

Contractive Fixed Point Theorems in Cone G- Metric Spaces

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Abstract:

In partially ordered cone G- metre spaces, we provide certain fixed point and coincidence theorems for mappings that meet contractive criteria under θ -maps.

Keywords: Cone G- metric space, Complete Cone G- metric space, Partial order Comparable elements

1. Introduction

A novel concept known as G-metric space was introduced in 2006 by Z. Mustafa and B. Sims as a generalised metric space [1]. Initiated in [2], fixed point theory in such spaces was investigated in [3]. Specifically, these studies developed the principle of Banach contraction mapping. Cone metric spaces are not a particularly new idea. Kurepa proposed the concept of metric spaces in 1934 [5], where the metric takes values in an ordered space. One can find examples of Huang-Zhang's definition [6] of a cone metric space in Chung's works [7]. These spaces were dubbed "cone-valued metric spaces" by Chung

In such spaces, additional fixed point solutions were achieved by a number of writers [9, 10]. Cone G-metric spaces, a generalisation of G-metric spaces and cone metric spaces, were recently introduced by Beg et al. [11]. They demonstrated a few fixed point theorems in terms of specific contractive requirements. Fixed points for ϕ -maps in G-metric spaces were studied by Shatanawi [4], and these fixed points are extended to cone G-metric spaces for two maps by Ozturk and Basarir [12]. Additionally, partially ordered G-metric spaces [15] and partially ordered cone metric spaces [14] have been studied with fixed point issues. In this research, we investigate common fixed point theorems in partially ordered cone G-metric spaces for mappings that meet contractive criteria associated with a nondecreasing θ -map [8,9]. Our findings are an ordered cone G-version extension of research by Ozturk and Basarir [12] and Shatanawi [4].

Preliminaries

Let P be a real Banach space and A be a subset of P . By φ we denote the zero element of P and by $\text{int} A$ the interior of A . The subset A is called an order cone if:

1. A is closed, nonempty and $A \neq \{ \varphi \}$;
2. $x, y \in A, x, y \geq 0, a, b \in A \Rightarrow xa + yb \in A$;
3. $a \in A$ and $-a \in A \Rightarrow a = \varphi$.

Definition 1 Let S be a nonempty set, P be a real Banach space and $A \subset P$ be an order cone. Suppose a mapping $T : S \times S \times S \rightarrow P$ satisfies

- (T1) $T(a,b,c) = \varphi$ if $a = b = c$;
- (T2) $\varphi < T(a,a,b)$ for all $a,b \in S$ with $a \neq b$;
- (T3) $T(a,a,b) \leq T(a,b,c)$ for all $a,b,c \in S$ with $c \neq b$;
- (T4) $T(a,b,c) = T(a,c,b) = T(b,c,a) = \dots$ (symmetry in all three variables);
- (T5) $T(a,b,c) \leq T(a,x,x) + T(x,b,c)$ for all $a,b,c,x \in S$ (rectangle inequality).

Then the function T is called a generalized cone metric on S and S is called a generalized cone metric space or, shortly, a cone G - metric space.

It is obvious that the concept of a cone G - metric space is more general than that of a G -metric space or a cone metric space. If $P = \mathbb{Q}$ and $A = [0, +\infty)$ then a cone G - metric space becomes a G -metric space.

Definition 2 Let (S, T) be a Cone G - metric space.

- (1) A sequence $\{a_n\}$ in S is said to converge to $a \in S$ if for every $z \in P$ with $\varphi \ll z$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $T(a_n, a_m, a) \ll z$.
- (2) A sequence $\{a_n\}$ in S is called a Cauchy sequence if for every $z \in P$ with $\varphi \ll z$ there is a positive integer N such that $T(a_n, a_m, a_\ell) \ll z$, for all $n, m, \ell \geq N$.
- (3) (S, T) is said to be complete if every Cauchy sequence in S is convergent in S .

Lemma 1 [11] Let S be a cone G - metric space over a normal cone, $a \in S$ and let $\{a_n\}$ be a sequence in S . Then the following are equivalent:

- (1) $\{a_n\}$ is convergent to a ;
- (2) $T(a_n, a_n, a) \rightarrow \varphi$ as $n \rightarrow \infty$;
- (3) $T(a_n, a, a) \rightarrow \varphi$ as $n \rightarrow \infty$;
- (4) $T(a_m, a_n, a) \rightarrow \varphi$ as $m, n \rightarrow \infty$.

