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Hydrodynamic Force Expression with Couple Stress Drag Over Sphere at Small Reynolds Numbers

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Abstract:

Introduction: The flow past a sphere has been an important problem since Stokes and the Navier Stokes (NS) equation is the basic governing equation for it. The main reason for its importance is that the NS equation is not yet solvable analytically and has been solved in the absence of either or both unsteady inertia and convective inertia. The hydrodynamic force on a spherical particle at low Reynolds numbers has been meticulously derived by Lovalenti and Brady [2] and Stokes [1], has derived the same for a couple stress fluid.

Objectives: In this problem we extend the works of Stokes [1] and Lovalenti and Brady [2] to obtain an expression for the hydrodynamic force on a couple stress flow over a rigid sphere at low Reynolds numbers.

Methods: We consider both the expressions for the hydrodynamic forces and combine them to get a new expression for the effects of couple stress fluids at low Reynolds numbers on a rigid sphere. Then the expression is validated by reproducing the results of both the authors.

Results: We consider both the extremes and obtain the plots of Stokes [1] and Lovalenti and Brady [2] and also obtain the transition from Newtonian fluid to Couple stress fluid at small Reynolds numbers. We note that the hydrodynamic force on the rigid sphere is the effect of viscous force and pressure and at zero couple stress! (an ideal situation) the fluid behaves as a Newtonian fluid with zero or constant flow when fluid is quiescent and as the couple stress parameters are introduces the force tend to grow from constant to exponential flow.

Conclusions: This work fills the gap in the literature for the expression for force on a couple stress fluid on a rigid sphere at small finite Reynolds number.

Keywords: Couple stresses, low Reynolds number, force, rigid spheres.

ϑ	Kinematic viscosity
Re_F	Amplitude of the periodic force
U_c	Characteristic particle slip velocity

$\boldsymbol{F}_{\boldsymbol{s}}^{\boldsymbol{H}_{\parallel}}(t) = -6\pi \boldsymbol{U}_{\boldsymbol{s}}.\boldsymbol{pp}$	Tangential part of Stokes' drag
$\boldsymbol{F}^{\boldsymbol{H}}$	Hydrodynamic force
a	Characteristic particle dimension
$Sl = \frac{a}{U_c}/\tau_c$	Strouhal number
τ_c	Characteristic time scale
$\overline{U_{px}}$	Mean of U_{px} and similarly for other corresponding values
$\boldsymbol{F}_{\boldsymbol{s}}^{\boldsymbol{H}}$	Pseudo steady Stokes' drag
$Re = aU_c/\vartheta$	Reynolds number
$\boldsymbol{F}_{\boldsymbol{s}}^{\boldsymbol{H}_{\perp}}(t) = -6\pi \boldsymbol{U}_{\boldsymbol{s}}.(\boldsymbol{\delta} - \boldsymbol{pp})$	Normal part of Stokes' Drag
$\boldsymbol{p} = \frac{\boldsymbol{Y}_{\boldsymbol{s}}(t) - \boldsymbol{Y}_{\boldsymbol{s}}(s)}{\lvert \boldsymbol{Y}_{\boldsymbol{s}}(t) - \boldsymbol{Y}_{\boldsymbol{s}}(s) \rvert}$	Unit vector in align with the x axis
$\boldsymbol{A}$	$A = \frac{Re}{2}\left(\frac{t-s}{ReSl}\right)^{1/2}\left(\frac{\boldsymbol{Y}_{\boldsymbol{s}}(t) - \boldsymbol{Y}_{\boldsymbol{s}}(s)}{t-s}\right)$
$\boldsymbol{U}_{\boldsymbol{p}}(t)$	Particle velocity
$\boldsymbol{U}_{\boldsymbol{s}}(t) = \boldsymbol{U}_{\boldsymbol{p}}(t) - \boldsymbol{U}^{\infty}(t)$	Stream velocity
$\boldsymbol{U}^{\infty}(t)$	Uniform velocity
$\boldsymbol{F}^{ext}$	External force
$\boldsymbol{\delta}$	idem tensor of order 2
m_p	Particle mass
$\boldsymbol{Y}_s$	Integrated particle displacement
$\boldsymbol{\alpha}$	Couple stress parameter

