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An In-Depth Exploration of The Properties and Theoretical Foundations of Neutrosophic Generalized Semipreclosed Sets in Neutrosophic Topological Spaces

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Abstract:

This paper goes deeply into various intrinsic characteristics of Neutrosophic generalized semipre closed sets. By meticulously investigating these sets, we hope to identify their essential traits and behaviors within the larger context of Neutrosophic set theory. Furthermore, our research looks at the complex linkages and interactions between Neutrosophic generalized semipre closed sets and other forms of Neutrosophic sets. Through this comparative research, we hope to highlight the links and distinctions that exist across these diverse types of Neutrosophic sets, thus contributing to a more thorough understanding of their respective functions and applications in the area.

Keywords: Neutrosophic subset, Neutrosophic topological space, Neutrosophic interior, Neutrosophic closure.

2000 AMS Subject Classification: 54A4, 08A72.

A. Introduction:

The concept of generalized closed sets in topology was notably advanced by Levine N., while Palaniyappan N. and Rao K.C. made significant contributions to the understanding of regular generalized closed sets. In the realm of Neutrosophic sets and Neutrosophic topological spaces, A.A. Salama and S.A. Alblowi have provided substantial insights.

Building on this extensive body of work, we have generalized the concept of sets to Neutrosophic topological spaces. In this paper, we present several interesting theorems and results on Neutrosophic generalized semipreclosed sets, contributing to the ongoing development and understanding of Neutrosophic set theory and its applications

The concept of a fuzzy subset was first introduced and thoroughly studied by L.A. Zadeh [15] in 1965, marking a significant milestone in the field of mathematical sciences. Since then, research activities in this area and its related domains have expanded, finding applications across various branches of science and engineering. In 1986, K. Atanassov introduced the concept of the intuitionistic fuzzy set, further enriching the fuzzy set theory. This foundational work has since been extended by numerous authors. The concept of Neutrosophic set, introduced by F. Smarandache [12], [13], serves as a mathematical tool for handling problems involving imprecise, indeterminate, and inconsistent data.

Our research in this paper has been motivated by several key works in the field. In 1968, C.L. Chang [4] introduced and studied fuzzy topological spaces, which generalize traditional topological spaces. This pioneering work has inspired many researchers to further develop the theory of fuzzy topological spaces. Among them, Andrijevic [1] introduced semipreclosed sets, and Dontchev [5] extended this concept to generalized semipreclosed sets in general topology. Subsequently, Saraf and Khanna [14] adapted these sets to fuzzy topological spaces, broadening their applicability.

Further contributions include the work of Tapas Kumar Mondal and S.K. Samantha [9], who introduced the topology of interval-valued fuzzy sets, and Bhattacharya B. and Lahiri B.K. [3], who developed the concept of semigeneralized closed sets in topology. Ganguly S. and Saha S. [6] explored fuzzy semipreopen sets in fuzzy topological spaces, while Indira R. et al. [7] investigated interval-valued fuzzy rw-closed and interval-valued fuzzy rw-open sets in interval-valued fuzzy topological spaces.

The concept of generalized closed sets in topology was notably advanced by Levine N. [8], while Palaniyappan N. and Rao K.C. [10] made significant contributions to the understanding of regular generalized closed sets. In the realm of Neutrosophic sets and Neutrosophic topological spaces, A.A. Salama and S.A. Alblowi [11] have provided substantial insights.

Building on this extensive body of work, we have generalized the concept of sets to Neutrosophic topological spaces. In this paper, we present several interesting theorems and results on Neutrosophic generalized semipreclosed sets, contributing to the ongoing development and understanding of Neutrosophic set theory and its applications.

B. Motivation:

The motivation behind this paper lies in the progressive evolution of mathematical concepts, particularly in the realm of topology and set theory. Inspired by seminal works by Levine N., Palaniyappan N., Rao K.C., A.A. Salama, and S.A. Alblowi, we embark on a journey to generalize the notion of sets to neutrosophic topological spaces.

Drawing from the foundational research of luminaries like L.A. Zadeh and K. Atanassov, who introduced fuzzy and intuitionistic fuzzy sets respectively, we recognize the significance of these frameworks in addressing uncertainties inherent in real-world data. Building upon the pioneering efforts of C.L. Chang, Andrijevic, Dontchev, Saraf, Khanna, Mondal, Samantha, Bhattacharya, Lahiri, Ganguly, Saha, Indira, and others in extending fuzzy set theory to various topological spaces, we aspire to expand the frontiers of knowledge in neutrosophic set theory.

