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Bounds on The Reduced Sombor Index of Graphs

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Abstract:

The reduced Sombor index is a modified version of the very famous Sombor Index for a graph G. In this article, the reduced Sombor index is studied on various classes of graphs and novel results on the bounds of the reduced Sombor index of graphs are obtained. The graphs with minimum reduced Sombor index are studied, and the graphs achieving certain bounds are found.

Keywords: Reduced Sombor index; topological index; graph invariant; trees; extremal problem; characterization.

1. Introduction

By a graph G, in this article, we mean an ordered pair (V_G, E_G), and the members of the sets V_G and E_G respectively are the vertices and edges of the graph. The set of vertices that are adjacent to a vertex u in G is denoted as $N_G(u)$ and called by "the open neighbourhood" of u in G. The term "closed neighbourhood" is $N_G[v] = N_G(v) \cup \{v\}$ and by a (v, w)-Path $v_1 v_2 \ldots w$ is a sequence of distinct members of the set V_G and the vertices v, w are usually known as the origin and the terminus of the path P respectively. The concept of distance between any two vertices $x, y \in V_G$ is usually defined as the length of the smallest (x, y)-path that exists in G. If $d_G(u) = 1$, then v is a pendant vertex and it is adjacent to a unique vertex in G, say u which is called a support vertex. For more on graphs and related works, the reader is referred to [1– 3].

The topological indices (also known as graph invariants) play a major role in the chemical graph theory because they are used to analyze the behaviour of the molecule structures and their inter-relationships. There are numerous topological indices available in the literature; a few of them are the Sombor index, Zagreb index, and so on. The topological indices were defined with minor and major modifications in the past and several classes of topological indices are available for the Sombor index and Zagreb index.

Given a graph, the Sombor (SO) index is defined (by Gutman[4]) as

$$SO(G) = \sum_{U,V \in E_G} \sqrt{d_G(u)^2 + d_G(v)^2}$$

The Sombor index, in recent years, received numerous attentions from academics and researchers throughout the globe [8–11]. For some recent surveys in Sombor index, one can refer to the articles [6, 7]. Chemical applications have been carried out in the articles [12, 13]. For various results and versions of Sombor index, one can refer [5, 14–17].

The reduced Sombor index is defined as

$$SO(G) = \sum_{u,v \in E_G} \sqrt{(d_G(u) - 1)^2 + (d_G(v) - 1)^2}$$

The reduced Sombor index is a recently introduced term and some of the works can be found in [18, 19]. The reduced Sombor index, for a wide collection of graphs, is studied in this article and characterized the graphs with minimum reduced Sombor index of connected and disconnected graphs.

2. Objectives

In this paper, the emphasis is on identifying the limits of the reduced Sombor index, a newly introduced topological measure applied in the exploration of chemical graph theory and related disciplines. The Sombor index is derived from the degrees of nodes in a graph, and the reduced Sombor index further elaborates on this idea. The researchers investigate the smallest and largest values of the reduced Sombor index over a wide range of graphs. By examining various graph configurations, they are able to define the specific types of graphs that reach these boundary values. This study enhances the understanding of how graph structure impacts the reduced Sombor index, offering valuable insights into its behaviour and usage in both theoretical and practical graph analyses. This structured approach not only provides thorough proofs for the claims and theorems but also uses logical reasoning to eliminate other possibilities, ensuring the reduced Sombor index bounds are well established for various graph types.

In future studies, attention can be directed toward trees showing either the minimal or maximal reduced Sombor index. Analyses may also investigate graphs with a secondary minimum or maximum. Furthermore, the constraints for trees based on the number of vertices and leaves could be assessed.

3. Methods

The initial phase of our study establishes a universal bound for any graph, proving that the reduced Sombor index (RSO) is always non-negative, with equality occurring only in specific cases like K_2 or a disjoint union of K_2 . For paths and stars with n vertices, we calculate the RSO by analyzing vertex degrees and structural properties. In paths, the linear structure simplifies the degree summation, while for stars, the central and leaf vertices have distinct degree contributions to the RSO. For wheel graphs, which consist of an outer cycle and a central hub vertex connected to all others, RSO calculation is based on the interaction between the central hub and the outer cycle, by summing the degree contributions from both the hub and outer vertices.

