Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 8s (2024)

On Discrete Harmonic Distributions related with Harmonic Mean Random Variable

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Article History:
Received: *20-05-2024*
Revised: *03-07-2024*
Accepted: *21-07-2024*

Abstract:
The main purpose of this paper is to introduce the concept of harmonically distributed discrete random variable. We define probability mass function such as harmonic mass function, exponential harmonic mass function, natural logarithmic harmonic mass function and their associated cumulative distribution functions. Existence the distribution is shown with some examples. Finally harmonic mass function is used to solve a run-time problem and a capacitance problem as an application of the distribution in the field of electrical engineering.

Keywords: Harmonic mean random variable, Harmonic mass function, Exponential harmonic mass function, Natural logarithmic harmonic mass function.

AMS 2010 Subject Classification: 62Exx

1. Introduction and Preliminary

In the theory of probability and statistics, the continuous harmonic distribution or harmonic law was studied by Etienne Halphen (1941) as a special case of generalized inverse Gaussian distribution family with $\gamma = 0$. The Geometry of statistical notions or the connections between Geometry and Statistics has existed from the beginning of both disciplines (Adler C. F. 1958 [1], and Fisher, J. B, 1978 [4]). The connection between arithmetic, geometric, and harmonic mean can be studied by the same approach of Geometry of Statistics. You may observe that some other investigations are based on the statistical properties of geometric notions (Ahangar R. R. 2010 [2], Hilbert, D. 1902 [5], Pearson, R. 2011 [10], Saville, J. D., and Wood R. G., (1977) [11]. Arithmetic mean as a great mode of investigation in statistics is used in many natural phenomena, but many researches prefer to use either geometric or harmonic mean in their investigations, (MacCluer, C. R. 2000 [7]). The arithmetic mean and the median may be the most popular and convenient measures that financial analysts use for their valuations. It is believed by many financial organizations and bankers that the Harmonic Mean provides better information for reasonable measure in investment strategy (Mathews and Gilbert- 2006, Meyer D., 1970 [8]). In slowly-decaying distributions, the harmonic mean often turns out to be a much better characterization than the arithmetic mean, which is a reciprocal transformation generally not even well-defined theoretically for these distributions (Pearson, R. 2011 [10]). Much research has been done and computational tools designed these days that are equipped to change the mode of computation in either arithmetic, geometric, or harmonic sense. The continuous harmonic density function, transformation (horizontal shift) of harmonic density function, harmonic density with stretch or contraction, and general harmonic function are studied by (Ahangar R. 2013

[3]. Later he developed the algorithm of continuous harmonic modelling and continuous harmonic regression.

The total capacitor (C_T) in an series circuit of n capacitance $C_i, i = 1,2,\cdots,n$ has a harmonic relation and the total resistance (R_T) in an parallel circuit of n resistance $R_i, i = 1,2,\cdots,n$ has a harmonic relation defined by

circuits	R_T	C_T
Series	$R_1 + R_2 + \cdots + R_n$	$\left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right)^{-1}$
Parallel	$\left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}\right)^{-1}$	$C_1 + C_2 + \cdots + C_n$

In order to reduce the voltage drop in the series circuit, we need to reduce the C_T and R_T in parallel circuit. According to recent study the drops C_T or R_T in the line is admissible upto 5% either in series or parallel respectively. Therefore a problem is there to find the probability of joining of $(k+1)^{th}$ capacitor after k^{th} capacitors in the series line shouldn't exceed 0.05. This concept encourages us to develop the discrete harmonic distribution.

To the best of the author's knowledge, no direct investigation exists to introduce discrete harmonic probability mass functions, moments, central moments and its applications. In this paper we develop the random variable for some types of discrete harmonic distribution and their cumulative distribution mass functions.