Our motivation is to contribute to the ongoing dialogue surrounding the theoretical foundations and practical applications of neutrosophic sets. By presenting novel theorems and results on neutrosophic generalized semipreclosed sets, we aim to enrich the understanding of neutrosophic set theory and its potential impact across diverse disciplines. Through our research, we hope to inspire further exploration and innovation in the burgeoning field of neutrosophic topology.

1.Preliminaries:

1.1 Definition:[8] Let X be any nonempty set. A mapping $A: X \to [0,1]$ is called a fuzzy subset (briefly, FSS) of X.

1.2 Definition: A Intuitionistic fuzzy subset (**IFS**) A of an Universal set X is defined as an object of the form A = { ⟨ x, $\mu_A(x)$, $\vartheta_A(x)$ ⟩ / x∈X }, where μ_A : X→[0, 1] and ϑ_A : X →[0, 1] define the degree of membership and the degree of non-membership of the element x in X respectively and for every x in X satisfying $0 \le \mu_A(x) + \vartheta_A(x) \le 1$.

1.3 Definition: A **Neutrosophic subset** (**NSS**) $\bar{A}$ of a set X is defined as an object of the form $\bar{A}$ = {⟨ x, $\mu_{\bar{A}}(x), \vartheta_{\bar{A}}(x), \gamma_{\bar{A}}(x)$⟩ / x∈X}, where $\mu_{\bar{A}}$:X→[0, 1] and $\vartheta_{\bar{A}}$:X→[0, 1] and $\gamma_{\bar{A}}$: X →[0, 1] define the degree of membership, degree of indeterminacy and the degree of non-membership of the element x in X respectively.

1.4 Definition: Let $\bar{A}$ and $\bar{B}$ be any two Neutrosophic subsets of a set X. We define the following relations and operations:

(i) $\bar{A} \subseteq \bar{B}$ if and only if $\mu_{\bar{A}}(x) \le \mu_{\bar{B}}(x)$ and $\vartheta_{\bar{A}}(x) \le \vartheta_{\bar{B}}(x)$ and $\gamma_{\bar{B}}(x) \le \gamma_{\bar{A}}(x)$ for all x in X.

(ii) $\bar{A} = \bar{B}$ if and only if $\mu_{\bar{A}}(x) = \mu_{\bar{B}}(x)$ and $\vartheta_{\bar{A}}(x) = \vartheta_{\bar{B}}(x)$ and $\gamma_{\bar{A}}(x) = \gamma_{\bar{B}}(x)$ for all x in X.

(iii) $(\bar{A})^c$ = { ⟨ x, $\gamma_{\bar{A}}(x), 1-\vartheta_{\bar{A}}(x), \mu_{\bar{A}}(x)$ ⟩ / x∈X }.

(iv) $\bar{A} \cap \bar{B}$ = { ⟨ x, min{ $\mu_{\bar{A}}(x)$, $\mu_{\bar{B}}(x)$ }, min{ $\vartheta_{\bar{A}}(x), \vartheta_{\bar{B}}(x)$ }, max { $\gamma_{\bar{A}}(x), \gamma_{\bar{B}}(x)$} ⟩ / x∈X }.

(v) $\bar{A} \cup \bar{B}$ = { ⟨ x, max{ $\mu_{\bar{A}}(x), \mu_{\bar{B}}(x)$ }, max{ $\vartheta_{\bar{A}}(x), \vartheta_{\bar{B}}(x)$ }, min{ $\gamma_{\bar{A}}(x), \gamma_{\bar{B}}(x)$ }⟩

/ x∈X }.

(vi) $\bar{0} = 0_M$ = { (a, 0, 0, 1) / a∈K } and $\bar{1} = 1_M$ = { (a, 1, 1, 0) / a∈K }.

1.5 Definition[8]: Let X be a set and ℑ be a family of Neutrosophic subsets of X. The family ℑ is called an Neutrosophic topology (NST) on X if ℑ satisfies the following axioms

(i) $\bar{0}, \bar{1} \in \Im$ (ii) If { $\bar{A}_i$; i∈I } ⊆ ℑ, then $\bigcup_{i\in I} \bar{A}_i \in \Im$

(iii) If $\bar{A}_1, \bar{A}_2, \bar{A}_3, \ldots \bar{A}_n \in \Im$, then $\bigcap_{i=1}^{i=n} \bar{A}_i \in \Im$.

The pair (X, ℑ) is called an Neutrosophic topological space (NSTS). The members of ℑ are called Neutrosophic open sets (NSOS) in X. A Neutrosophic subset $\bar{A}$ in X is said to be Neutrosophic closed set (NSCS) in X if and only if $(\bar{A})^c$ is a NSOS in X.

