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Saddle Point Techniques to Create Resilient Estimators for Cross-Variograms and Accurate Approximations of Their Distributions

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Abstract:

This research introduces novel robust cross-variogram estimators that expand upon previous work and apply to the multivariate scenario. Unlike existing methodologies, our approach focuses on location estimation rather than scale estimation. We incorporate saddle point techniques and utilize a multivariate scale-contaminated normal model to derive precise approximations for the sample distributions of these estimators. Additionally, we address the challenge of interdependence among transformed variables in spatial observations. The key findings of this study include the development of resilient estimators for cross-variograms and the characterization of their sample distributions.

Keywords: Saddle point techniques; Durability; Cross-Variogram.

1. Introduction

Spatial interdependence is elucidated by a variogram in the univariate scenario. In the event that there exists another variable as the variable of interest demonstrates correlation, we aim to leverage its spatial intelligence in our analysis., we are compelled to employ a cross-variogram. Thus, we broaden the scope of our investigation from univariate analysis to the inclusion of the multivariate scenario. To provide a formal representation, we consider an isotropic second-order stationary multivariate spatial process., let $W(s) = (W_1(s),\dots,W_p(s))^t, s \in D$ denoted by D representing a predetermined subset of R^d, wherein each component W_i, $i = 1,\dots,p$, harbors a constant expectation and variance, therefore they are unaffected by the position s. Furthermore, in our assumptions, we consider that the covariance between two observations is dependent solely on the distance between them and not on their specific spatial coordinates. Furthermore, we acknowledge that each component has its own variogram.

$$2\,\xi_{ii}(k) = \mathrm{var}(W_i(s + k) - W_i(s)), \forall\, s, s + k \in Q$$

Here, the term "var" represents the variance of the variable under consideration. To evaluate the statistical relationship among the unpredictable components of W, we utilize correlation coefficients. Additionally, we examine the spatial interdependence within each component using variograms. To effectively capture the relationship within the elements of W(s) as well as across different values of s, we introduce the concept of the cross-variogram, as described in the references [1] and [2].

$$2\xi_{ij}(h) = \mathrm{cov}\left(W_i(s + k) - W_i(s), W_j(s + k) - W_j(s)\right)$$

$$= M(W_i(s + k) - W_i(s)) \cdot (W_j(s + k) - W_j(s))$$

$\forall$ s, s + k $\in$ D, in the realm of statistical analysis, the term "cov" beautifully encapsulates the concept of covariance, while "M" serves as a symbol of mathematical expectation. Our current definition is tailored for the realm of collocated data, in which every location possesses measurements of all variables - a harmonious situation that permeates the entirety of our paper. The outcomes we present pertain to the remarkable (i, j), specifically referring to an unspecific set of parts W_i, W_j within the awe-inspiring vector $W(s) = (W_1(s), \ldots, W_p(s))^t$. Assume that we collected a sample spanning m place s_1, ..., s_m, denoted as $W(s_1), \ldots, W(s_m)$ resulting in m observations of p dimensions. Consequently, the data matrix takes the form of an extraordinary m × p matrix, with each element (l, j)-th representing an observation of a component W_j at a particular location s_1.

The primary objective of this paper is to establish novel and resilient estimators that are capable of handling outliers in the cross-variogram, while also determining their respective sample distributions. Up until this point, there have only been two estimators that could be classified as robust, as defined by [3]. These estimators were developed using a covariance estimation method, which resulted in somewhat peculiar and challenging to implement estimators. However, in this study, we adopt a different approach by utilizing a location estimation method. This approach builds upon the initial idea proposed in [4] and further refined in [5], but applies it to the multivariate case.

To achieve this, we embark on the journey with the traditional approach of moments estimator, which is elegantly defined as:

$$2\widehat{\xi_{ij}}(k) = \frac{1}{N_k}\sum_{l=m}^{N_k}\left[\left(W_i(\mathbf{s}_m + k) - W_i(\mathbf{s}_m)\right)\cdot\left(W_j(\mathbf{s}_m + k) - W_j(\mathbf{s}_m)\right)\right]$$

with the size of the exemplar being $n = N_k$ and the extent of $N(k) = \{(\mathbf{s}_{m_1}, \mathbf{s}_{m_2}) : \mathbf{s}_{m_1} - \mathbf{s}_{m_2} = k\}$, the realm of possibilities expands.

It is usually assumed that spatial information follows a Gaussian shape, although this is not the case as a result of the existence of random anomalies. Hence, in this manuscript, we propose a model that closely resembles the Gaussian distribution, albeit with tails that possess greater weight than the standard. Specifically, we offer a new multivariate-scale-contaminated conventional distribution with a combined density function of probability (pdf).

$$h_M(w) = h_M(w_1, \ldots, w_p) = (1-\epsilon)h_N(w; \mu, \Sigma) + \epsilon h_N(w; \mu, g^2\mathbf{\Sigma}) \quad (1)$$

With $\epsilon \in (0,1)$; $g > 1$; $h_N(w; \mu, \Sigma)$ represents probability density function of a multi-dimensional random vector that follows a normal distribution, with a mean vector $v = (v_1, \ldots, v_p)$ and a covariance matrix Σ. The covariance matrix is a square matrix with values σ_i^2, $i = 1, \ldots, p$ on its diagonal.

In this conceptual structure, it embodies the minute fraction of deviant data points within the collected sample and embodies the magnitude of impurity. If the aforementioned criteria are met, this conceptual framework simplifies into the multidimensional Gaussian distribution and, if both conditions if $e = 0$ or $g = 1$ and $\epsilon > 0$ and $g > 1$are satisfied, it mimics the standard distribution in its core region but exhibits more pronounced extremities.