1. Introduction

Solutions containing long-chain polymers, biological fluids, suspensions and colloidal fluids, liquid crystals, viscoelastic materials and other complex geometry structures and the complex fluids can exhibit couple stress behaviour [3 - 7]. The entangled polymer chains introduce microscale interactions that affect the overall rheological properties of the fluid[3]. Biological fluids, such as synovial fluid in joints, can exhibit non-Newtonian behaviour with microstructure effects. Blood is a complex fluid with suspended particles, cells, and plasma constituents. The interactions between these components at the microscale can be significant. In the context of blood flow, the micro-rotations can represent the rotation of individual blood cells or the effects of cell-cell interactions. The additional terms in the constitutive equations of couple stress fluid account for the microscale contributions to the stress tensor. Understanding blood flow as a couple stress fluid can have implications for studying hemo - dynamics, especially in micro vessels and regions with complex geometries. It may provide insights into phenomena such as margination (concentration of blood cells near vessel walls) and rheological changes under pathological conditions [8-17]. The presence of cells, proteins, and other microstructures contributes to the overall rheological response. Fluids containing suspended particles or colloids interact between particles and the surrounding fluid introduce microscale effects that influence the overall flow characteristics. Liquid crystal fluids, which are composed of anisotropic molecules due to the alignment and rotation of the molecules in response to flow exhibits couple stress behaviour. The couple stress fluids exhibit more complex flow patterns and vortical structures compared to Newtonian fluids, particularly in regions with high shear rates[17]. The microstructural effects in couple stress fluids contribute to a viscoelastic response, meaning that these fluids may exhibit both viscous and elastic behaviour. There are various engineering applications of couple stress fluids [19-43] obtained by understanding the rheological behaviour of couple stress fluids, including biomedical engineering, polymer processing, and the design of devices involving microfluidics. The study of couple stress fluids contributes to the development of materials with tailored rheological properties for specific applications, such as enhanced lubrication or controlled drug delivery in medical devices. Hence, the couple stress fluids provide a more refined description of fluid behaviour by considering microstructural effects. The study of these fluids is essential in understanding and predicting the flow characteristics of complex fluids encountered in various natural and engineered systems. The couple stress fluids are a kind of micropolar fluids and also a non – Newtonian fluid that accounts for the effects of microstructure in the fluid. In the couple stress fluid model additional internal micro – rotations and micro – deformations are considered. This microstructure introduces an additional complexity to fluid dynamics and its behaviour is characterized by a couple stress tensor. This tensor describes the internal moments or torques within the fluid. Couple stress fluids incorporate microscale effects within the fluid. These microstructures represent the inherent rotational or deformational aspects at the microscopic level. The distinguishing feature of couple stress fluids is the inclusion of a couple stress tensor in the stress-strain relationship. This tensor describes the internal moments or torques within the fluid. In addition to the conservation of mass and momentum equations, couple stress fluids require additional equations governing the evolution of microstructure, introducing new parameters related to micro-rotation. These governing equations was first modelled by V K Stokes and is given by [1].

Understanding Sphere Flow in Couple Stress Fluids at Low Reynolds Number

The behaviour of fluids can be described using a concept known as couple stress fluids, which take into account the microstructure and the intermolecular interactions within the fluid. In couple stress fluids, the stress at a point is not only dependent on the rate of strain but also on the rate of rotation at that point. This unique characteristic of couple stress fluids leads to deviations from the predictions of classical fluid mechanics. When considering low Reynolds number in fluid mechanics, the inertial forces are negligible compared to viscous forces, resulting in a laminar flow regime. This has significant implications for the behaviour of fluids around objects such as spheres, as the flow patterns and drag forces experienced by the sphere are greatly influenced by the low Reynolds number conditions.