1.6 Definition: Let (X, ℑ) be an NSTS and $\bar{A}$ be an NSS in X. Then the Neutrosophic interior and Neutrosophic closure are defined by $nsint(\bar{A}) = \cup \{ \bar{G} : \bar{G}$ is an NSOS in X and $\bar{G} \subseteq \bar{A} \}$, $nscl(\bar{A}) = \cap\{ \bar{K} : \bar{K}$ is an NSCS in X and $\bar{A} \subseteq \bar{K} \}$. For any NSS $\bar{A}$ in (X, ℑ), we have $nscl(\bar{A}^c) = (nsint(\bar{A}))^c$ and $nsint(\bar{A}^c) = (nscl(\bar{A}))^c$.

1.7 Definition: An NSS $\bar{A}$ of an NSTS (X, $\Im$) is said to be a

(i) Neutrosophic regular closed set (NSRCS for short) if $\bar{A} = nscl\left(nsint(\bar{A})\right)$

(ii) Neutrosophic semiclosed set (NSSCS for short) if $nsint\left(nscl(\bar{A})\right) \subseteq \bar{A}$

(iii) Neutrosophic preclosed set (NSPCS for short) if $nscl\left(nsint(\bar{A})\right) \subseteq \bar{A}$

(iv) Neutrosophic α closed set (NS**α**CS for short) if $nscl\left(nsint\left(nscl(\bar{A})\right)\right) \subseteq \bar{A}$

(v) Neutrosophic β closed set (NS**β**CS for short) if $nsint\left(nscl\left(nsint(\bar{A})\right)\right) \subseteq \bar{A}$.

1.8 Definition: An NSS $\bar{A}$ of an NSTS (X, $\Im$) is said to be an

(i) Neutrosophic generalized closed set (NSGCS for short) if $nscl(\bar{A}) = \bar{U}$, whenever $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ is an NSOS

(ii) Neutrosophic regular generalized closed set (NSRGCS for short) if $nscl(\bar{A}) \subseteq \bar{U}$, whenever $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ is an NSROS.

1.9 Definition: A NSS $\bar{A}$ of an NSTS (X, $\Im$) is said to be an

(i) Neutrosophic semipreclosed set (NSSPCS for short) if there exists an NSPCS $\bar{B}$ such that $nsint(\bar{B}) \subseteq \bar{A} \subseteq \bar{B}$

(ii) Neutrosophic semipreopen set (NSSPOS for short) if there exists an NSPOS $\bar{B}$ such that $\bar{B} \subseteq \bar{A} \subseteq nscl(\bar{B})$.

1.10 Definition: Two NSSs $\bar{A}$ and $\bar{B}$ are said to be not q- coincident if and only if $\bar{A} \subseteq \bar{B}^{c}$.

1.11 Definition: Let $\bar{A}$ be an NSS in an NSTS (X, $\Im$). Then the Neutrosophic semipre interior of $\bar{A}$ ($nsspint(\bar{A})$ for short) and the Neutrosophic semipre closure of $\bar{A}$ ($nsspcl(\bar{A})$ for short) are defined by $nsspint(\bar{A}) = \cup \{\bar{G} : \bar{G}$ is an NSSPOS in X and $\bar{G} \subseteq \bar{A}\}$, $nsspcl(\bar{A}) = \cap \{\bar{K} : \bar{K}$ is a NSSPCS in X and $\bar{A} \subseteq \bar{K}\}$. For any NSS $\bar{A}$ in $(X, \Im)$, we have $nsspcl(\bar{A}^{c}) = (nsspint(\bar{A}))^{c}$ and $nsspint(\bar{A}^{c}) = (nsspcl(\bar{A}))^{c}$.

1.12 Definition: A NSS $\bar{A}$ in NSTS (X, $\Im$) is said to be a Neutrosophic generalized semipreclosed set (NSGSPCS for short) if $nsspcl(\bar{A}) \subseteq \bar{U}$ whenever $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ is an NSOS in (X,$\Im$).

1.13 Example: Let $X = \{a, b\}$ and $\bar{G} = \{\langle a, 0.5, 0.6, 0.5\rangle, \langle b, 0.4, 0.5, 0.6\rangle\}$. Then $\Im = \{\bar{0}, \bar{G}, \bar{1}\}$is an NST on X and the NSS $\bar{A} = \{\langle a, 0.4, 0.4, 0.6\rangle, \langle b, 0.2, 0.3, 0.7\rangle\}$ is a NSGSPCS in (X,$\Im$).