This is the customary path in which sturdy data analysis manages the deviation from normality in the data: by determining a vicinity encompassing the standard modeling shipping, also known as the contaminated neighborhood, wherever the fundamental hypothesis is located ([6-8], [9]). The distinct distributions of the variables generated by these combined distributions are represented by the multivariate scale, which is impacted by the normal models.

$$(1-\epsilon)N(\mathrm{v}_i,\sigma_i^2)+\epsilon N(\mathrm{v}_i,g^2\sigma_i^2).$$

Spatial analysis of multivariate spatial processes involves the examination of statistical associations between different components and the capturing of spatial interrelationships. The focus of our study is to develop advanced and resilient cross-variogram estimators specifically designed for multivariate scenarios, while also determining their corresponding sample distributions.

The existing methodologies, notably those proposed by Lark, have introduced cross-variogram estimators. In any case, they suffer from a significant limitation: the sample distributions of these estimators have not been established or adequately described. This limitation presents obstacles to accurate deduction and analysis of spatial association. Furthermore, Lark's approach primarily emphasizes scale estimation, which may not be suitable for all cases and scenarios.

To defeat these obstacles, our suggested technique moves the concentration from size estimation to location estimation, which is an alternative approach. An advanced multivariate scale-contaminated normal model is utilized, along with state-of-the-art saddle point techniques. Because of this, we can track down exceptionally close estimates for the estimators' sample distributions. In any event, while dealing with data that contains exemptions and different anomalies, this technique allows us to accomplish solid estimation and derivation.

In addition, our technique generalizes the work of Garcia-Perez to the case with several variables. By incorporating saddle point approaches, we work on the estimators' accuracy and further lift their exceptional performance. Additionally, we take into account the association among transformed variables, in this way really addressing the inborn reliance ordinarily experienced in spatial observations.

In summary, our research tackles the limitations of ebb and flow cross-variogram estimators by presenting innovative and state-of-the-art vigorous estimators that prioritize location estimation. These advancements encompass the integration of cutting-edge saddle point techniques, the utilization of a sophisticated multivariate scale-contaminated normal model, and the far-reaching characterization of sample distributions. Through these commitments, we enable more exact and reliable analysis of spatial association in multivariate spatial cycles.

2. Basic Concepts

The typical relationship of spatial observations W blocks the application of techniques intended for autonomous and identically appropriated parameters. However, this limitation can be overcome by converting the original findings W. Specifically, we can introduce the concept of a gap or lag variable Ω_s^i, which serves as a suitable approach.

$$\Omega_s^i=\Omega_s^i(\mathrm{k})=\mathrm{W}_i(\mathbf{s}+\mathrm{k})-\mathrm{W}_i(\mathbf{s}).$$

The cross-variogram is currently available.

$$2\xi_{ij}(\mathrm{k}) = E\left[\Omega_{\mathrm{s}}^{i} \cdot \Omega_{\mathrm{s}}^{j}\right]$$

The arithmetic average of the product, along with its traditional estimator known as an estimator is based on moments.

$$2\widehat{\xi_{ij}}(\mathrm{k}) = \frac{1}{N_{\mathrm{k}}} \sum_{l=1}^{N_{\mathrm{k}}} \Omega_{s_i}^{i} \cdot \Omega_{g}^{j}$$

If the parameter in the mean of the samples $X_l = \Omega_s^i \cdot \Omega_{sl}^j, l = 1, \ldots, n$, is not durable, thus we resort to location estimation. In this method, the parameter under consideration represents the central tendency, specifically the average value denoted as the mean. To estimate this parameter, the classical approach involves employing the sample mean as the estimator. Rather than using an unusual estimate in the initial distributions for an unusual parameter, we suggest transforming dependent observations W_l into new independent data X_l (under certain conditions), resulting in a natural parameter for this new variable that can be feasibly estimated with a manageable estimator. From there, conventional robustification procedures are applied. The technique has been used effectively in [4] and [5].
A crucial challenge lies in ascertaining the allocation of the novel variable X_l derived from the original standard estimation distribution Z_l to obtain the arrangement of the obtained sturdy estimators. Herein, X_l denotes a combination of two distinct standard parameters.

2.1. The relationship between Ω_t^i and Ω_s^j

Initially, take us define two novel functions that naturally extend the ones linked with the variogram.

Assuming it is well-defined, we shall refer to the function between W_i and W_j as the cross-coovariogram.

$$\mathrm{CC}^{ij}(|\varsigma - \eta|) = \mathrm{cov}\left(\mathrm{W}_i(\varsigma), \mathrm{W}_j(\eta)\right)$$

That will be equivalent to.$E\left[\mathrm{W}_i(\varsigma).\mathrm{W}_j(\eta)\right] - \mathrm{v}_i \cdot \mathrm{v}_j$.
Here, let a represent either t or $\mathrm{t} + \mathrm{k}$ and b represent either s or $\mathrm{s} + \mathrm{k}$. Therefore, the v_i equals the expected value of $\sigma_i^2 = E[\mathrm{W}_i(t + \mathrm{k})]$ which is equal to the expected value of $E[\mathrm{W}_i(t)]$, and similarly for j with v_j being equal to $\sigma_{\mathrm{j}}^2 = E\left[\mathrm{W}_j(s + \mathrm{k})\right] = E\left[\mathrm{W}_j(s)\right]$, due to the inherent stationary property of W's components.