2. Harmonic Mean RV (HMRV) and Discrete Harmonic Distribution

For any real $(n+1)$ real values $a_0, a_1, \cdots, a_n$, harmonic convexity of $A = \{a_0, a_1, \cdots, a_n\}$ is

$$HC(A) = \left[\sum_{i=0}^{n} (\alpha_i / a_i)\right]^{-1} = \frac{a_0 a_1 \cdots a_n}{\sum_{i=0}^{n} \alpha_i a_0 a_1 \cdots a_{i-1} a_{i+1} \cdots a_n}$$

satisfying $\sum_{i=0}^{n} \alpha_i = 1$. Taking $\alpha_i = \frac{1}{n+1}$, we have

$$HC(A) = \left[\sum_{i=0}^{n} (\alpha_i / a_i)\right]^{-1} = \left[\sum_{i=0}^{n} (\frac{1}{(n+1)a_i}\right]^{-1} = (n+1)\frac{a_0 a_1 \cdots a_n}{\sum_{i=0}^{n} a_0 a_1 \cdots a_{i-1} a_{i+1} \cdots a_n}.$$

Let X be the harmonic mean random variable consists of successive harmonic mean values of a_i's, i.e., $X = \{x_0, x_1, \cdots, x_n\}$ where

$$x_k = \left[\sum_{i=0}^{k} (1/a_i)\right]^{-1} = \frac{a_0 a_1 \cdots a_k}{\sum_{i=0}^{k} a_0 a_1 \cdots a_{i-1} a_{i+1} \cdots a_k} \qquad (2.1)$$

for $k = 0,1,\cdots,n$, then we have

$$x_0 = a_0, x_1 = \frac{a_0 a_1}{a_0 + a_1}, x_2 = \frac{a_0 a_1 a_2}{a_1 a_2 + a_0 a_2 + a_0 a_1}, \cdots, x_n = \frac{a_0 a_1 \cdots a_n}{a_1 a_2 \cdots a_n + a_0 a_2 \cdots a_n + a_0 a_2 \cdots a_{n-1}}.$$

Consider $p(x) = \frac{C}{x}$ such that $\sum_{i=0}^{n} p(x_i) = 1$ for some $C > 0$. Assume that

$$F(x_0) = p(x_0) = \frac{C}{a_0}$$

$$F(x_1) = p(x_0) + p(x_1) = C\left[\frac{1}{a_0} + \frac{a_0 + a_1}{a_0 a_1}\right] = C\frac{a_0 + 2a_1}{a_0 a_1} = C\left[\frac{1}{a_1} + \frac{2}{a_0}\right]$$

$$F(x_2) = p(x_0) + p(x_1) + p(x_2)$$
$$= C\left[\frac{1}{a_0} + \frac{a_0 + a_1}{a_0 a_1} + \frac{a_1 a_2 + a_0 a_2 + a_0 a_1}{a_0 a_1 a_2}\right]$$
$$= C\frac{a_0 a_1 + 2a_0 a_2 + 3a_1 a_2}{a_0 a_1 a_2} = C\left[\frac{1}{a_2} + \frac{2}{a_1} + \frac{3}{a_0}\right].$$

In general, for $k = 0,1,\cdots,n$, we have

$$F(x_k) = C\left[\frac{1}{a_k} + \frac{2}{a_{k-1}} + \frac{3}{a_{k-2}} + \cdots + \frac{k-1}{a_2} + \frac{k}{a_1} + \frac{k+1}{a_0}\right] = C\sum_{i=0}^{k} \frac{i+1}{a_{k-i}}.$$

Now

$$1 = p(x_0) + p(x_1) + p(x_2) + \cdots + p(x_n)$$
$$= C\left[\frac{1}{a_0} + \frac{a_0 + a_1}{a_0 a_1} + \frac{a_1 a_2 \cdots a_n + a_0 a_2 \cdots a_n + a_0 a_2 \cdots a_{n-1}}{a_0 a_1 \cdots a_n}\right]$$
$$= C\left[\frac{1}{a_n} + \frac{2}{a_{n-1}} + \frac{3}{a_{n-2}} + \cdots + \frac{n-1}{a_2} + \frac{n}{a_1} + \frac{n+1}{a_0}\right] = C\sum_{i=0}^{n} \frac{i+1}{a_{n-i}}$$
$$\Rightarrow C = \left(\sum_{i=0}^{n} \frac{i+1}{a_{n-i}}\right)^{-1} \qquad (2.2).$$

Hence

$$p(x) = \left(\sum_{i=0}^{n} \frac{i+1}{a_{n-i}}\right)^{-1} \frac{1}{x}, x = x_0, x_1, \cdots, x_n. \qquad (2.3)$$

is a special type of harmonic pmf in the harmonic set.