1.14 Definition: Let $\alpha, \beta, \chi \in [0,1]$. A Neutrosophic point (NSP for short), written as $\bar{p}_{\ (\alpha,\beta,\chi)}$ is defined to be an NSS of X is given by

$$\bar{p}_{\ (\alpha,\beta,\chi)}(x) = \begin{cases} (\alpha,\beta,\chi) & \text{if} \quad \text{x = p} \\ (0,0,0\ ) & \text{otherwise.} \end{cases}$$

2. Some properties:

2.1 Theorem: In an NSTS (X, $\Im$), each NSCS is an NSGSPCS in (X, $\Im$).

Proof: Let $\bar{A}$ in (X,$\Im$) be an NSCS. Let us assume that in (X, $\Im$) $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ is an NSOS. According to hypothesis, therefore $nsspcl(\bar{A}\ ) \subseteq nscl(\bar{A}\ ) = \bar{A} \subseteq \bar{U}$. Thus, in (X, $\Im$), $\bar{A}$ is an NSGSPC

2.2 Remark: The following example shows that the converse of the preceding theorem need not be true.

llustration: Let X= {a, b} and $\bar{G} = \{ \langle\ a, 0.5, 0.6,\ 0.5\rangle, \langle\ b, 0.4, 0.5, 0.6\ \rangle \}$ be the example. Then, on X, $\Im = \{ \bar{0}, \bar{G}\ , \bar{1}\}$ is an NST. An NSS in X is denoted by $\bar{A} = \{ \langle a, 0.4, 0.5, 0.6\ \rangle, \langle\ b, 0.2, 0.3, 0.7, \rangle \}$. In X, $\bar{A}$ is not an NSCS, but it is an NSGSPCS.

2.3 Theorem: Each NSGSPCS in (X, $\Im$) is an NSRCS in the NSTS.

Proof: Theorem 2.1 makes it clear that every NSRCS is an NSCS.

2.4 Remark: As the following example illustrates, the above theorem's converse need not be true.

llustration: For illustration, let X= {a, b} and $\bar{G} = \{ \langle\ a, 0.4, 0.8, 0\ \rangle, \langle\ b, 0.3, 0.6, 0\ \rangle \}$ be the example. Then, on X, $\Im = \{\bar{0}, \bar{G}\ , \bar{1}\}$ is an NST. Assume that an NSS in X is

$\bar{A} = \{ \langle\ a, 0.3, 0.6, 0\ \rangle, \langle\ b, 0.2, 0.4, 0\ \rangle \}$. In X, $\bar{A}$ is therefore an NSGSPCS but not an NSRCS.

2.5 Theorem: A NSGSPCS in (X, $\Im$) is an NSGCS for every NSGCS in an NSTS (X, $\Im$).

Proof: Assume that $\bar{A}$ is an NSGCS in NSTS (X, $\Im$). Next, suppose that $\bar{A} \subseteq \bar{U}$ and that $\bar{U}$ is an NSO in (X, $\Im$). By hypothesis, $\bar{A}$ is an NSGSPCS in X since $nsspcl(\bar{A}\ ) \subseteq nscl(\bar{A}\ )$

And $nscl(\bar{A}\ ) \subseteq \bar{U}$

2.6 Remark: The following example shows that the converse of the preceding theorem need not be true.

llustration: Let X = {a, b} and $\bar{G} = \{ \langle\ a, 0.5, 0.6, 0.5\rangle, \langle\ b, 0.4, 0.5, 0.6\ \ \rangle \}$ be the example. Then, on X, $\Im = \{ \bar{0}, \bar{G}\ , \bar{1}\}$ is an NGT. An NSS in X is represented by $\bar{A} = \{ \langle\ a, 0.4, 0.5, 0.6\ \rangle, \langle\ b, 0.2, 0.3, 0.7\ \rangle \}$. In X, $\bar{A}$ is therefore an NSGSPCS but not an NGGCS.

2.7 Theorem: states that each NSSPCS in an NSTS (X, $\Im$) is also an NSGSPCS in (X, $\Im$). **Proof:** Let $\bar{A}$ be an NSSPCS in X as proof. Assume that in (X, $\Im$), $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ is an NSOS. We then have $nsspcl(\bar{A}\ ) \subseteq \bar{U}$ since $nsspcl(\bar{A}\ ) = \bar{A}$. Thus, in (X, $\Im$), A ∨ is an NSGSPCS.

2.8 Remark: As the following example shows, the preceding theorem's converse need not hold true.

llustration: Let X = {a, b} and $\bar{G} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.6, 0.7, 0.4 \rangle \}$ be the example. Then, on X, $\Im = \{\bar{0}, \bar{G}, \bar{1}\}$ is an NST. Consider the following NSS in X: $\bar{A} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.7, 0.8, 0.3 \rangle \}$. In X, $\bar{A}$ is not an NSSPCS, but it is an NSGSPCS.