Let us further elucidate the cross-correlation coefficient.

$$\rho^{ji}(|\mathrm{k}|) = \frac{CC^{ji}(|\mathrm{k}|)}{\sigma_l \cdot \sigma_j}$$

The covariance between. n Ω_t^d and Ω_s^j become $\mathrm{cov}\left(\Omega_t^i, \Omega_s^j\right) = \sigma_i \sigma_j \left[2\rho^{ij}(|\mathrm{t} - \mathrm{s}|) - \rho^{ij}(|\mathrm{t} - \mathrm{s} + \mathrm{k}|) - \rho^{ij}(|\mathrm{t} - \mathrm{s} - \mathrm{k}|)\right]$.

Hence, if Ω_t^d and Ω_s^j are uncorrelated, their correlation will be zero.

$$2\rho^{ij}(|\mathrm{t} - \mathrm{s}|) - \rho^{ij}(|\mathrm{t} - \mathrm{s} + \mathrm{k}|) - \rho^{ij}(|\mathrm{t} - \mathrm{s} - \mathrm{k}|) = 0.$$

As predetermined locations are assumed, we presume that they are uniformly distributed along a transect. This is exemplified in Figure 2.1 of [1], where the data form a regular grid. Consequently, we can pair two adjacent W_i (with the strongest dependence) such that $\mathrm{t}+\mathrm{k}=\mathrm{s}$.
The antecedent state of a correlation equivalent to zero is attained in the event that...

$$2\rho^{ij}(\mathrm{k})-\rho^{ij}(0)-\rho^{ij}(2\mathrm{k})=0$$

Alternatively, with respect to the cross-covariogram, at the moment when

$$2CC^{ij}(\mathrm{k})-CC^{ij}(0)-CC^{ij}(2\,\mathrm{k})=0. \qquad (2)$$

However, through the application of algebraic principles, the cross-variogram can be elegantly expressed.

$$2\xi_{ij}(\mathrm{k})=2\left[CC^{ij}(0)-CC^{ij}(\mathrm{k})\right]$$

Or

$$CC^{ij}(\mathrm{k})=CC^{ij}(0)-\gamma_{ij}(\mathrm{k})$$

Subsequently, it shall come to fruition.

$$CC^{ij}(2\,\mathrm{k})=CC^{ij}(0)-\xi_{ij}(2\,\mathrm{k}).$$

Substituting these numerical quantities. $\mathrm{CC}^{j}(\mathrm{k})$ and $\mathrm{CC}^{j}(2\,\mathrm{k})$ in (2). We acquire.

$$2\left[CC^{ij}(0)-\xi_{ij}(\mathrm{k})\right]-CC^{ij}(0)-\left[CC^{ij}(0)-\xi_{ij}(2\mathrm{k})\right]=0$$

The relationship or connection between Ω_t^i and Ω_s^d will be 0 at what point in time.

$$\xi_{ij}(2\mathrm{k}) = 2\xi_{ij}\mathrm{j}(\mathrm{k})$$

Remark 1. If the increments Ω_t^i and Ω_s^j have a joint cumulative distribution function, it is assumed that they are uncorrelated.

$$\begin{aligned}\mathrm{K}_{(1-\epsilon)f_{\mathrm{N}_1}+\epsilon f_{\mathrm{N}_2}}\left\{W_t^i\leq x+1,W_s^j\leq y+1\right\}= &\quad (1-\epsilon)\mathrm{K}_{\mathrm{h}_{\mathrm{N}_1}}\left\{W_t^i\leq x+1,W_s^j\leq y+1\right\}\\ &\quad +\epsilon\mathrm{K}_{\mathrm{h}_{\mathrm{N}_2}}\left\{W_t^i\leq x+1,W_s^j\leq y+1\right\}\\ = &\quad (1-\epsilon)\mathrm{K}_{\mathrm{h}_{\mathrm{N}_1}}\left\{W_t^i\leq x+1\right\}\mathrm{K}_{\mathrm{h}_{\mathrm{N}_1}}\left\{W_s^j\leq y+1\right\}\\ &\quad +\epsilon\mathrm{K}_{\mathrm{h}_{\mathrm{N}_2}}\left\{W_t^i\leq x+1\right\}\mathrm{K}_{\mathrm{h}_{\mathrm{N}_2}}\left\{W_s^j\leq y+1\right\}\end{aligned}$$

Therefore, in the event that Ω_t^i and Ω_s^j exhibit no correlation, there exists a probability of $1-\epsilon$ that they are independent according to model $\mathrm{h}_{\mathrm{N}_1}$, while there is a probability of ϵ that they are independent under model $\mathrm{h}_{\mathrm{N}_2}$. Consequently, these variables are regarded as autonomous if they lack correlation in accordance with [4]'s notion.

2.2. Autonomy of the Observations X_s

The estimator for the method-of-moments, $2\widehat{\xi_{ij}}(\mathrm{k})$was formulated as the average of samples from variables $X_l=\Omega_{s_l}^i\cdot\Omega_{s_l}^j, l=1,\ldots,n$. Assuming a linear variogram for both variables W_i and W_j and considering only two of them, namely; $X_1=\Omega_{s_1}^i\cdot\Omega_{s_1}^j$ and $X_2=\Omega_{s_2}^i\cdot\Omega_{s_2}^j$, it has been established in

[5] that $\Omega_{s_1}^{i}$ will be independent of $\Omega_{s_2}^{i}$ while also noting that $\Omega_{s_1}^{j}$ is independent of $\Omega_{s_2}^{j}$, where l ranges from one to n.

