G}_{j-1}) + G(\mathcal{G}_{j-1}, \mathcal{G}_j, \mathcal{G}_j)$$

Letting $i \rightarrow +\infty$ and considering eqn (8) and (9), we get

$$\lim_{i,j \rightarrow +\infty} G(\mathcal{G}_i, \mathcal{G}_j, \mathcal{G}_j) = 0$$

Case4: i and j are both odd integer, applying the rectangular inequality of the G -metric, we have

$$G(\mathcal{G}_i, \mathcal{G}_j, \mathcal{G}_j) \leq G(\mathcal{G}_i, \mathcal{G}_{j-1}, \mathcal{G}_{j-1}) + G(\mathcal{G}_{j-1}, \mathcal{G}_j, \mathcal{G}_j)$$

On permitting $i \rightarrow +\infty$ and in view of eqn (8) and (9)

$$\lim_{i,j \rightarrow +\infty} G(\mathcal{G}_i, \mathcal{G}_j, \mathcal{G}_j) = 0$$

After putting everything together, we may decide that

$$\lim_{i,j \rightarrow +\infty} G(\mathcal{G}_i, \mathcal{G}_j, \mathcal{G}_j) = 0$$

Thus, we conclude that $\{\mathcal{G}_i\}$ is a G -Cauchy sequence in Ω then. $\alpha, \beta - G$ -Completeness of metric space (Ω, G) ensure that there is $\mathcal{G} \in \Omega$ so that $\mathcal{G}_i \rightarrow \mathcal{G}$ using $\alpha, \beta - G$ -Continuous of the mappings H and J , we deduce that $\mathcal{G}_{2i+1} = H\mathcal{G}_{2i} \rightarrow H\mathcal{G}$ and $\mathcal{G}_{2i+2} = J\mathcal{G}_{2i+1} \rightarrow J\mathcal{G}$. By unique of limit, we obtain $H\mathcal{G} = J\mathcal{G} = \mathcal{G}$. Thus, \mathcal{G} is a fixed point of H and J .

EXAMPLE:

Define $G : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ By

$$G(\mathcal{G}, \nu, \rho) = \begin{cases} \max\{\mathcal{G}, \nu, \rho\}, & \text{if } \mathcal{G} \neq \nu \neq \rho \\ 0, & \text{if } \mathcal{G} = \nu = \rho \end{cases}$$

Let H, J be two self-mappings on $[0, +\infty)$ defined by $H\mathcal{G} = \frac{1}{2} \text{Sin}^2 \mathcal{G}$ and $J\mathcal{G} = \frac{1}{4} \text{Sin}^2 \mathcal{G}$. In

addition, define the function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ by $\psi(\mathcal{G}) = \frac{\mathcal{G}}{1 + \mathcal{G}}$.

Furthermore, we define the function $\alpha, \beta : \Omega \times \Omega \times \Omega \rightarrow [0, +\infty)$ by

$$\alpha(\mathcal{G}, \nu, \rho) = \begin{cases} e^{\mathcal{G} + \nu + \rho} & \text{if } \mathcal{G}, \nu, \rho \in [0, 1] \\ 0, & \text{if } \mathcal{G} > 1 \text{ or } \nu > 1 \text{ or } \rho > 1 \end{cases}$$

And

$$\alpha(\vartheta, \nu, \rho) = \begin{cases} e^\vartheta & \text{if } \vartheta, \nu, \rho \in [0,1] \\ 1, & \text{if } \vartheta > 1 \text{ or } \nu > 1 \text{ or } \rho > 1 \end{cases}$$

Then

1. ψ is a perfect function
2. there exists $\vartheta_0 \in \Omega$

$$\alpha(H\vartheta_0, H^2\vartheta_0, H^2\vartheta_0) \geq \beta(H\vartheta_0, H^2\vartheta_0, H^2\vartheta_0) \text{ and}$$

$$\alpha(H^2\vartheta_0, H\vartheta_0, H\vartheta_0) \geq \beta(H^2\vartheta_0, H\vartheta_0, H\vartheta_0)$$

3. (H, J) is a pair of $(\alpha, \beta) - G$ - admissibility
4. H and J are $(\alpha, \beta) - G$ - Continuous
5. Ω, G is an $\alpha, \beta - G$ - Complete metric space
6. (H, J) is an $G - (\alpha, \beta, \psi) -$ Contraction

Proof: It is an easy matter to see eqn's (1)-(3). Prove eqn (4), let (ϑ_i) be any sequence in $[0, +\infty)$ such that $\vartheta_i \rightarrow \vartheta \in [0, +\infty)$ and $\alpha(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1}) \geq \beta(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1})$ for all $i \in \mathcal{N}$. Thus $\vartheta_i \in [0,1]$ for all $i \in \mathcal{N}$

if $\vartheta_i = \vartheta$ for any except a finite number, we infer that $H\vartheta_i \rightarrow H\vartheta$ as $i \rightarrow +\infty$. If $\vartheta_i \neq \vartheta$ for any save that has an infinite number, we observe that $\vartheta=0$. Hence, $\vartheta_i \rightarrow 0$ in $([0,1], |\cdot|)$. Therefore $\max\{\frac{1}{2} \sin^2 \vartheta_i, 0, 0\} \rightarrow 0 = H\vartheta$ in $([0, +\infty), G)$. That is H is $(\alpha, \beta) - G$ - Continuous.