2.9 Theorem: Each NSαCS within an NSTS (X, ℑ) corresponds to an NSGSPCS within (X, ℑ). **Proof:** Theorem 2.7 makes it clear that every NSαCS is also an NSSPCS.

2.10 Remark: The following example shows that the converse of the preceding theorem need not be true.

llustration: Let X = {a, b} and $\bar{G} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.6, 0.5, 0.4 \rangle \}$ be the example. Then, on X, $\Im = \{ \bar{0}, \bar{G}, \bar{1} \}$ is an NST. Assume that an NSS in X is $\bar{A} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.7, 0.5, 0.3 \rangle \}$. In (X, ℑ), $\bar{A}$ is therefore an NSGSPCS but not an NSαCS.

2.11 Theorem: states that any NSβCS in an NSTS (X, ℑ) is an NSGSPCS in (X, ℑ). **Proof:** Let $\bar{A}$ be an NSβCS in X as proof. Suppose that in (X, ℑ), $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ is an NSOS. We have $ns\beta cl(\bar{A}) \subseteq \bar{U}$ since $ns\beta cl(\bar{A}) = \bar{A}$. Thus, in (X, ℑ), $\bar{A}$ is an NSGSPCS.

2.12 Remark: As the following example illustrates, the above theorem's converse need not be true.

llustration: Let X = {a, b} and $\bar{G} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.6, 0.5, 0.4 \rangle \}$ be the example. Then, on X, $\Im = \{ \bar{0}, \bar{G}, \bar{1} \}$ is an NST. Assume that an NSS in X is $\bar{A} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.7, 0.5, 0.3 \rangle \}$. In such case, $\bar{A}$ in X is an NSGSPCS but not an NSβCS.

2.13 Theorem: Each NSGSPCS in (X, ℑ) is an NSSCS in the NSTS (X, ℑ). **Proof:** Let A ∨ in (X, ℑ) be an NSSCS. According to theorem 2.7, $\bar{A}$ is an NSGSPCS in (X, ℑ) since every NSSCS is an NSSPCS.

2.14 Remark: As the following example illustrates, the above theorem's converse need not be true.

llustration: let X = {a, b} and $\bar{G} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.6, 0.5, 0.4 \rangle \}$ be the example. Then, on X, $\Im = \{ \bar{0}, \bar{G}, \bar{1} \}$ is an NST. Assume that the NSS in X is $\bar{A} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.7, 0.5, 0.3 \rangle \}$. In X, $\bar{A}$ is an NSGSPCS, but it is not an NSSCS.

2.15 Theorem: Each NSGSPCS in (X, ℑ) is an NSPCS in an NSTS. **Proof:** Theorem 2.7 makes it clear that every NSPCS is an NSSPCS.

2.16 Remark: As the following example illustrates, the above theorem's converse need not be true.

llustration: Let X = {a, b} and $\bar{G} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.6, 0.5, 0.4 \rangle \}$ be the example. Hence, an NST on X is given by $\Im = \{ \bar{0}, \bar{G}, \bar{1} \}$. Consider the following NSS in X.

$\bar{A} = \{ \langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.7, 0.5, 0.3 \rangle \}$. In X, $\bar{A}$ is an NSGSPCS, but it is not an NSPCS.

2.17 Remark: In an NSTS (X,ℑ), the union of any two NSGSPCS is not an NSGSPCS in (X,ℑ).

llustration: X = {a, b} and $\bar{A}_1 = \{\langle a, 0.7, 0.5, 0.3 \rangle, \langle b, 0.8, 0.5, 0.2 \rangle\}$ and $\bar{A}_2 = \{\langle a, 0.6, 0.5, 0.4\rangle, \langle b, 0.7, 0.5, 0.3 \rangle\}$. should be taken into consideration. Hence, an NST on X is given by $\Im = \{\bar{0}, \bar{A}_1, \bar{A}_2, \bar{1}\}$. Assume that there are two NSSs in X, $\bar{A} = \{\langle a, 0.6, 0.5, 0.4 \rangle, \langle b, 0.4, 0.5, 0.3 \rangle\}$ and

$\bar{B} = \{\langle a, 0.4, 0.5, 0.4 \rangle, \langle b, 0.8, 0.5, 0.2 \rangle\}$

Since $\bar{A} \cup \bar{B} = \{\langle a, 0.6, 0.5, 0.4 \rangle, \langle b, 0.8, 0.5, 0.2\rangle\} \subseteq \bar{A}_1$, then $\bar{A}$ and $\bar{B}$ are NSGSPCS, but $\bar{A} \cup \bar{B}$ is not in X. $nsspcl(\bar{A} \cup \bar{B}) = \bar{1} \not\subset \bar{A}_1$.