Example 2.1 If a runner runs as per the time (in seconds) given in the time space $T = \{t_0, t_1, \cdots, t_n\}$, $t_i \neq 0$ with relative speed has a harmonic random variable $X_T = \{x: x = x(t), t \in T\} = \{x_0, x_1, \cdots, x_n\}$ obtained by equation (2.1). The runner has a relative distance covered in time $t \in A$

has the harmonic pmf measured by (2.3). Thus for $a = 2,3,5$, the equation (2.1) gives harmonic *rv* X_T with images $x = 2, \frac{6}{5}, \frac{30}{31}$. Since equation (2.2) gives $C = \frac{30}{71}$, we have the harmonic pmf $p(x) = \frac{30}{71x}$, $x = 2$, $\frac{6}{5}$, $\frac{30}{31}$ obtained by (2.3) satisfies $\sum_{i=0}^{n} p(x_i) = 1$. Hence the random variable X_T has the pmf values are

$$p(2) = \frac{15}{71}, p(6/5) = \frac{25}{71}, \text{and } p(30/31) = \frac{31}{71}$$

and cdf values are

$$F(2) = \frac{15}{71}, F(6/5) = \frac{40}{71}, \text{and } F(30/31) = 1.$$

Example 2.2 In a series with three capacitors $C_1 = 2\mu F$, $C_2 = 4\mu F$ and $C_3 = 5\mu F$ are connected, then *rv* X_C measures total capacitance after connecting to the next capacitor, then $X_C = \{x_1, x_2, x_3\}$ where $x_1 = 2$, $x_2 = \left(\frac{1}{2} + \frac{1}{4}\right)^{-1} = 4/3$ and $x_3 = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{5}\right)^{-1} = 20/19$. The harmonic pmf of X_C is $p(x) = \frac{5}{11x}$ for $x = 2, 4/3,\ 20/19$, i.e., $p(2) = \frac{5}{22}$, $p(4/3) = 15/44$ and $p(20/19) = 19/44$ and the cdf of X_C are $F(2) = \frac{5}{22}$, $F(4/3) = 25/44$ and $F(20/19) = 1$.

3. Discrete Exponential Harmonic Distribution

Let $\mathbb{Z}_m^n$ be the set of integers lie in between m and n for $0 \le m \le n-1$ and $K = \{x \neq 0 : x = k \exp\left(\frac{1}{k-1}\right), k \ge 1\}$ satisfying x_k.

Let X be a discrete random variable defined by $X = \{z_x \in \mathbb{R} : x = 0,1,\cdots,n\}$ and the discrete harmonic probability mass function (dhpmf) be

$$p(x; z) = \frac{C}{(n+1)z_x}, x = 0,1,\cdots,n, \text{and, } 0 \text{ otherwise,}$$

for some $C > 0$ satisfying $0 \le p(x; z) \le 1$ and $\sum_{x=0}^{n} p(x; a) = 1$. Since $0 \neq x \in K$, we obtain $x = k\exp\left(\frac{1}{k-1}\right)$, i.e., $x^k = kx$ for some $k > 0$ and $x > 0$.