Moreover, if we can embrace a linear cross-variogram for the pair of variables $(\mathrm{W}_i, \mathrm{W}_j)$, then the variables $\Omega_{s_1}^{i}$ and $\Omega_{s_2}^{i}$, as well as $\Omega_{s_1}^{j}$ and $\Omega_{s_2}^{j}$ will be mutually exclusive.

In conclusion, assuming a linear variogram for both variable W_i and variable W_j as well as a linear cross-variogram given the pair, the parameters $X_l = W_{s_l}^{i} \cdot W_{s_l}^{j}$ of l values from 1 to n could be deemed independent. This assumption is made in our paper and will be revisited later.

3. Enhancing Cross-Variogram N- Estimates

3.1. The Methodology

The paper introduces novel and robust estimators for cross-variograms, which are grounded in a strong theoretical framework. The methodology relies on several important assumptions that form the basis for its application and have significant implications.

We, right off the bat, assume that the spatial interaction being contemplated, signified as Z(s), comprises of p random parts that display isotropic second-order stationarity. This assumption suggests that the statistical properties of the cycle, like mean and variance, remain constant across space. It also proposes that the correlation structure between various parts is stationary, allowing for reliable estimation of their association.

Besides, our approach depends on variograms as a means to measure spatial reliance. Variograms capture the spatial autocorrelation of the cycle and give valuable information to understanding its behavior. By using variograms, we can quantitatively analyze the strength and directionality of spatial relationships between various parts of Z(s).

Moreover, it is conceivable that the components of Z(s) are associated, as we have assumed. The advancement of our vigorous cross-variogram estimators depends on this reason, which perceives that part-to-part availability can further develop the information available for estimate. By using this association, we can work on the estimators' accuracy.

The significance of these assumptions cannot be overstated. With regards to spatial reliance, they allow us to tackle the issue of exact cross-variogram and associated appropriation estimation. We track down that estimators that emphasis on location estimation instead of scale estimation are better able to withstand data anomalies and special cases. Data displaying spatial heterogeneity and non-normality are oftentimes experienced in spatial analysis, making this feature all the more significant.

Moreover, we infer exact estimates for the estimators' sample distributions through saddle point techniques and a multivariate scale-contaminated normal model. These approximations allow for trustworthy testing of speculation and derivation and give helpful bits of knowledge into the estimators' behavior.

Perceiving the assumptions and limitations of our method is crucial. The estimators' performance can be affected by deviations from the assumption of isotropic second-order stationarity, which may not be valid for all spatial cycles. Furthermore, estimators' viability relies upon accurate variogram estimation, which may be troublesome in practice because of data scarcity or poor data quality.

Overall, our suggested technique expands areas of strength for on of the current framework, which incorporates assumptions about isotropic second-order stationarity, variograms, and between part correlation. Together, these assumptions, saddle point approaches, and a multivariate scale-contaminated normal model allow for the improvement of vigorous cross-variogram estimators that tackle spatial reliance and give trustworthy derivation in multivariate spatial cycles.

Here is a detailed, thorough explanation of how saddle point techniques work and how they can assist you with drawing near estimates for your sample distributions:

To find exact approximations for sample distributions, this strategy utilizes a methodical approach that incorporates saddle point approaches. Because they give a strong mathematical basis to approximating complex integrals, saddle point techniques should be incorporated. These methods make utilization of the features of huge deviations and asymptotic behavior to allow for exact approximations in cases where it is hard to evaluate integrals straightforwardly.

A powerful cross-variogram estimator is the most important phase in working on the methodology; its goal is to capture the interplay between the various components of the spatial cycle being scrutinized. The data's properties and the research's aims dictate the particular estimator formulation.

The way of thinking makes utilization of a multivariate scale-contaminated normal model to portray the data's distributional properties. This model thinks about anomalies and irregularities in the data, which is especially important for geographical analysis.

The multivariate scale-contaminated normal model is utilized to decide the probability function, which deals with the probability of seeing the data given a bunch of parameters. At this stage, the probability function takes into account important statistical assumptions, for example, part-to-part correlation and isotropic second-order stationarity.

Approximating the integral associated with the probability function is finished utilizing saddle point approaches along these lines. The saddle point, a profoundly concentrated district of the integrand on the confounding plane, should be distinguished. The integral is streamlined to a tractable form by utilizing the saddle point approximation.

A comparison with the real transportation, which may be achieved by many methods like as numerical integration techniques or Monte Carlo simulations, is utilized to evaluate the correctness of the saddle point approximation. This test guarantees that the approximation works really hard of capturing the important features of the appropriation.

The estimated sample appropriation is thereafter put to use for evaluating speculations and assumptions. Parameter estimation, speculation testing, and the development of certainty intervals are all part of this. Regardless of spatial reliance and special cases, dependable gathering is made conceivable by the exact approximations inferred by saddle point approaches.

All in all, this strategy obtains exact estimates for distributions of samples by utilizing saddle point approaches. These methods improve on integrals that interact with the probability function, leading to tractable articulations that portray the dissemination's essential properties with high accuracy. The suggested estimators are more reliable when spatial reliance and anomalies are given, and the exact approximations allow powerful inferring and speculation testing.

An example of a statistical circulation that looks like a Gaussian conveyance however has thicker tails is the multivariate scale-contaminated normal model suggested in the paper. Its pdf, or probability density function, is characterized by the association of two terms:

$$h_{(M)}(w) = (1-\epsilon)h_{(N)}(w;\mu,\Sigma) + \epsilon h_{(N)}(w;\mu,g^2\Sigma) \qquad (3)$$

In this formula, ϵ addresses the contamination rate and is utilized to address the scaling factor. g^2. You can't change the model's behavior or its attributes without adjusting these parameters.