2.18 Remark: An NSGSPCS in (X, $\Im$) is not the intersection of two NSGSPCS in an NSTS.

llustration: Consider the example, let $X = \{a, b\}$ and

$\bar{G} = \{\langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.6, 0.5, 0.4 \rangle\}$. Then $\Im = \{\bar{0}, \bar{G}, \bar{1}\}$ is an NST on X. Let $\bar{A} = \{\langle a, 0.5, 0.6, 0.5 \rangle, \langle b, 0.7, 0.5, 0.3 \rangle\}$ and

$\bar{B} = \{\langle a, 0.6, 0.5, 0.4\rangle, \langle b, 0.6, 0.5, 0.4 \rangle\}$ be NSS in X. Then $\bar{A}$ and $\bar{B}$ are NSGSPCS but $\bar{A} \cap \bar{B}$ is not an NSGSPCS in X, since

$\bar{A} \cap \bar{B} = \{\langle a, 0.5, 0.5, 0.5 \rangle, \langle b, 0.6, 0.5, 0.4 \rangle\} \subseteq \bar{G}$ but $nsspcl(\bar{A} \cap \bar{B}) = \bar{1} \not\subset \bar{G}$.

2.19 Theorem: Let (X,$\Im$) be an NSTS. Then for every $\bar{A} \in$ NSGSPC(X) and for every $\bar{B} \in$NSS(X), $\bar{A} \subseteq \bar{B} \subseteq nsspcl(\bar{A})$ implies$\bar{B} \in$ NSGSPC(X).

Proof: Let $\bar{B} \subseteq \bar{U}$ and $\bar{U}$ be a NSOS in (X,$\Im$). Then since $\bar{A} \subseteq \bar{B}, \bar{A} \subseteq \bar{U}$. By hypothesis, $\bar{B} \subseteq nsspcl(\bar{A})$.Therefore $nsspcl(\bar{B}) \subseteq nsspcl(nsspcl(\bar{A})) = nsspcl(\bar{A}) \subseteq \bar{U}$, since $\bar{A}$ is a NSGSPCS in (X,$\Im$). Hence $\bar{B} \in$ NSGSPC(X).

2.20 Theorem: If and only if $\bar{A}$ is not q- coincident $\bar{F} \Longrightarrow nsspcl(\bar{A})$ not q-coincident $\bar{F}$,for each NSCS $\bar{F}$, of X, then an NSS $\bar{A}$ of an NSTS (X,$\Im$) is an NSGSPCS in (X,$\Im$). **Proof: Necessity**: Assume that A¯is not q-coincident $\bar{F}$, and that $\bar{F}$, is an NSCS in (X,$\Im$). By definition 1.10$\bar{A} \subseteq \bar{F}^c$, where an NSOS in (X,$\Im$) is represented by$\overline{F}^c$. Then, theoretically, $nsspcl(\bar{A}) \subseteq \bar{F}^c$. Therefore, according to definition 1.10 once more $nsspcl(\bar{A})$ is not q-coincident $\bar{F}$.

Sufficiency: Assume that $\bar{A} \subseteq \bar{U}$ and that $\bar{U}$ is an NSOS in (X,$\Im$). Therefore, $\bar{A} \subseteq (\bar{U}^c)^c$and $\bar{U}^c$ are NSCSs in (X,$\Im$). According to the theory, $\bar{A}$ is not q-coincident $\bar{U}^c$. Not q-coincident $\bar{U}^c$, but rather$\Longrightarrow$ $nsspcl(\bar{A})$. Therefore, $ivifspcl(\bar{A}) \subseteq (\bar{U}^c)^c = \bar{U}$ by definition 1.10. Consequently, $nsspcl(\bar{A}) \subseteq \bar{U}$. Therefore $\bar{A}$ in (X, $\Im$) is an NSGSPCS.

2.21 Theorem: Let $(X, \Im)$ be a NSTS. Then every NSS in $(X, \Im)$ is a NSGSPCS in $(X, \Im)$ if and only if NSSPO(X) = NSSPC(X).