Letting $z_x = a^x$ for some parameter $a > 0$, $x = 0,1,2,\cdots,n$, we obtain

$$p(x; a, n) = \frac{C}{(n+1)a^x}, x = 0,1,\cdots,n, \text{and, } 0 \text{ otherwise,}$$

Considering it as a legitimate pmf, we have unity property, i.e.,

$$1 = \sum_{x=0}^{n} p(x; a, n) = \sum_{x=0}^{n} \frac{C}{(n+1)a^x}$$

$$= \frac{C}{(n+1)}\left(1 + \frac{1}{a} + \frac{1}{a^2} + \cdots + \frac{1}{a^{n-1}} + \frac{1}{a^n}\right) = \frac{Ca^{-n}}{(n+1)}(1 + a + a^2 + \cdots + a^n)$$

implying $C = (n+1)\left(\frac{a-1}{a-a^{-n}}\right)$ for $a > 1$. Thus fixed fixed parameters $n = 2,3,\cdots$ and $a = 2,3,\cdots$, the dhpmf is obtained as

$$dh(x;a,n) = P(X = x) = \left(\frac{a-1}{a-a^{-n}}\right)\frac{1}{a^x}, x = 0,1,\cdots,n; \;\; and \text{ 0 otherwise}$$

and the corresponding discrete harmonic probability distribution function (DHPDF) is given by

$$DH(x;a,n) = P(X \leq x) = \sum_{x=0}^{n} dh(y;a) = \left(\frac{a-1}{a-a^{-n}}\right)\sum_{y=0}^{x}\frac{1}{a^y}, x \leq n$$

$$= \left(\frac{a-1}{a-a^{-n}}\right)a^{-x}\left(\frac{a^{x+1}-1}{a-1}\right) = \frac{a-a^{-x}}{a-a^{-n}} \text{ for } x = 0,1,\cdots,n.$$

We define the discrete complementary harmonic probability distribution function (*DCHPDF*) as

$$CH(x;a,n) = P(X \geq x) = 1 - P(X \leq x-1) = 1 - \frac{a-a^{-(x-1)}}{a-a^{-n}} = \frac{a^{-(x-1)}-a^{-n}}{a-a^{-n}}$$

for $x = 0,1,\cdots,n$. Thus for $a = 2$, the dhpmf is

$$dh(x;a) = \left(\frac{1}{2-2^{-n}}\right)\frac{1}{2^x}, x = 0,1,2,\cdots n; \text{ and 0 otherwise}$$

and the *DHPDF* is

$$DH(x;2,n) = \frac{2-2^{-x}}{2-2^{-n}}, x = 0,1,2,\cdots,n.$$

For $a = 3$, the dhpmf is

$$dh(x;3,3) = \left(\frac{2}{3-3^{-n}}\right)\frac{1}{3^x}, x = 0,1,2,\cdots,n; \text{ and 0 otherwise}$$

and the *DHPDF* is

$$DH(x;3,n) = \frac{3-3^{-x}}{3-3^{-n}}. x = 0,1,2,\cdots,n.$$

The accompanying table is given for $a = 2$ and $n = 3$:

$a = 2, n = 3$				
	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$dh(x;2,3)$	0.5333	0.2667	0.1333	0.0667
$DH(x;2,3)$	0.5333	0.8000	0.9333	1.0000
$a = 2, n = 3$				
	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$dh(x;3,3)$	0.6750	0.2250	0.0750	0.0250
$DH(x;3,3)$	0.6750	0.9000	0.9750	1.0000

The accompanying table is given for $a = e$ and $n = 1,2,3$:

$a = e, n = 1$				
	$x = 0$	$x = 1$		
$dh(x; e, 1)$	0.7311	0.2689		
$DH(x; e, 1)$	0.7311	1.0000		
$a = e, n = 2$				
	$x = 0$	$x = 1$	$x = 2$	
$dh(x; e, 2)$	0.6652	0.2447	0.0009	
$DH(x; e, 2)$	0.6652	0.9100	1.0000	
$a = e, n = 3$				
	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$dh(x; e, 3)$	0.6439	0.2369	0.0871	0.0321
$DH(x; e, 3)$	0.6439	0.8808	0.9679	1.0000