In graphs with cycles, the lower bound of RSO is achieved when all vertices are on a single unique cycle, derived from analyzing vertex degree distributions and their cyclical nature. Additionally, we provide exact formulas for the RSO in complete graphs and complete bipartite graphs, depending on the number of vertices and their partitions. For a complete graph with n vertices, the Reduced Sombor index (RSO) is given by $RSO(G) = n(n-1)(n-2)\sqrt{2}$. For a complete bipartite graph, the RSO is calculated as $RSO(G) = pq\sqrt{(p-1)^2+(q-1)^2}$, where $|X| = p$ and $|Y| = q$ represent the two vertex partitions. These results are derived from analyzing the degree distributions in these regular structures, which facilitate the RSO calculations.

For connected graphs with weak support vertices, we demonstrate that $RSO(G) \geq m$, where m is the number of edges and q is the number of weak support vertices. This finding stems from analyzing how pendant vertices next to weak support vertices affect the graph's structure and RSO value. Additionally, we establish a theorem identifying graphs with an RSO of exactly zero through a case-by-case analysis of connected and disconnected graphs, utilizing vertex degree properties. We also prove that for graphs with $n \geq 3$ vertices, $RSO(G) \geq n-1$, achieving equality only if the graph is a path or a star, based on graph diameters.

To establish the results and bounds, we utilized several key graph theory techniques. The degree of each vertex is crucial for determining the Reduced Sombor index (RSO). By analyzing vertex degrees across different graph structures, we derived precise bounds and values for the RSO. We conducted detailed case-by-case examinations of specific graph types—such as paths, cycles, trees, and stars to understand how their unique properties influence the RSO. Proof by contradiction helped us eliminate graph configurations that did not meet specific RSO conditions, proving the uniqueness of certain structures for given bounds. Additionally, we applied logical reasoning regarding graph connectivity and diameter to identify necessary and sufficient conditions for specific RSO values. These techniques allowed us to establish rigorous bounds on the RSO for a wide variety of graphs.

4. Results

Finding bounds of topological indices are always has been interesting and studied by researchers in the literature. For results on the bounds of various topological indices, the reader may refer [4]. For a wide collection of graphs, the bounds on the reduced Sombor index are observed in this section. These results are also used in the subsequent results where we found the maximum and minimum values of the reduced Sombor index of graphs. Let us start with the following proposition:

Observation 1 For any graph G, $RSO \geq 0$.

Proposition 2 If G has a path on n vertices, then, $RSO(G) = n - 1$.

Proposition 3 If G is a star on n vertices where $n \geq 4$, then, $RSO(G) = n - 1$.

Proposition 4 If G is a wheel on n vertices where $n \geq 4$, then, $RSO(G) = (n-1)(1+\sqrt{2})$

Proposition 5 If G is a graph with at least one cycle, then, $RSO(G) \geq 3\sqrt{2}$.

Proposition 6 If G is a graph with a cycle of length k. Then, $RSO(G) \geq k\sqrt{2}$ with equality if and only if every vertex of G lies in the unique cycle of cycle.

Proposition 7 If G is a complete graph, then, $RSO(G) = n(n-1)(n-2)\sqrt{2}$..

Proposition 8 If $G = (X, Y)$ is a complete bipartite graph with $|X| = p$ and $|Y| = q$, then $RSO(G) = pq\sqrt{(p-1)^2+(q-1)^2}$.

Proposition 9 If G is a connected graph with q weak support vertices, $RSO(G) \geq (m-q)$, where m is the number of edges in G.