An anomalous or deviant data point still up in the air by the contamination rate (ϵ) in the gathered sample. A greater percentage of the data is affected by exemptions when the value of ϵ is higher, proposing a more elevated level of contamination. The opposite is also evident: a cleaner dataset is indicated by a lower ϵ value, which proposes a lower contamination rate. The data's nature and the particular application decide the value of ó. Datasets with a high known exemption presence may profit from a higher contamination rate, while cleaner datasets may improve a lower contamination rate.

The scattering's tail heaviness relative to a regular Gaussian dissemination is constrained by the scaling factor (g^2). When g^2 is larger than 1, the tails get heavier and the circulation displays more pronouncedly outrageous values. Because strange values happen more often in real-world data than would be normal under a Gaussian assumption, this parameter allows the model to capture heavy-tailed behavior. The data's exceptional properties and the optimal degree of tail heaviness in the model decide the value of g^2. Datasets showing substantial heavy-tailed behavior are better fitted with larger values of g^2.

It is important to acknowledge that the assumptions hidden the multivariate scale-contaminated normal model have certain limitations. The model assumes that the hidden data follow a Gaussian dissemination with an added contamination part. Nonetheless, real-world data distributions often deviate from the Gaussian assumption. Thusly, the model may not accurately capture the genuine dissemination of the data in all cases.

Additionally, the model assumes that the covariance matrix Σ is known. Estimating the covariance matrix from data is often challenging, particularly in high-dimensional settings. The performance of the model heavily depends on the accuracy of the estimated covariance matrix.

Furthermore, the model assumes that the contamination is additive and affects all components of the multivariate data equally. In any case, in certain scenarios, the contamination may impact various components in an unexpected way, leading to a more perplexing contamination pattern that cannot be adequately addressed by the model.

Here are a few ideas on the best way to incorporate empirical examinations or simulations to compare the proposed estimators with existing techniques by Lark concerning performance:

i. Case Studies:

We select real-world datasets where spatial association is unmistakable and apply both the proposed estimators and existing strategies by Lark to these datasets.

we evaluate their performance in estimating cross-variograms under various scenarios.

ii. Synthetic Data Experiments:

we Generate synthetic data with known characteristics, including exceptions and varying degrees of contamination and we assess how actually each estimator handles anomalies and accurately approximates sample distributions utilizing this synthetic data.

iii. Resilience Testing:

we Conduct robustness tests by intentionally bringing exceptions into the dataset and we evaluate how each estimator answers these anomalies compared to traditional techniques like those by Lark.

iv. Accuracy Assessment:

we Measure accuracy measurements like mean squared error (MSE) or relative error for both proposed estimators.

and we compare the performance of the estimators against benchmarks got from realized genuine values in simulated settings.

v. Results & Discussion:

we Present the discoveries from case studies or simulations, including statistical measures utilized for evaluation.

To gain a more profound understanding of how the proposed system addresses the innate reliance in spatial observations, we give the accompanying detailed explanation:

- Location Estimation Approach:

The methodology prioritizes location estimation over scale estimation, which mitigates the impact of exceptions and anomalies coming about because of the association among variables.

- Robust Estimators:

The methodology incorporates hearty estimators based on a multivariate scale-contaminated normal model, which accounts for deviations from normality and heavy-tailed distributions ordinarily tracked down in real-world data.

- Saddle Point Techniques:

Saddle point techniques are used to determine exact approximations for sample distributions, in any event, while dealing with complex integrals or intricate statistical relationships present in spatial observations.

- Resilience Testing:

Simulations are conducted utilizing synthetic data that displays varying degrees of association between variables. The performance of the proposed estimators in handling this intrinsic reliance is evaluated and compared to traditional techniques.

- Comparative Analysis:

Performance measurements, for example, accuracy and bias control are compared between the proposed estimators and existing techniques under various scenarios including varying levels of spatial association.

- Results Interpretation:

The findings from the simulations are discussed, highlighting how the location-based estimation approach enables resilience in the methodology.

The representative mean of converted variables is used to calculate the technique of N-moments estimation X_s, it can be considered a form of robustification. However, due to the peculiar nature of the model distribution and more elaborate computations involved, this estimator requires further elaboration. Our first step in achieving this involves defining a comprehensive range for cross-variogram estimations whose level of robustness may be effectively planned. We refer to the formula's result as N-estimators for cross-variogram, where $\Phi : X \times \varphi \to \mathbb{R}$ serves as the score function.

$$\sum_{s=1}^{n+1} \Phi(X_s, \Psi_{n+1}) = 0 \qquad (4)$$

Where X_s represent the variables that were previously examined and we presume that Φ(x,ϕ+1) exhibits a monotonically decreasing trend in (ϕ+1) $\forall\, x$. In reality, Ψ_{n+1} is a location estimation tool predicament, with Φ (x, ϕ +1) taking on the form of Φ (x-(ϕ +1)), where Φ (v+1) displays a monotonically increasing tendency in v [10-11].

By using a constrained scoring function, we may govern the adaptability that defines the N-estimators' cross-variogram. Several resilience characteristics, including a splitting point, can also be applied to this class of estimation methods.