4. Discrete Natural Logarithmic Harmonic Distribution

For any real $x > 1$, we have $\ln x > 0$. Let $a_0 \geq a_k$ for $k = 1,2,\cdots,n$ such that the greatest integer of a_0 is at least $n + 1$. Let X be the harmonic mean random variable with images $x_k, k = 0,1,2,\cdots,n$ obtained by the relation given in (2.1). We have

$$x_0 = a_0, x_1 = \frac{a_0 a_1}{a_0 + a_1}, x_2 = \frac{a_0 a_1 a_2}{a_1 a_2 + a_0 a_2 + a_0 a_1}, \cdots,$$

$$x_n = \frac{a_0 a_1 \cdots a_n}{a_1 a_2 \cdots a_n + a_0 a_2 \cdots a_n + a_0 a_2 \cdots a_{n-1}},$$

so $x_k > 1$ for each k as $[a_0] \geq n + 1$. The logarithmic harmonic pmf of X is defined by

$$p(x; n) = \begin{cases} \frac{C}{(n+1)\ln x}, & \text{if } x = x_0, x_1, \cdots, \quad x_n; \\ 0, & \text{otherwise,} \end{cases}$$

for some $C > 0$ satisfying $0 \leq p(x; n) \leq 1$ and $\sum_{x=0}^{n} p(x; n) = 1$ where

$$\ln x_k = \ln\left(\frac{a_0 a_1 \cdots a_n}{\left(\sum_{i=0}^{k} a_0 a_1 \cdots a_{i-1} a_{i+1} \cdots a_k\right)}\right) = \sum_{i=0}^{k} \ln a_i - \ln\left(\sum_{i=0}^{k} a_0 a_1 \cdots a_{i-1} a_{i+1} \cdots a_k\right).$$

for for $k = 0,1,\cdots,n$.

$$1 = \sum_{k=0}^{n} p(y_k; n) = \frac{C}{(n+1)} \sum_{k=0}^{n} \frac{1}{\ln x_k}$$

$$= \frac{C}{(n+1)} \left[\frac{1}{\ln x_0} + \frac{1}{\ln x_1} + \frac{1}{\ln x_2} + \cdots + \frac{1}{\ln x_{n-1}} + \frac{1}{\ln x_n}\right]$$

$$\Rightarrow C = (n+1)\left[\frac{1}{\ln x_0}+\frac{1}{\ln x_1}+\frac{1}{\ln x_2}+\cdots+\frac{1}{\ln x_{n-1}}+\frac{1}{\ln x_n}\right]^{-1}.$$

Thus the logarithmic harmonic pmf of X is

$$p(x) = lh(x;n) = \begin{cases} \dfrac{k}{\ln x}, & if\ \ x = x_0, x_1, \cdots, x_n; \\ 0, & \text{otherwise,} \end{cases}$$

where

$$k = \left[\frac{1}{\ln x_0}+\frac{1}{\ln x_1}+\frac{1}{\ln x_2}+\cdots+\frac{1}{\ln x_{n-1}}+\frac{1}{\ln x_n}\right]^{-1},$$

$$\ln x_0 = \ln a_0$$

$$\ln x_1 = \ln a_0 + \ln a_1 - \ln(a_0 + a_1)$$

$$\ln x_2 = \ln a_0 + \ln a_1 + \ln a_2 - \ln(a_0a_1 + a_0a_2 + a_1a_2)$$

and so on,

$$\ln x_n = \sum_{k=0}^{n} \ln a_k - \ln\left(\sum_{k=0}^{n} a_0a_1\cdots a_{k-1}a_{k+1}\cdots a_n\right).$$

provided $x_k > 1$ for each $k = 0,1,2\cdots$ and cdf of X is $LH(x;n) = P(X \le x)$ for all $x = x_0, x_1, \cdots, x_n$ given by

$$LH(x_0;n) = p(x_0;n) = \frac{k}{\ln x_0},$$

$$LH(x_1;n) = p(x_0;n) + p(x_1;n) = k\left[\frac{1}{\ln x_0}+\frac{1}{\ln x_1}\right]$$