Proof: Let G be a connected graph with q weak support vertices, say $u_1, u_2, \ldots, u_q$. Then, there are q pendant vertices adjacent to these weak support vertices, say $v_1, v_2, \ldots, v_q$. Then, out of the m-edges of G, these q edges $u_1v_1, u_2v_2, \ldots u_qv_q$ are adding a zero for each, 3 and the remaining $(m-q)$ edges add at least one for each implies that $RSO(G) = m-q$. But, there may be edges (out of the $(m-q)-edges$) adding more than 1 also to $RSO(G)$, for example, if G have a consecutive 3 adjacent vertices u, v, w such that $d(v) \geq 3$, then the edges uv, vw both adds at least 2 to $RSO(G)$. Thus, $RSO(G) \geq (m - q)$. □

Theorem 10 If G is a graph with the number of vertices $n \geq 2, RSO(G) \geq 0$ and the equality is true if and only if $G = K_2 \; or \; G = mK_2$.

Proof: The converse of the result is true as $G = K_2 \; or \; G = mK_2$ are graphs with RSO(G) = 0. To prove the necessary part, first assume that RSO(G) = 0. Let x be a vertex of G such that $d(x) \geq 2$. Let y be a vertex incident with x in G. Irrespective of the degree of y, the edge xy adds up a 1 to the $RSO(G)$, which implies that $RSO(G) \geq 1$, which is a contradiction to our assumption. Thus, $d(x)$ cannot be more than one. Thus, any vertex of G can have only degree one.

Case-1: G is connected.

Claim: $G = K_2$.

It is enough to prove that G has only two vertices. Suppose that G let u, v, w be 3 arbitrary vertices of G such that u and v are adjacent and v and w are adjacent. Since G is a connected graph, this assumption is possible, otherwise G would be disconnected. Thus, $d(v) \geq 2$, implying that $RSO \geq 2$, a contradiction. Thus, G has only two vertices, which means, $G = K_2$.

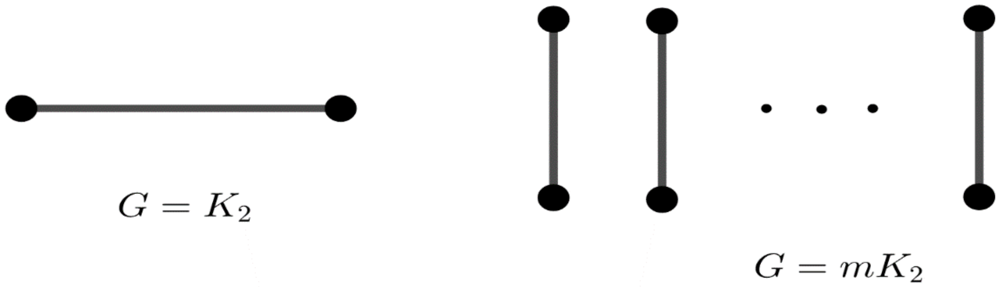


Figure 1: Graphs G = K_2 and G = mK_2

Case-2: G is disconnected.

Claim: $G = mK_2$.

Since G is disconnected, G must be the union of a finite number of connected components, say $G_1, G_2, \ldots, G_m$. For each of these connected component, by Case-1, we have that $G_i = K_2$.for $i = 1, 2, \ldots, m$. Thus, $G = mK_2$.

By Case-1 and Case-2, the theorem is proved.

Theorem 11 If G is a connected graph with the number of vertices $n \geq 3, RSO(G) \geq n - 1$ and the equality is true if and only if G is a path on n vertices, or G is a star.

Proof: Let G be a graph with at least 3 vertices. If G is a path on n vertices, then by Proposition-2, $RSO(G) = n - 1$, or if G is a star on n vertices, then by Proposition-3, $RSO(G) = n - 1$. Thus, the sufficient part is true.

To prove the other part, let G be a graph with n vertices and $RSO(G) = n - 1$. If G is a cycle, then $RSO(G) \geq n\sqrt{2} > n - 1$. Thus, G cannot even contain a cycle. Since G is a connected graph, we must have that G is a tree. We shall prove the result based on the diameter of G.

Claim-1: If $diam(G) = 2$, then G is a star.

Assume that $diam(G) = 2$. Then, the path joining any two vertices is of length two. Since the graph is connected, there must be a vertex x which is the center of all paths of length two. Otherwise, each pair of vertices are connected by a distinct path of length two which implies that G is the union of copies of P_3. Then, G is disconnected, a contradiction.