Proof: Necessity: Suppose that every NSS in $(X, \Im)$ is a NSGSPCS in $(X, \Im)$. Let $\bar{U} \in NSO(X)$. Then $\bar{U} \in NSSPO(X)$ and by hypothesis, $nsspcl(\bar{U}) \subseteq \bar{U} \subseteq nsspcl(\bar{U})$. This implies $nsspcl(\bar{U}) = \bar{U}$. Therefore $\bar{U} \in NSSPC(X)$. Hence $NSSPO(X) \subseteq NSSPC(X)$. Let $\bar{A} \in NSSPC(X)$. Then $\bar{A}^c \in NSSPO(X) \subseteq NSSPC(X)$. That is $\bar{A}^c \in NSSPC(X)$. Therefore $\bar{A} \in NSSPO(X)$. Hence $NSSPC(X) \subseteq NSSPO(X)$. Thus $NSSPO(X) = NSSPC(X)$.

Sufficiency: Suppose that $NSSPO(X) = NSSPC(X)$. Let $\bar{A} \subseteq \bar{U}$ and $\bar{U}$ be a NSOS in $(X, \Im)$. Then $\bar{U} \in NSSPO(X)$ and $nsspcl(\bar{A}) \subseteq nsspcl(\bar{U}) = \bar{U}$, since $\bar{U} \in$ NSSPC(X), by hypothesis. Therefore $\bar{A}$ is an NSGSPCS in X.

2.22 Theorem: If $\bar{A}$ is a NSOS and a NSGSPCS in $(X, \Im)$, then $\bar{A}$ is a NSSPCS in $(X, \Im)$.

Proof: Since $\bar{A} \subseteq \bar{A}$ and $\bar{A}$ is a NSOS in $(X, \Im)$, by hypothesis, $nsspcl(\bar{A}) \subseteq \bar{A}$. But $\bar{A} \subseteq nsspcl(\bar{A})$. Therefore $nsspcl(\bar{A}) = \bar{A}$. Hence $\bar{A}$ is a NSSPCS in $(X, \Im)$.

2.23 Theorem: Let $\bar{A}$ be a NSGSPCS in $(X, \Im)$ and $\bar{p}_{(\alpha,\beta,X)}$ be an NSP in X such that $\bar{p}_{(\alpha,\beta,X)} q\, nsspcl(\bar{A})$. Then $nscl(\bar{p}_{(\alpha,\beta,X)})\, q\, \bar{A}$.

Proof: Let $\bar{A}$ be an NSGSPCS in $(X, \Im)$ and let $\bar{p}_{(\alpha,\beta,X)} q\, nsspcl(\bar{A})$. If $nscl(\bar{p}_{(\alpha,\beta,X)})$ not q-coincident $\bar{A}$, then by definition 1.10, $\bar{A} \subseteq \left(nscl(\bar{p}_{(\alpha,\beta,X)})\right)^c$, where $\left(nscl(\bar{p}_{(\alpha,\beta,X)})\right)^c$ is a NSOS in $(X, \Im)$. Then by hypothesis, $nsspcl(\bar{A}) \subseteq \left(nscl(\bar{p}_{(\alpha,\beta,X)})\right)^c \subseteq \left(\bar{p}_{(\alpha,\beta,X)}\right)^c$. Therefore by definition 1.10, $\bar{p}_{(\alpha,\beta,X)}$ not q-coincident $nsspcl(\bar{A})$, which is a contradiction to the hypothesis. Hence $nscl(\bar{p}_{(\alpha,\beta,X)})\, q\bar{A}$.

2.24 Theorem: For any NSS $\bar{A}$ in a NSTS $(X, \Im)$, the following conditions are equivalent:

(i) $\bar{A}$ is a NSOS and a NSGSPCS in $(X, \Im)$

(ii) $\bar{A}$ is a NSROS in $(X, \Im)$.

Proof: (i) $\Rightarrow (ii)$ Let $\bar{A}$ be a NSOS and a NSGSPCS in a NSTS $(X, \Im)$. Then $nsspcl(\bar{A}) \subseteq \bar{A}$. Since $nsspcl(\bar{A})$ is NSSPCS, by definition 1.7, there exists a NSPCS $\bar{B}$ such that $nsint(\bar{B}) \subseteq nsspcl(\bar{A}) \subseteq \bar{B}$ and $nscl(nsint(\bar{B})) \subseteq \bar{B}$.

Now $nsint\left(nscl\left(nsint\left(nsspcl(\bar{A})\right)\right)\right) \subseteq nsint(nscl(nsint(\bar{B}))) \subseteq nsint(\bar{B}) \subseteq nsspcl(\bar{A})$.