Governing Equations

In the study of couple stress fluids, the governing equations play a crucial role in understanding the behaviour of such fluids, especially at low Reynolds numbers. The specific equations that apply to low Reynolds number flow in couple stress fluids include the momentum balance equation, which incorporates the effects of couple stresses, and the angular momentum balance equation, which accounts for the microstructure of the fluid. These equations, when applied to low Reynolds number flow, offer insights into the intricate interplay between viscous forces, couple stresses, and inertial forces, providing a comprehensive framework for analysing the dynamics of such fluids in the presence of a sphere.

In the study of couple stress fluids, it is crucial to derive and understand the governing equations that describe their behaviour, particularly at low Reynolds numbers. The low Reynolds number regime is characterized by the dominance of viscous forces over inertial forces, making it essential to account for the effects of microstructural phenomena such as couple stresses. Two fundamental equations that play a key role in describing low Reynolds number flow in couple stress fluids are the momentum balance equation and the angular momentum balance equation.

The motivation and objective of the research

In this work the viscous force and the inertial force is taken care of by the particle Reynolds number and the couple stresses are taken care by the couple stress parameter given by Stokes [1]. This work is a first attempt to combine the hydrodynamic force expression given by Lovalenti and Brady [2] and the drag expression for the spherical particle in couple stress fluid given by Stokes [1]. The motivation for this work is that the expression for long time limit is given by Lovalenti and Brady [2] and the drag expression for various radius are given by Stokes [50, 1]. Until now the attempt to combine these were not found in the literature. Hence the curious question answered here is what is the hydrodynamic force over a sphere at low Reynolds numbers in the presence of couple stresses.

Formulation of the problem

The Lovalenti and Brady (1993) formalism for the hydrodynamic force on a rigid sphere undergoing arbitrary time – dependent motion in an arbitrary time dependent uniform flow field at small Reynolds numbers is given by the expression as

$$\begin{aligned} F^H(t) &= \frac{4\pi}{3} ReSl\dot{U}^{\infty}(t) - 6\pi U_s(t) - \frac{2\pi}{3} ReSl\dot{U}_s(t) \\ &\quad + \frac{3}{8}\left(\frac{ReSl}{\pi}\right)^{1/2} \left\{ \int_{-\infty}^{t} \left[\frac{2}{3} F_s^{H_{||}}(t) - \left\{ \frac{1}{|A|^2} \left(\frac{\pi^{1/2}}{2|A|} erf(|A|) - exp(-|A|^2) \right) \right\} F_s^{H_{||}}(s) \right. \right. \\ &\quad \left. + \frac{2}{3} F_s^{H_\perp}(t) - \left\{ exp(-|A|^2) - \frac{1}{2|A|^2} \left(\frac{\pi^{1/2}}{2|A|} erf(|A|) - exp(-|A|^2) \right) \right\} F_s^{H_\perp} \right] \\ &\quad \left. \times \frac{2ds}{(t-s)^{3/2}} \right\} + o(Re). \end{aligned} \quad (1)$$

Here, $U_s = U_p - U^{\infty}$ is the slip velocity of the fluid. U_p is the velocity of the particle. $\boldsymbol{U_s}$ has been non-dimensionalized by U_c. The acceleration terms $\dot{U}_s$ and $\dot{U}^{\infty}$ are non - dimensionalized by U_c/τ_c, where