Definition 3 Let S be a nonempty set. Then (S, T, \preceq) is called an ordered cone G -metric space if:

- (i) (S, T) is a cone G - metric space,
- (ii) (S, \preceq) is a partially ordered set.

Let (S, \preceq) be a partially ordered set.

Then $a,b \in S$ are called comparable if $a \preceq b$ or $b \preceq a$ holds.

In [13], Nashine and Samet introduced the following concept.

Let S be a non-empty set and let $Q : S \rightarrow S$ be a given mapping. For every $a \in S$, we denote by $Q^{-1}(a)$ the subset of S defined by $Q^{-1}(a) = \{ p \in S : Qp = a \}$.

Definition 4 Let (S, \preceq) be a partially ordered set and let $G, T, Q : S \rightarrow S$ be given mappings such that $GS \subseteq QS$ and $XS \subseteq QS$. We say that X and G are weakly increasing with respect to Q if for all $a \in S$, we have: $Ga \preceq Xb, \forall b \in Q^{-1}(Ga)$ and $Xa \preceq Gb, \forall b \in Q^{-1}(Xa)$. If $G = X$, we say that G is weakly increasing with respect to Q .

Definition 5 ([9,10]). Let A be an order cone. A non decreasing function $\theta : P \rightarrow P$ is called a θ -map if:

- (i) $\theta(\varphi) = \theta$ and $\varphi < \theta(\omega) < \omega$ for $\omega \in A \setminus \{ \varphi \}$,
- (ii) $\omega \in \text{in } A$ implies $\omega - \theta(\omega) \in \text{in } A$,
- (iii) if $\omega \in A \setminus \{ \varphi \}$ and $z \in \text{in } A$, then there exists $n_0 \in \mathbb{N}$ such that $\theta^n(\omega) \ll z$ for each $n \geq n_0$.

Theorem 1 Let (S, \preceq) be a partially ordered set, A be an order cone and let T be a cone G -metric on S . Let $G, Q : S \rightarrow S$ betwo mappings such that

$$T(Ga, Gb, Gc) \preceq \varphi(T(Qa, Qb, Qc)) \tag{1}$$

for all $a,b,c \in S$ with $Qa \succeq Qb \succeq Qc$, where θ is a θ -map. We suppose the following:

- (i) G is weakly increasing with respect to Q ;
- (ii) QS is a complete subspace of S ;
- (iii) S is regular.

Then G and Q have a coincidence point.

Proof. Let a_0 be an arbitrary point in S . Since $GS \subseteq QS$ (by Definition 4), we can construct a sequence $\{a_n\}$ in S defined by $Qa_{n+1} = Ga_n, \forall n \in \mathbb{N}_0$.

Now, since $a_1 \in Q^{-1}(Ga_0)$ and $a_2 \in Q^{-1}(Ga_1)$, using that G is weakly increasing with respect to Q , we obtain that $Qa_1 = Ga_0 \preceq Ga_1 = Qa_2 \preceq Ga_2 = Qa_3$.

Continuing this process, we get that $Qa_1 \preceq Qa_2 \preceq Qa_3 \preceq \dots \preceq Qa_n \preceq Qa_{n+1} \preceq \dots$.

We will prove that $\{Qa_n\}$ is a Cauchy sequence in $(Q(S), T)$. We distinguish two cases.

First case. There exists $n \in \mathbb{N}$ such that $Qa_n = Qa_{n+1}$. Using the considered contractive condition, we get $Ga_n = Ga_{n+1}$, that is, $Qa_{n+1} = Qa_{n+2}$. So, for every $m \geq n$, we have $Qa_m = Qa_n$. This implies that $\{Qa_n\}$ is a Cauchy sequence.