To prove (5), let $\{\vartheta_i\}$ be a G -Cauchy sequence in $([0, +\infty), G)$ such that $\alpha(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1}) \geq \beta(\vartheta_i, \vartheta_{i+1}, \vartheta_{i+1})$ then $\vartheta_i \in [0,1]$ for all $i \in \mathcal{N}$ if there exists $\vartheta \in [0,1]$ such that $\vartheta_i = \vartheta$ for all but finitely many, then $\vartheta_i \rightarrow \vartheta$ as $i \rightarrow +\infty$ now suppose the elements of $\{\vartheta_i\}$ are distinct for all but finitely many. Given $\varepsilon > 0$ since $\{\vartheta_i\}$ is a G -Cauchy sequence in $([0, +\infty), G)$, then there exists $i_0 \in \mathcal{N}$ such that $\max\{\vartheta_i, \vartheta_j, \vartheta_j\} < \varepsilon$ for all $j > i \geq i_0$ Therefore, $\max\{\vartheta_i, 0, 0\} < \varepsilon$ for $i \geq i_0$ so, $\vartheta_i \rightarrow 0$ in $([0, +\infty), G)$

Thus, $([0, +\infty), G)$ is an $\alpha, \beta - G$ - Complete metric space

To prove (6), let $\vartheta, \nu, \rho \in \Omega$ such that $\alpha(\vartheta, \nu, \rho) \geq \beta(\vartheta, \nu, \rho)$. then $\vartheta, \nu, \rho \in [0,1]$ so,

$$\begin{aligned}
 \psi(G(H\vartheta, J\nu, J\rho)) &= \psi\left(G\left(\frac{1}{2}\text{Sin}^2\vartheta, \frac{1}{4}\text{Sin}^2\nu, \frac{1}{4}\text{Sin}^2\rho\right)\right) \\
 &= \psi\left(\max\left\{\frac{1}{2}\text{Sin}^2\vartheta, \frac{1}{4}\text{Sin}^2\nu, \frac{1}{4}\text{Sin}^2\rho\right\}\right) \\
 &= \frac{\max\left\{\frac{1}{2}\text{Sin}^2\vartheta, \frac{1}{4}\text{Sin}^2\nu, \frac{1}{4}\text{Sin}^2\rho\right\}}{1 + \max\left\{\frac{1}{2}\text{Sin}^2\vartheta, \frac{1}{4}\text{Sin}^2\nu, \frac{1}{4}\text{Sin}^2\rho\right\}} \\
 &= \frac{\max\left\{\text{Sin}^2\vartheta, \frac{1}{2}\text{Sin}^2\nu, \frac{1}{2}\text{Sin}^2\rho\right\}}{2 + \max\left\{\text{Sin}^2\vartheta, \frac{1}{2}\text{Sin}^2\nu, \frac{1}{2}\text{Sin}^2\rho\right\}} \\
 &\leq \frac{4}{5} \left(\frac{\max\{\text{Sin}^2\vartheta, \text{Sin}^2\nu, \text{Sin}^2\rho\}}{1 + \max\{\text{Sin}^2\vartheta, \text{Sin}^2\nu, \text{Sin}^2\rho\}} \right) \\
 &\leq \frac{4}{5} \left(\frac{\max\{\vartheta, \nu, \rho\}}{1 + \max\{\vartheta, \nu, \rho\}} \right) = \frac{4}{5} \psi(G(\vartheta, \nu, \rho)) \\
 &\leq \max\left\{\frac{4}{5}\psi(G(\vartheta, \nu, \rho)), \frac{4}{5}\psi(G(\vartheta, H\vartheta, H\vartheta)), \right. \\
 &\quad \frac{4}{5}\psi(G(\nu, J\nu, J\nu)), \frac{4}{5}\psi(G(\rho, J\rho, J\rho)), \frac{4}{5}\psi(G(\nu, H\rho, H\rho)), \\
 &\quad \frac{4}{5}\psi(G(\rho, H\vartheta, H\vartheta)), \frac{4}{5}\psi(G(\vartheta, J\vartheta, J\vartheta)), \frac{4}{15}\psi(G(\vartheta, J\nu, J\nu)), \\
 &\quad \left. \frac{4}{15}\psi(G(\nu, J\rho, J\rho)), \frac{4}{15}\psi(G(\rho, J\vartheta, J\vartheta))\right\}
 \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
 \psi(G(J\vartheta, H\nu, H\rho)) &\leq \max\left\{\frac{4}{5}\psi(G(\vartheta, \nu, \rho)), \frac{4}{5}\psi(G(\vartheta, J\vartheta, J\vartheta)), \frac{4}{5}\psi(G(\nu, H\vartheta, H\vartheta)), \right. \\
 &\quad \frac{4}{5}\psi(G(\nu, J\vartheta, J\vartheta)), \frac{4}{5}\psi(G(\nu, H\rho, H\rho)), \frac{4}{5}\psi(G(\rho, H\vartheta, H\vartheta)), \\
 &\quad \frac{4}{5}\psi(G(\rho, J\vartheta, J\vartheta)), \frac{4}{15}\psi(G(\vartheta, H\nu, H\nu)), \frac{4}{15}\psi(G(\nu, H\rho, H\rho)), \\
 &\quad \left. \frac{4}{15}\psi(G(\rho, J\vartheta, J\vartheta))\right\}
 \end{aligned}$$

Hence, H and J Satisfy definition 3.4 for $k = \frac{4}{5}$ Therefore, H, J Satisfy all the conditions of theorem 3.6 . Therefore H and J have a common fixed point.

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