τ_c is the characteristic timescale. U^{∞} is the velocity of the fluid as $|r| \to \infty$. Re is the Reynolds number, defined as Re = aU_c/ν based on a characteristic slip velocity, U_c, a denotes the characteristic particle dimension and ν is the kinematic viscosity of the fluid. $F_s^{H_{||}} = -6\pi U_s.\underline{pp}$ and$F_s^{H_{\perp}} = -6\pi U_s.(\delta - \underline{pp})$, where δ is the idem tensor of order 2 and unit vector$\underline{p} = \frac{Y_s(t)-Y_s(s)}{|Y_s(t)-Y_s(s)|}$, here $Y_s(t) - Y_s(s)$ is the integrated displacement of the particle relative to the fluid from time s to the current time t. Sl is the Strouhal number and $A = \frac{Re}{2}\left(\frac{t-s}{ReSl}\right)^{1/2}\left(\frac{Y_s(t)-Y_s(s)}{t-s}\right)$. F^H is scaled by $\mu a U_c$.

We use equation (1) to obtain the equation governing the unidirectional motion of a sphere in a quiescent fluid, starting with zero velocity at time t = 0, with $U_s = U_p$ where U_p is the velocity of the particle, scaled with respect to the size of the particle and the frequency of the external periodic force, ω, i.e., we take U_c=$a\omega$ and $U^{\infty} = 0$. Under these conditions, equation (3.2.1) reduces to

$$F^H(t) = -6\pi U_p(t) - \frac{2\pi}{3} ReSl\dot{U}_p(t) + \frac{3}{8}\left(\frac{ReSl}{\pi}\right)^{1/2} \int_0^t \left[\frac{-8\pi U_p(t)ds}{(t-s)^{3/2}} - \left\{\frac{1}{|A|^2}\left(\frac{\pi^{1/2}}{2|A|} erf(|A|) - exp(-|A|^2)\right)\right\}\frac{-12\pi U_p(s)ds}{(t-s)^{3/2}}\right] \quad (2)$$

We note that the integral in equation (2) contains a singularity at s = t. In order to take account of this singularity, the integral was split into the intervals $[0, t-\varepsilon]$ and $[t-\varepsilon, t]$ for a small positiveε. Thus we get two expressions for the integral.

Where, $A = \frac{Re}{2}\left(\frac{t-s}{ReSl}\right)^{1/2}\left(\frac{Y_s(t)-Y_s(s)}{t-s}\right)$

That is we get,

$$F^H(t) = -6\pi U_p(t) - \frac{2\pi}{3} ReSl\dot{U}_p(t) + \frac{3}{8}\left(\frac{ReSl}{\pi}\right)^{1/2}(P+Q) \quad (3)$$

where

$$P = \int_0^{t-\varepsilon}\left[\frac{-8\pi U_p(t)ds}{(t-s)^{3/2}} - \left\{\frac{1}{|A|^2}\left(\frac{\pi^{1/2}}{2|A|} erf(|A|) - exp(-|A|^2)\right)\right\}\frac{-12\pi U_p(s)ds}{(t-s)^{3/2}}\right] \quad \text{(3a) and}$$

$$Q = \int_{t-\varepsilon}^{t}\left[\frac{-8\pi U_p(t)ds}{(t-s)^{3/2}} - \left\{\frac{1}{|A|^2}\left(\frac{\pi^{1/2}}{2|A|} erf(|A|) - exp(-|A|^2)\right)\right\}\frac{-12\pi U_p(s)ds}{(t-s)^{3/2}}\right] \quad \text{(3b)}$$

Denoting the components of $\boldsymbol{F^H(t)}, \boldsymbol{U_p(t)}, \dot{U}_p$, $\boldsymbol{A}$, $\boldsymbol{P}$ and $\boldsymbol{Q}$ along the direction of the force field by $F^H(t)$, $U_p(t)$, $\dot{U}_p$, A, P and Q and then transforming the integral with respect to A, we get

$$Q = \int_0^{c\sqrt{\varepsilon}} \frac{8\pi U_p^2(t)\, Re\, d\, A}{(Re\, S\, l)^{1/2} A^2} - \int_0^{c\sqrt{\varepsilon}} \frac{1}{|A^2|}\left(\frac{\sqrt{\pi}}{2|A|} erf(|A|) - exp(-|A^2|)\right) \frac{12\pi U_p^2(t)\, Re\, d\, A}{(Re\, S\, l)^{1/2} A^2} \qquad (3c)$$

where $c = \frac{ReU_p(t)}{2\sqrt{ReSl}}$.