Second case. The successive terms of $\{Qa_n\}$ are different. From (1), we have

$$\begin{aligned} T(Qa_n, Qa_{n+1}, Qa_{n+1}) &= T(Ga_{n-1}, Ga_n, Qa_n) \\ &\leq \theta(T(Qa_{n-1}, Qa_n, Qa_n)) \\ &\leq \theta^2(T(Qa_{n-2}, Qa_{n-1}, Qa_{n-1})) \dots \end{aligned}$$

$$\leq \theta^n(T(Qa_0, Qa_1, Qa_1)).$$

Fix $z, \varphi \ll z$. According to property (iii) of function θ , there is $n_0 \in \mathbb{N}$ such that $\theta^n(T(Qa_0, Qa_1, GQa_1)) \ll z$ for $n \geq n_0$. We get that $T(Qa_n, Qa_{n+1}, Qa_{n+1}) \ll z$ for $n \geq n_0$. In a similar way, there is $N_1 \in \mathbb{N}$ such that

$$T(Qa_m, Qa_{m+1}, Qa_{m+1}) < z - \theta(c) \quad \text{for all } m \geq N_1. \quad (2)$$

We claim that

$$T(Qa_n, Qa_m, Qa_m) \ll z \quad \forall m > n \geq N_1 \quad (3)$$

and prove it by induction on m . The inequality (3) holds for $m = n + 1$ by using (2) and the fact that $z - \theta(z) < z$. Assume that (3) holds for $m = d$. For $m = d + 1$,

$$\begin{aligned} T(Qa_n, Qa_{d+1}, Qa_{d+1}) &\leq Q(Qa_n, Qa_{n+1}, Qa_{n+1}) + T(Qa_{n+1}, Qa_{d+1}, Qa_{d+1}) \\ &\ll z - \theta(z) + \theta(T(Qa_n, Qa_d, Qa_d)) \\ &\ll z - \theta(z) + \theta(z) = z. \end{aligned}$$

By induction on m , we conclude that (3) holds for all $m > n \geq N_1$. Now axiom (T5) of G -metric implies that

$$T(a_m, a_n, a_\ell) \leq T(a_m, a_n, a_n) + T(a_n, a_n, a_\ell) \ll 2z$$

holds for $m, n, \ell \geq N_1$. Hence $\{Qa_n\}$ is a G -Cauchy sequence in (QS, T) which is complete by assumption. Then, there exist $p = Qc, c \in S$ such that

$$\lim_{n \rightarrow \infty} Qa_n = p = Qc. \quad (4)$$

Since $\{Qa_n\}$ is a non-decreasing sequence and S is regular, it follows from (4) that $Qa_n \leq Qc$ for all $n \in \mathbb{N}$. Assume $Qa_n \neq Qc$.

Fix $z, \varphi \ll z$, and choose a natural number n such that

$$T(Qa_n, Qa_n, Qc) \ll \frac{z}{2} \quad \text{and} \quad T(Qa_{n+1}, Qc, Qc) \ll \frac{z}{2}.$$

Hence, we can apply the considered contractive condition to obtain

$$\begin{aligned} T(Gc, Qc, Qc) &\leq T(Gc, Ga_n, Ga_n) + T(Ga_n, Qc, Qc) \\ &\leq \theta(T(Qa_n, Qa_n, Qc)) + T(Qa_{n+1}, Qc, Qc) \quad (\text{by (1)}) \\ &< T(Qa_n, Qa_n, Qc) + T(Qa_{n+1}, Qc, Qc) \\ &\ll \frac{z}{2} + \frac{z}{2} = z. \end{aligned}$$

Since $z \in A$ is arbitrary, it follows that $T(Gc, Qc, Qc) = \varphi$ which by axiom (T2) implies that $Gc = Qc$. Then c is a coincidence point for the mappings G and Q .

Corollary 1 Let (S, \leq) be a partially ordered set, A be an order cone and suppose there is a metric T on S such that (S, T) is a complete cone G - metric space. Let $G : S \rightarrow S$ be a mapping such that

$$T(Ga, Gb, Gc) \leq \theta (T(a,b,c))$$

holds for all $a,b,c \in S$ with $a \geq b \geq c$ where θ is a θ -map. We suppose the following:

- (i) $Ga \leq G(Ga)$ for all $a \in S$;
- (ii) S is regular.

Then G has a fixed point.

Theorem 2 Let (S, \leq) be a partially ordered set, A be an order cone and suppose there is a cone G - metric T on S such that (S, T) is a complete cone G - metric space. Let $G, Q : S \rightarrow S$ be nondecreasing mappings such that for all $a,b,c \in S$ with $Qa \geq Qb \geq Qc$ there exists

$$\Theta(a,b,c) \in \{T(Qa, Qb, Qc), T(Qa, Ga, Ga), T(Qb, Gb, Gb), T(Ga, Gb, Gc)\}$$

such that

$$T(Ga, Gb, Gc) \leq \theta (\Theta(a,b,c)),$$

where θ is a θ -map. We suppose the following:

- (i) G is weakly increasing with respect to Q ,
- (ii) S is regular.