$$LH(x_2;n) = p(x_0;n) + p(x_1;n) + p(x_2;n) = k\left[\frac{1}{\ln x_0}+\frac{1}{\ln x_1}+\frac{1}{\ln x_2}\right]$$

$$LH(x_3;n) = p(x_0;n) + p(x_1;n) + p(x_2;n) + p(x_3;n)$$

$$= k\left[\frac{1}{\ln x_0}+\frac{1}{\ln x_1}+\frac{1}{\ln x_2}+\frac{1}{\ln x_3}\right]$$ and so on.

Example 4.1 Let $n = 4$. For $a_0 = 5, a_1 = 5.1, a_2 = 5.2, a_3 = 5.3, a_4 = 5.4$, we have $x_0 = 5, x_1 = 2.52475, x_2 = 1.69956, x_3 = 1.28689, x_4 = 1.03923$. Therefore

$$lh(x;4) = \begin{cases} \dfrac{k}{\ln x}, & if\ \ x = x_0, x_1, \cdots, x_n; \\ 0, & \text{otherwise,} \end{cases}$$

where

$$k = \left[\frac{1}{\ln x_0} + \frac{1}{\ln x_1} + \frac{1}{\ln x_2} + \frac{1}{\ln x_3} + \frac{1}{\ln x_4}\right]^{-1} = 0.0298$$

which is approximated to 4 significant digit form. Thus

$$\ln x_0 = 1.60944; \ln x_1 = 0.92614; \ln x_2 = 0.53037; \ln x_3 = 0.25223; \ln x_4 = 0.03848,$$

implying the pmf of X is

$$p(x_0; 5) = \frac{k}{\ln x_0} = 0.01852; p(x_1; 5) = \frac{k}{\ln x_1} = 0.03218; p(x_2; 5) = \frac{k}{\ln x_2} = 0.05619$$

$$p(x_3; 5) = \frac{k}{\ln x_3} = 0.11815; p(x_4; 5) = \frac{k}{\ln x_4} = 0.77443.$$

and cdf of X is $LH(x; n) = P(X \le x)$ for all $x = x_0, x_1, \cdots, x_n$ given as follows.

$$LH(x_0; 4) = p(x_0; 4) = \frac{k}{lnx_0} = 0.01853,$$

$$LH(x_1; 4) = p(x_0; 4) + p(x_1; 4) = k\left[\frac{1}{\ln x_0} + \frac{1}{\ln x_1}\right] = 0.05073$$

$$LH(x_2; 4) = p(x_0; 4) + p(x_1; 4) + p(x_2; 4) = k\left[\frac{1}{\ln x_0} + \frac{1}{\ln x_1} + \frac{1}{\ln x_2}\right] = 0.10695$$

$$LH(x_3; 4) = p(x_0; 4) + p(x_1; 4) + p(x_2; 4) + p(x_3; 4)$$

$$= k\left[\frac{1}{\ln x_0} + \frac{1}{\ln x_1} + \frac{1}{\ln x_2} + \frac{1}{\ln x_3}\right] = 0.22518$$

$$LH(x_4; 4) = p(x_0; 4) + p(x_1; 4) + p(x_2; 4) + p(x_3; 4) + p(x_4; 4)$$

$$= k\left[\frac{1}{\ln x_0} + \frac{1}{\ln x_1} + \frac{1}{\ln x_2} + \frac{1}{\ln x_3} + \frac{1}{\ln x_4}\right] = 1.00012.$$

The distribution value at $x \ge x_4$ is approximated to $1 - \epsilon$ where $\epsilon = 0.00055$ because of truncational errors present in x_k's and pmf values in x_k's.

Future Scope

Our next aim to study the mean, variance, moment, skewness, Kurtosis, moment generating functions of the discrete harmonic mean distributed random variables with the harmonic pmf, exponential harmonic pmf, natural logarithmic harmonic pmf in future.

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