Note that, Q vanishes as ε tends to zero; i.e., as s→t, $\frac{1}{|A^2|}\left(\frac{\sqrt{\pi}}{2|A|} erf(|A|) - exp(-|A^2|)\right) \to \frac{2}{3}$.

Hence the two singular terms cancel each other as s→t, and thus we obtain an expression for the hydrodynamic force on a couple stressed sphere in a quiescent fluid as:

$$F^H(t) = -6\pi U_p(t)(2+\alpha) - \frac{2\pi}{3} ReSl\dot{U}_p(t) + \frac{3}{8}\left(\frac{ReSl}{\pi}\right)^{1/2} \left\{ \int_0^{t-\varepsilon} \left[\left\{ \frac{1}{|A|^2} \left(\frac{\pi^{1/2}}{2|A|} erf(|A|) - exp(-|A|^2) \right) \right\} \frac{12\pi U_p(s) ds}{(t-s)^{3/2}} \right] + 16\pi U_p(t) \left[\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{\varepsilon}} \right] \right\} \qquad (4)$$

Here α is the couple stress parameter, equation included from the works of Stokes [1].

Method used to validate the expression:

The program to solve the integral were written using GNU OCTAVE and the results of Lovalenti and Brady [1] and Stokes' [2] were reproduced. Hence, we claim that the programs are robust and valid.

Results:

The difference between the inclusion of couple stresses is that there is a decay in the drag ratio when the radius of sphere increases as seen in figure (1), where as in the absence of couple stress the Reynolds number is attained eventually at large times as in figure (2). Further the inclusion of both leads to the convergence of the curves of the first figures. That is all the curves of the figure(1) irrespective of the μ value merge with the last curve which is due to the inclusion of resistance to the change in the change in the motion of the fluid.

Conclusion:

This work is the combination of expressions given by V K Stokes and Lovalenti and Brady, in which we extended the couple stress fluid hydrodynamic force at zero Reynolds numbers to small and finite Reynolds numbers. It is observed that the combination of both couple stress parameters and low Reynolds numbers gives linearly decaying trajectory at large times. This is an expected result as there is a fore – aft symmetry in couple stresses and as the resistance to change in motion is applied on the couple forces of the fluids, they tend to nullify their effects and tend to zero. The authors in spite of performing the numerical experiments several times the expression for hydrodynamic force gave only the linear trend at large times.

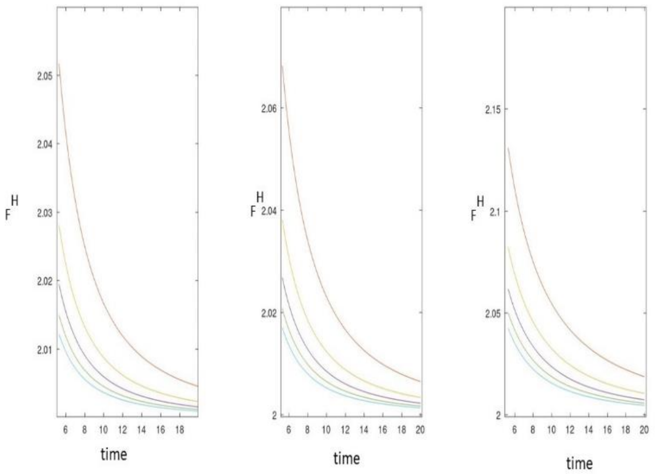

Figure 1: The hydrodynamic force versus time at zero Reynolds numbers.