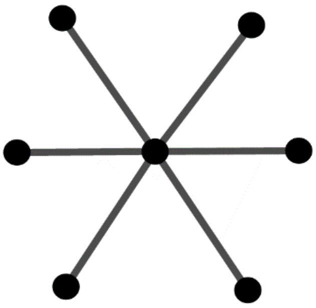
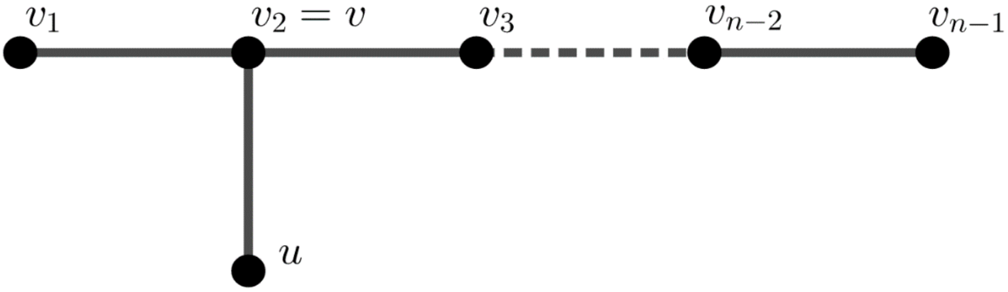


Figure 2: A star

Thus, there exists a center vertex of all paths of length two. Now, if $x_1, x_2, \ldots, x_{n-1}$ are the vertices connected to x, we claim that none of these vertices $x_i x_j, 1 \leq i \leq (n - 2), j = i + 1$ are connected. Otherwise, it forms a wheel, and by Proposition-4, $RSO(G) \geq (n - 1)$, a contradiction. Thus, x is the center vertex and all other $x_i{}'$s, $1 \leq i \leq (n - 1)$ are adjacent to x. This implies that G is a star.

Claim-2: If $diam(G) \geq 3$, then G is a path.

Assume now that $diam(G) \geq 3$. We claim that all vertices of G belong to a unique diametrical path $P : v_1 v_2, \ldots, v_n$ of G. Suppose there exists a vertex, say u which is not on the diametrical path but adjacent to a vertex, say v, which is lying on the diametrical path. Without loss of generality, let v be a support vertex of G. Assume that the diametrical path is given by $v_1 v_2 = v, \ldots v_{n-1}$. Thus, the vertex $v_2 = v$ is the only vertex that has $d(v) = 3$ and all other vertices have degree either one or two.



For this structure, the reduced Sombor index is given by

$$\begin{aligned}
RSO(G) &= \sqrt{(d(v_1)-1)^2+(d(v_2)-1)^2} + \sqrt{(d(v_2)-1)^2+(d(u)-1)^2} \\
&\quad + \sqrt{(d(v_{n-2})-1)^2+(d(v_{n-1})-1)^2} + \sqrt{(d(v_2)-1)^2+(d(v_3)-1)^2} \\
&\quad + \sum_{i=3}^{n-2}\sqrt{(d(v_i)-1)^2+(d(v_{i+1})-1)^2} \\
&= \sqrt{(1-1)^2+(3-1)^2} + \sqrt{(3-1)^2+(1-1)^2} + \sqrt{(2-1)^2+(1-1)^2} \\
&\quad + \sqrt{(3-1)^2+(2-1)^2} + \sqrt{(2-1)^2+(2-1)^2} \\
&= \sqrt{2^2} + \sqrt{2^2} + \sqrt{1^2} + \sqrt{5} + (n-4)\sqrt{1^2} \\
&= 2 + 2 + 1 + \sqrt{5} + (n-4) \\
&> n-1
\end{aligned}$$

The choice of v being adjacent to any other vertex on the diametrical path results in the same $RSO(G)$, which implies that v cannot be adjacent to a vertex on the diametrical path. So, there is no other vertex adjacent to a vertex w which is adjacent to a vertex on the diametrical path. Thus, there cannot be two different diametrical paths in G. Thus, G itself is a path. Thus, the theorem is proved.

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