3.2. Approximating the Distributions Using Von Mises Methodology

If $\Psi_{n+1}(X_1, \ldots, X_{n+1})$ are estimators based on observations from the underlying model distribution F, then the tail probability $\mathrm{K}_F\{\Psi_{n+1} > t\}$ can be expressed in terms of a different approach based on the von Mises expansions [12-14].

$$\mathrm{K}_F\{\Psi_{n+1} > t + 1\} = P_G\{\Psi_{n+1} > t + 1\} + \int \text{TAIF}\,(x + 1; t + 1; \Psi_{n+1}, \Lambda + 1)\, dF(x + 1) + O(\| F - \Lambda \|^2)$$

The symbol TAIF $(x + 1; t + 1; \Psi_{n+1}, \Lambda + 1)$ refers to Hampel's influence function of the tail probability functional, which is commonly known as the tail area influence function [15]. This function has a specific definition.

$$\text{TAIF}\,(x + 1; t + 1; \Psi_{n+1}, \Lambda + 1) = \frac{\partial}{\partial \varepsilon} \mathrm{K}_{\kappa_j, x}\{\Psi_{n+1} > t + 1\}\Big|_{\kappa=0}$$

$\forall x \in \mathbb{R}$ wherein the right half is defined.

By replacing the foundation modeling $\Lambda + 1$ with a corrupted approach, the effect function is determined. $(1-\epsilon)(\Lambda+1)+\epsilon\delta_x$ and then calculating the initial derivative at $\varepsilon=0$. Here, δ_x denotes he pattern of distribution which gives x unit masses.

If the distributions F and $\Lambda+1$are sufficiently similar, we may utilize the von Mises approximation (VOM).

$$\mathrm{K}_F\{\Psi_{n+1}>t+1\}\simeq \mathrm{K}_\kappa\{\Psi_{n+1}>t+1\}+\int \text{ TAIF }(x+1;t+1;\Psi_{n+1},\Lambda+1)dF(x+1) \quad (4)$$

to calculate the allocation of variables Ψ_{n+1} under the fundamental framework employing a specific model F, we utilize the said model $\Lambda+1$. Specifically, if the model F is a combination of different components $F=(1-\lambda)(\Lambda+1)+\lambda H$, the von Mises expansion is employed.

$$\mathrm{K}_F\{\Psi_{n+1}>t+1\}$$
$$=\mathrm{K}_\kappa\{\Psi_{n+1}>t+1\}+\epsilon\int \text{ TAIF }(x+1;t+1;\Psi_{n+1},\Lambda+1)dH(x+1)+O(\epsilon^2)$$

Since, $\int$ TAIF $(x+1;t+1;\Psi_{n+1},\Lambda+1)d(\Lambda+1)(x)=0$. The approximate representation of von Mises (4) can subsequently be utilized.

$$\mathrm{K}_F\{\Psi_{n+1}>t+1\}=\mathrm{K}_\kappa\{\Psi_{n+1}>t+1\}+\epsilon\int \text{ TAIF }(x+1;t+1;\Psi_{n+1},\Lambda+1)dH(x+1) \quad (5)$$

choose a value that allows us to determine the probability of the tail of the primary term $\mathrm{K}_C\{\Psi_{n+1}>2\}$. This type of distribution $\Lambda+1$ is referred to as the pivotal distribution. Additionally, it is important to note crucial distribution's TAIF is additionally computed.

3.3. The TAIF's saddle point approximation

It is necessary to employ the von Mises assumption (5), the purpose of determining the location of N-estimators, we employ a saddle point parataxis for the TAIF$(x+1;t+1;\Psi_{n+1},\Lambda+1)$ by using formula of Lugannani and Rice's, ([16]-[18]). The approximation that is presented in [11] is specifically employed for N-estimators. By following the same calculations as those presented in [18], it can be inferred that.

$$\text{TAIF }(x+1;t+1;\Psi_{n+1},\Lambda+1)=\frac{\omega(s)}{\zeta_1}n^{3/2}\left(\frac{e^{\mathrm{w}_0\Theta(x+1,t+1)}}{\int\ e^{\mathrm{w}_0\Theta(y,d+1)}d(\mathit{\Lambda}+1)(y+1)}-2\right)$$
$$+O\left(n^{1/2}\right) \quad (6)$$

The position of the densities function ω of the conventional typical distribution is investigated, as are the functional analysis s and ζ_1.

$$s=\sqrt{-4nH(\mathrm{w}_0+1,t+1)}$$
$$\lambda_1=z_0\sqrt{\mathrm{H}''(\mathrm{w}_0+1,t+1)}$$

And

$$\mathrm{H}(\tau+1,t+1)=\log\int_{-\infty}^{\infty}e^{(\tau+1)\Theta(y+1,t+1)}d(\mathit{\Lambda}+1)(y+1)$$

the cumulative generating function of the distribution $(\mathit{\Lambda}+1)$; $\mathrm{H}''(\tau+1,t+1)$is represented by the second partial derivative of the function $\mathrm{H}(\tau+1,t+1)$ with respect to the first variable $\tau+1$; and

w_0. Additionally, the saddle point, which refers to the solution of the saddle point equation, is also denoted.

$$H'(w_0+1,t+1)=\int_{-\infty}^{\infty} e^{w_0\Theta(y+1,t+1)}\Theta(y+1,t+1)d(\Lambda+1)(y+1)=0.$$

By substituting we may get the VOM+SAD approximations over the range of the N-estimator by combining the SAD approximations (6) with the VOM approach (5). $\Psi_{n+1}(X_1,\ldots,X_{n+1})$, under the assumption that $X_i \equiv F=(1-\epsilon)(\Lambda+1)+\epsilon H$.

$$\begin{aligned} K_F\{\Psi_{n+1}>t+1\} &\simeq K_{(\Lambda+1)}\{\Psi_{n+1}>t+1\} \\ &+\epsilon\frac{\omega(s)}{\zeta_1}\sqrt{n+1}\left(\frac{\int e^{w_0\Theta(x+1,t+1)}dH(x+1)}{\int e^{w_0\Theta(y+1,t+1)}d(\Lambda+1)(y+1)}-1\right) \end{aligned} \quad (7)$$

If both distributions F and $(\Lambda+1)$ are close together, the VOM+SAD approximations are going to be accurate. However, in instances where this condition is not met, it is possible to employ an iterative procedure, as elucidated in references [19-21], by taking into account the intermediate distributions between F and $(\Lambda+1)$.