Then G and Q have a coincidence point.

Proof. Let a_0 be an arbitrary point in S . Since $GS \subseteq QS$ (by Definition 4), we can construct a sequence $\{a_n\}$ in S defined by:

$$Sa_{n+1} = Ga_n, \quad \forall n \in \mathbb{N}.$$

Now, since $a_1 \in Q^{-1}(Ga_0)$ and $a_2 \in Q^{-1}(Ga_1)$, using that G is weakly increasing with respect to Q , we obtain that

$$Qa_1 = Ga_0 \leq Ga_1 = Qa_2 \leq Ga_2 = Qa_3.$$

Continuing this process, we get that

$$Qa_1 \leq Qa_2 \leq Qa_3 \leq \dots \leq Qa_n \leq Qa_{n+1} \leq \dots.$$

If there exists $n_0 \in \{1, 2, \dots\}$ such that $\Theta(a_{n_0}, a_{n_0-1}, a_{n_0-1}) = \theta$

then it is clear that $Qa_{n_0-1} = Qa_{n_0} = Ga_{n_0-1}$ and so we are finished.

Now we can suppose

$$\Theta(a_n, a_{n-1}, a_{n-1}) > \varphi$$

for all $n \geq 1$.

Assume $Qa_n \neq Qa_{n-1}$, for each $n \in \mathbb{N}$. Thus for $n \in \mathbb{N}$, we have

$$T(Qa_n, Qa_{n+1}, Qa_{n+1}) = T(Ga_{n-1}, Ga_n, Ga_n) \leq \theta (\Theta(a_{n-1}, a_n, a_n))$$

where

$$\Theta(a_{n-1}, a_n, a_n) \in \{T(Qa_{n-1}, Qa_n, Qa_n), T(Qa_{n-1}, Ga_{n-1}, Ga_{n-1}), T(Qa_n, Ga_n, Ga_n), T(Ga_{n-1}, Qa_n, Qa_n)\}$$

$$= \{T(Qa_{n-1}, Qa_n, Qa_n), T(Qa_{n-1}, Qa_n, Qa_n), T(Qa_n, Qa_{n+1}, Qa_{n+1}), T(Qa_n, Qa_n, Qa_n)\}$$

$$= \{T(Qa_{n-1}, Qa_n, Qa_n), T(Qa_n, Qa_{n+1}, Qa_{n+1}), \varphi \}.$$

· If $\Theta(a_{n-1}, a_n, a_n) = T(Qa_n, Qa_{n+1}, Qa_{n+1})$, then

$$T(Qa_n, Qa_{n+1}, Qa_{n+1}) \leq \theta (T(Qa_n, Qa_{n+1}, Qa_{n+1}))$$

and by the property of θ we have

$$T(Qa_n, Qa_{n+1}, Qa_{n+1}) < T(Qa_n, Qa_{n+1}, Qa_{n+1})$$

which is impossible.

· If $\Theta(a_{n-1}, a_n, a_n) = \varphi$, then

$$T(Qa_n, Qa_{n+1}, Qa_{n+1}) \leq \varphi (\theta) < \varphi$$

which is a contradiction. Therefore, $\Theta(a_{n-1}, a_n, a_n) = T(Qa_{n-1}, Qa_n, Qa_n)$, and then

$$T(Qa_n, Qa_{n+1}, Qa_{n+1}) \leq \theta (T(Qa_{n-1}, Qa_n, Qa_n)).$$

Thus for $n \in \mathbb{N}$, we have

$$T(Qa_n, Qa_{n+1}, Qa_{n+1}) = T(Ga_{n-1}, Ga_n, Ga_n)$$

$$\leq \theta (T(Qa_{n-1}, Qa_n, Qa_n))$$

$$\leq \theta^2(T(Qa_{n-2}, Qa_{n-1}, Qa_{n-1}))$$

$$\leq \theta^n(T(Qa_0, Qa_1, Qa_1)).$$

By an argument similar to that in the proof of theorem, one can show that $\{Qa_n\}$ is a Cauchy sequence. Since S is

G -complete, Qa_n is convergent to $p \in S$. Now we show that $Qp = Gp$.