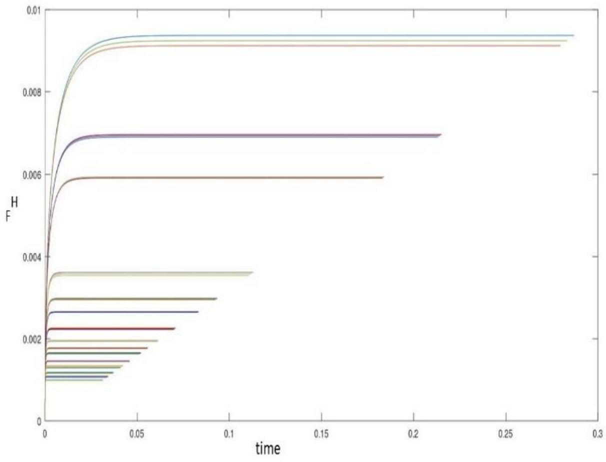

Figure 2: The hydrodynamic force with respect to time at zero couple stress parameter at small Reynolds numbers showing self – similar patterns

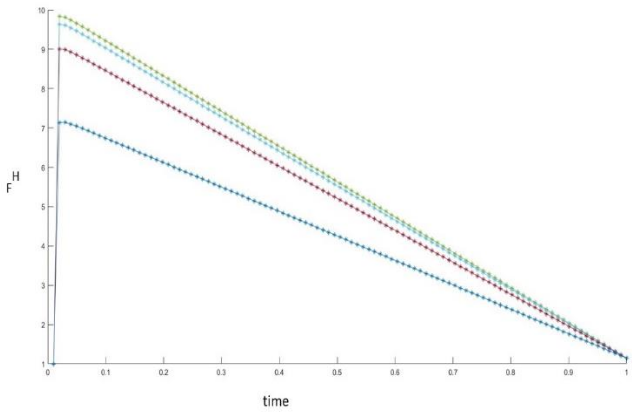

Figure 3: the combination of both Re and couple stress parameter, showing the linear decay of hydrodynamic force at large times.

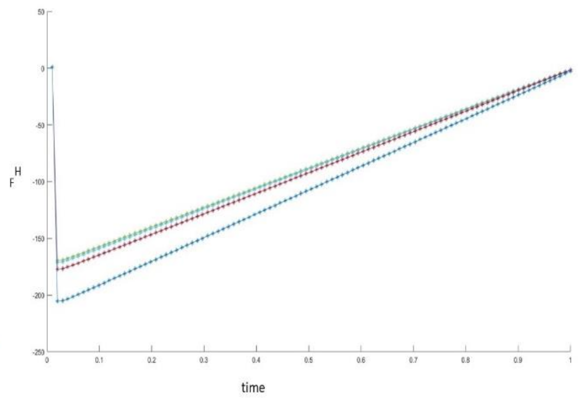

Figure 4: the combination of both Re and couple stress parameter, showing the linear decay of hydrodynamic force at large times, for Re < α.

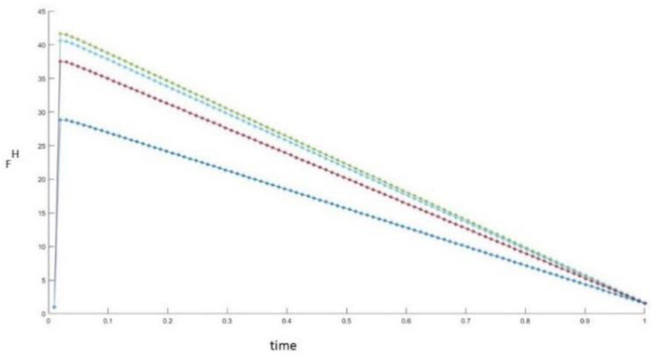

Figure 5: the combination of both Re and couple stress parameter, showing the linear decay of hydrodynamic force at large times, Re > α.

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