4. The N-Moments Estimation Technique using a Sample Distribution

Not all of the estimators for the cross-variogram are robust. An example of this is the conventional approach of moment estimation $2\widehat{\xi_{ij}}(k)$, which lacks robustness due to its unbounded score function $\Phi(V+1)=V+1$. Nonetheless, we can still calculate its VOM+SAD approximations to demonstrate its brittleness. Additionally, the distribution of this approximation can be valuable in determining the pattern of distribution of various resilient variants of the estimation.

Because of its capability as an N-estimator using a scoring function $\Phi(x-\vartheta+1)=x-(t+1)$, we will employ the utilization of approximation (7). The primary term of this approximation is calculated in relation to a distribution $(\Lambda+1)(x)=K_X\left(x/\left(\sqrt{2\xi_{ii}(k)}\sqrt{2\xi_{jj}(k)}\right)\right)$, where K_X denotes the cumulative distribution function corresponding to the probability density function p_X. Consequently, the principal term in (7) defined as:

$$\begin{aligned} K_{(\Lambda+1)}\{2\widehat{\gamma_{ij}}(k)>t+1\} &= K\left\{\frac{1}{N_h}\sum_{s=1}^{N_h} X_s>t+1\right\}=K_{(\Lambda+1)}\left\{\frac{1}{N_k}\sum_{s=1}^{N_k}\Omega_s^j\Omega_s^j>t+1\right\} \\ &= K_{(\Lambda+1)}\left\{\frac{1}{N_k}\sum_{s=1}^{N_k}\frac{\Omega_s^i}{\sqrt{2\xi_{ii}(k)}}\frac{\Omega_s^j}{\sqrt{2\gamma_{jj}(k)}}>\frac{t+1}{\sqrt{2\xi_{ii}(k)}\sqrt{2\xi_{jj}(k)}}\right\} \\ &= \int_d^{\infty} k_{\overline{X}}(x)dx \end{aligned}$$

The distribution of the estimator obtained through the method of N-moments approach using the sample data is characterized by $d=(t+1)/(p\ 2\xi_{ii}(k)q\ 2\xi_{jj}(k))$ the probability density function (PDF) denoted by f(x), as defined in (4). This is because the previous terminal likelihood matches the sample's mean with a terminal value. obtained from multiplying two independent standard normal distributions.

4.1. Implementation of the Theoretical Outcomes through Simulations.

We can observe the high level of accuracy exhibited by the VOM+SAD approximations in relation to the Technique of N-moments estimation through comprehensive simulation research, even when dealing with a relatively small sample size of n = 4. In this study, we specifically focused on a bivariate normal distribution with a mean vector of (0, 0) and a covariance matrix that resulted in marginal variances of 0.49 and 0.81, as well as a covariance of 0.5. To thoroughly investigate the performance of the approximation, we considered four distinct scenarios: no contamination, contamination with an e value of 0.07. Under these controlled conditions, we were able to generate Figure 1, which clearly demonstrates the exceptional accuracy of the VOM+SAD approximations, particularly in the tails of the distribution. These tails are of utmost importance when conducting tests and constructing confidence intervals.

In Table 1, we have included certain values pertaining to Figure 1. These values represent the VOM+SAD approximations as well as the accurate values obtained through simulation.

If we calculate the relative errors of the approximation, expressed as a percentage, using the data provided in this table, as is customary, we can determine the level of accuracy.

$$100 \times \frac{|\text{Exact} - \text{Approx}|}{1 - \text{Exact}}$$

we have derived Table 2, which presents remarkably the approximate results have modest relative mistakes. This is one of the most noticeable advantages of using saddle point estimations. [14].

	$\epsilon = 0$	$\epsilon = 0.07$	$\epsilon = 0.3$
t = 0.2	0.32622	0.32527	0.32652
t = 0.4	0.24191	0.24381	0.32646
t = 0.6	0.09927	0.09733	0.08125
t = 0.8	0.05839	0.04216	0.03192
t = 1.0	0.021844	0.01937	0.01143

Table 2. The VOM+SAD approximation's relative errors are expressed as percentages.

4.2 Dependability of the Technique of N-Moments-Estimation

As we raise e or g, we can observe in Figure 2 that the pattern of distribution of the Ns-moments moment estimation technique changes is method becoming less robust. The R software used to produce create this figure can be found in the supplementary material.

4.3. Discussion

Robustness in the context of cross-variogram estimation pertains to the capacity of estimators to effectively handle outliers or data points that deviate significantly from the expected statistical model. The primary objective of robust estimators is to minimize the impact of outliers during the estimation process, ensuring that the resulting estimates are less influenced by extreme observations.

To gain insights into how these estimators handle outliers and assess their level of robustness, let's delve into the conventional approach known as the moments estimator, as briefly mentioned in the document. The moments estimator calculates the cross-variogram by averaging the product of differences between observations at different locations. However, this estimator is susceptible to outliers since it assigns equal weight to all observations, including the outliers.

In contrast, the research introduces novel and resilient estimators specifically designed to address the outlier issue in cross-variogram estimation. These estimators employ a location estimation method rather than relying on covariance estimation. While the document does not provide detailed information about these estimators, they build upon previous methodologies and incorporate saddle point techniques.