Since $\{Qa_n\}$ is a nondecreasing sequence and $Qa_n \rightarrow p$, by regularity of S

we have $Qa_n \leq p$ for all n . If $Qa_n = p$ for some n , then, by construction, $Qa_{n+1} = p$ and p is a fixed point. So we assume that $Qa_n \neq p$. Then, for $n \in \mathbb{N}$, we have

$$\begin{aligned} T(Qp, Qp, Gp) &\leq T(Qp, Qp, Qa_n) + T(Qa_n, Qa_n, Gp) \\ &= T(Qp, Qp, Qa_n) + T(Ga_{n-1}, Ga_{n-1}, Gp) \\ &\leq T(Qp, Qp, Qa_n) + \theta (\Theta(a_{n-1}, a_{n-1}, p)) \end{aligned}$$

where

$$\Theta(a_{n-1}, a_{n-1}, p) \in \{T(Qa_{n-1}, Qa_{n-1}, Qp), T(Qa_{n-1}, Ga_{n-1}, Ga_{n-1}), T(Qa_{n-1}, Ga_{n-1}, Ga_{n-1}), T(Ga_{n-1}, Qa_{n-1}, Qp)\}$$

$$= \{T(Qa_{n-1}, Qa_{n-1}, Qp), T(Qa_{n-1}, Qa_n, Qa_n), T(Qa_n, Qa_{n-1}, Qp)\}.$$

Fix $z, \varphi \ll z$. Choose a natural number N_1 such that $T(Qp, Qp, Qa_n) \ll \frac{z}{2}$ and

$$T(Qa_{n-1}, Qa_{n-1}, Qp) \ll \frac{z}{2}, \text{ for all } n \geq N_1. \text{ We}$$

investigate these situations as follows:

Case 1. If $\Theta(a_{n-1}, a_{n-1}, p) = T(Qa_{n-1}, Qa_{n-1}, Qp)$, then we have

$$\begin{aligned} T(Qp, Qp, Gp) &\leq T(Qp, Qp, Qa_n) + \theta(T(Qa_{n-1}, Qa_{n-1}, Qp)) \\ &< T(Qp, Qp, Qa_n) + T(Qa_{n-1}, Qa_{n-1}, Qp) \\ &\ll \frac{z}{2} + \frac{z}{2} = z. \end{aligned}$$

Case 2. If $\Theta(a_{n-1}, a_{n-1}, p) = T(Qa_{n-1}, Qa_n, Qa_n)$, then we have

$$\begin{aligned} T(Qp, Qp, Qp) &\leq T(Qp, Qp, Qa_n) + \theta(T(Qa_{n-1}, Qa_n, Qa_n)) \\ &< T(Qp, Qp, Qa_n) + T(Qa_{n-1}, Qa_n, Qa_n) \ll z. \end{aligned}$$

Case 3. If $\Theta(a_{n-1}, a_{n-1}, p) = T(Qa_n, Qa_{n-1}, Qp)$, then we have

$$\begin{aligned} T(Qp, Qp, Gp) &\leq T(Qp, Qp, Qa_n) + \theta(T(Qa_n, Qa_{n-1}, Qp)) \\ &< T(Qp, Qp, Qa_n) + T(Qa_n, Qa_{n-1}, Qp) \\ &\leq T(Qp, Qp, Qa_n) + T(Qa_n, Qa_{n-1}, Qa_{n-1}) + T(Qa_{n-1}, Qa_{n-1}, Qp) \\ &\ll z \end{aligned}$$

whenever $n \in \mathbb{N}$. Thus in all cases $T(Qp, Qp, Gp) \ll z$ for arbitrary $Z \in A$. It follows that $T(Qp, Qp, Gp) = \theta$ which implies that $Gp = Qp$.

Then p is a coincidence point for the mappings G and Q .

Conclusion

We examine common fixed point theorems in partially ordered cone G-metric spaces for mappings that meet contractive criteria associated with a nondecreasing θ -map [8,9]. The work by Shatanawi [4] and Ozturk and Basarir [12] is extended to our results in an ordered cone G-version.

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