Now $nsint(nscl(nsint(\bar{A}))) \subseteq nsint(nscl(nsint(nsspcl(\bar{A})))) \subseteq nsspcl(\bar{A})$. Therefore $\bar{A} \cup nsint(nscl(nsint(\bar{A}))) \subseteq nsspcl(\bar{A}) \subseteq \bar{A}$. This implies that $nsint(nscl(nsint(\bar{A}))) \subseteq \bar{A}$. Since $\bar{A}$ is a NSOS, $nsint(\bar{A}) = \bar{A}$. Therefore $nsint(nscl(\bar{A}) \subseteq \bar{A}$. Since $\bar{A}$ is an NSOS, it is a NSPOS. Hence $\bar{A} \subseteq nsint(nscl(\bar{A}))$. Therefore $\bar{A} = nsint(nscl(\bar{A})$. Hence $\bar{A}$ is a NSROS in $(X, \Im)$.

(ii)⇒ (ii) Let $\bar{A}$ be a NSROS in $(X, \Im)$. Therefore $\bar{A} = nsint(nscl(\bar{A}))$. Since every NSROS is a NSOS, $\bar{A}$ is a NSOS and $\bar{A} \subseteq \bar{A}$. This implies that $nsint(nscl(\bar{A})) \subseteq \bar{A}$. That is $nsint\left(nscl\left(nsint(\bar{A})\right)\right) = nsint(nscl(\bar{A})) \subseteq \bar{A}$. Thus $\bar{A}$ is a NSβCS. Hence by theorem 2.11, $\bar{A}$ is a NSGSPCS in $(X, \Im)$.

2.25 Theorem: For a NSOS $\bar{A}$ in $(X, \Im)$, the following conditions are equivalent:

(i) $\bar{A}$ is a NSCS in $(X, \Im)$,

(ii) $\bar{A}$ is a NSGSPCS and a NSQ-set in $(X, \Im)$.

Proof: $(i) \Longrightarrow (ii)$ Since $\bar{A}$ is a NSCS, it is a NSGSPCS in $(X, \Im)$. Now $nsint(nscl(\bar{A})) = nsint(\bar{A}) = \bar{A} = nscl(\bar{A}) = nscl(nsint(\bar{A}))$,by hyp-othesis. Hence $\bar{A}$ is a NSQ-set in $(X, \Im)$.

$(ii) \Longrightarrow (i)$ Since $\bar{A}$ is a NSOS and a NSGSPCS in $(X, \Im)$, by theorem 2.24, $\bar{A}$ is a NSROS in $(X, \Im)$. Therefore $\bar{A} = nsint(nscl(\bar{A})) = nscl(nsint(\bar{A})) = nscl(\bar{A})$, by hypothesis. Hence $\bar{A}$ is a NSCS in $(X, \Im)$.

2.26 Theorem: Let $(X, \Im)$ be a NSTS. Then for every $\bar{A}$ ∈NSSPC(X) and for every NSS $\bar{B}$ in X, $nsint(\bar{A}) \subseteq \bar{B} \subseteq \bar{A}$ implies $\bar{B} \in NSGSPC(X)$.

Proof: Let $\bar{A}$ be a NSSPCS in X. Then by definition 1.7, there exists a NSPCS, say $\bar{C}$ such that $nsint(\bar{C}) \subseteq \bar{A} \subseteq \bar{C}$. By hypothesis, $\bar{B} \subseteq \bar{A}$. Therefore $\bar{B} \subseteq \bar{C}$. Since $nsint(\bar{C}) \subseteq \bar{A}$, $nsint(\bar{C}) \subseteq nsint(\bar{A})$ and $nsint(\bar{C}) \subseteq \bar{B}$. Thus $nsint(\bar{C}) \subseteq \bar{B} \subseteq \bar{C}$ and by definition 1.7, $\bar{B} \in NSSPC(X)$. Hence by Theorem 2.7, $\bar{B} \in NSGSPC(X)$.

Conclusion :

This study examines the features of Neutrosophic Generalized Semipreclosed Sets in Neutrosophic Topological Space. By diving into these sets' traits and behaviors, the study hopes to provide the groundwork for future research and advancement in this sector. The theorems offered herein are not only useful for advancing theoretical understanding, but they also serve as the foundation for practical applications.

The detailed analysis reveals how Neutrosophic Generalized Semipreclosed Sets can be leveraged to extend various aspects of topological theory. Specifically, the study explores the potential to apply these sets to functions, including open maps, closed maps, and homeomorphisms. This extension of the theory offers a robust framework for investigating the continuity and compatibility of functions within Neutrosophic Topological Spaces.

Through this work, the utility of Neutrosophic Generalized Semipreclosed Sets in broadening the scope of topological studies is demonstrated, paving the way for new discoveries and applications in the realm of Neutrosophic topology. By establishing these foundational principles, the aim is to inspire further research that will continue to expand and refine the understanding of Neutrosophic topological structures and their practical implications.

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