The robustness of these estimators can be evaluated based on their ability to deliver reliable estimates even when outliers are present. Robust estimators should exhibit resilience to the influence of outliers, resulting in more accurate and stable outcomes compared to non-robust estimators. In essence, these estimators should generate estimates that are less affected by extreme values, thereby producing more robust and dependable cross-variogram estimates.

Formally defining robustness with regards to cross-variogram estimation includes quantifying an estimator's resistance to exceptions and its ability to give steady and accurate estimates even within the sight of outrageous observations. This can be assessed by examining the breakdown point, which addresses the proportion of exceptions that an estimator can handle before its performance significantly deteriorates. A powerful estimator ought to have a high breakdown point, indicating its capability to accommodate a substantial percentage of exceptions without compromising the quality of estimation.

To obtain a more advanced paraphrase, please give the particular sentence or passage you would like me to rephrase.

A thorough examination utilizing statistical measures would normally be conducted to compare the performance of the suggested estimators to that of existing approaches, like those by Lark. A couple of moves toward compare these estimators are as per the following:

1. Statistical Measures:

- Accuracy: Evaluate each estimator's performance by calculating their relative error or mean squared error (MSE).

- Bias: Decide the degree of bias by contrasting the anticipated values with actual or established standards.

2. Computational Efficiency:

- The computational efficiency of any approach ought to be evaluated as far as its execution time and asset use.

3. Empirical Data/Simulation Results:

It is essential to have access to empirical data or simulation brings about order to conduct an exhaustive comparison utilizing the statistical measures mentioned before.

4. Conduct Comparative Analysis:

Utilizing these statistical measurements, compare the accuracy, computational efficiency, and bias control of Lark's approaches with those of the proposed estimators.

5. Consult Domain Experts/Literature Search:

It very well might be useful to counsel topic specialists or undertake additional literature searches on the off chance that more information about clear frameworks is required for this comparison than what is given in this paper.

By following to these methods and taking into account suitable statistical measurements in addition to empirical data/simulations when they are accessible, we can gain knowledge into the performance characteristics of the proposed estimators and how they compare to existing methods, like Lark's.

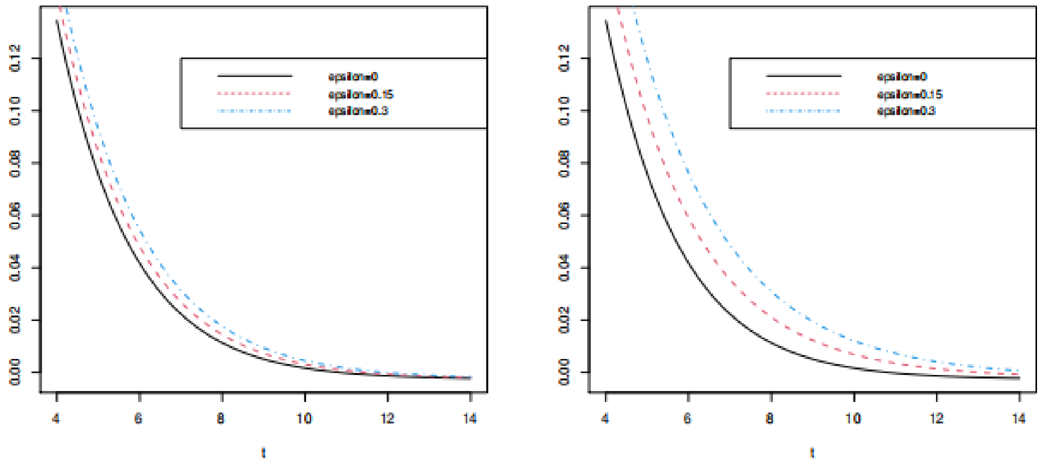

Figure 2. With a sample size of N_k = 3, the method-of-moments estimator 2cij(h) has a tail distribution.

Additionally, there are two underlying models for three: $(2 - e)N(0,2) + 2\,e\,N(0,1.33)$ and $(2 - e)N(0,2) + 2e\,N(0,1.56)$,various contamination levels e.

Conclusions

1. Main Findings and Contributions:

New robust estimators for cross-variograms are presented in this review, with an emphasis on location estimation in multivariate settings. A multivariate scale-contaminated normal model and saddle point techniques are utilized in the process to track down exact approximations for sample distributions. This technique works on the accuracy and adaptability of estimators by addressing the reliance among transformed variables in spatial observations.

2. Practical Implications:

If applied to the investigation of spatial data, the suggested techniques have real-world results. at it comes to geographical data analysis, they improve at dealing with exemptions and anomalies. These techniques further develop enlistment capacities by estimating cross-variograms more reliably and zeroing in on spatial dependency.

3. Limitations:

The suggested methodology has a handful of potential drawbacks that ought to be noted. Certain scenarios may be restricted in their applicability because of assumptions like isotropic second-order stationarity and the prerequisite for exact covariance matrix estimation. Another factor that could affect the model's validity is in the event that the actual data doesn't follow a normal distribution.

4. Future Research Areas:

Various ideas for future research are made in the research: Applications Across Domains: To evaluate these estimators' suitability in various domains, investigate their utilization outside geo statistics, for example in environmental monitoring or financial modeling.

To work on the estimator's performance in various scenarios, utilizing more sophisticated statistical methods or take a gander at different models would be beneficial. This would be a methodological enhancement.

Testing for Robustness: Validate the estimators' reliability across different scenarios by conducting complete robustness testing utilizing real-world datasets with varying levels of sophistication.

Conflicts Of Interest: The authors declare no conflict of interest

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