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Toeplitz Matrices whose Elements are Coefficients of new Subclasses of Analytical Functions

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Abstract:
In this study, we explore Toeplitz matrices composed of coefficients from new subclasses and establish upper limits for the initial four determinants like of these matrices. Our findings are innovative and unique, with the only similar results being in recent works by Thomas and Halim [1], which pertain to starlike and close - to - convex functions, and by Radhika et al. [2], focusing on functions with bounded boundary rotation. Along with we have determined the Zalcman, Generalized Zalcman conjecture and Krushkal inequalities for some parameters. Keywords: Star-like function, Convex function, Coefficient bounds, Univalent functions, Toeplitz matrices, Hankel determinants, Zalcman conjecture, Generalized Zalcman conjecture and Krushkal inequalities.

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1. Introduction

Hankel matrices (and their determinants) hold significant importance in various mathematical fields and find numerous practical uses. A closely related concept to Hankel determinants is the Toeplitz determinants. Essentially, a Toeplitz matrix can be likened to an inverted Hankel matrix, as Hankel matrices have constant entries along their reverse diagonal, while Toeplitz matrices maintain constant entries along their diagonal. A comprehensive overview of the applications of Toeplitz matrices in both pure and applied mathematics can also be located in reference [7]. They possess excellent computational properties and are compatible with a wide range of algorithms and determinant computations.

Let $\mathcal{A}$ signify the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} d_n z^n. \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, represent by $\mathcal{S}$ the class of all functions in $\boldsymbol{\mathcal{A}}$ which are univalent in $\mathbb{D}$ and normalized by $f(0) = 0 = f'(0) - 1$. Also, an significant class of functions will be called $\mathcal{P}, \mathcal{P}$ defines the family of functions ϕ with the limitations that the image domain of ϕ (ϕ *is a c*onvex function *with* $Re(\phi) > 0$ in $\mathbb{D}$.) is symmetric along the real axis and star - like about $\phi(0) = 1$ with $\phi'(0) > 0$. We say that for $f_1, f_2 \in \mathcal{A}$, an f_1 is subordinate to f_2 and write $f_1(z) \prec f_2(z)$, if and only if there exists w, analytic in $\mathbb{D}$, such that $w(0) = 0, |w(z)| < 1$ for $|z| < 1$ and $f_1(z) = f_2(w(z))$. In particular, if f_2 is univalent in $\mathbb{D}$, then we have the following equivalence:

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(|z| < 1) \subset f_2(|z| < 1). \tag{1.2}$$

Two of the most important and well - investigated subclass of univalent functions are the class $\mathcal{S}^*(\alpha)$ is the class star - like functions of order $\alpha, (0 \leq \alpha < 1)$ is defined by

$$\mathcal{S}^*(\alpha) = \left\{ f \in \boldsymbol{\mathcal{A}} : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ (z \in \mathbb{D}) \right\}. \tag{1.3}$$

The class $\mathcal{K}(\alpha) \subset \mathcal{S}$ of convex functions of order $\alpha, (0 \leq \alpha < 1)$ is defined by

$$\mathcal{K}(\alpha) = \left\{ f \in \boldsymbol{\mathcal{A}} : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ (z \in \mathbb{D}) \right\}. \tag{1.4}$$

The class $\mathcal{V}(\alpha) \subset \mathcal{S}$ of closed - to - convex functions of order $\alpha, (0 \leq \alpha < 1)$ is defined by

$$\mathcal{V}(\alpha) = \left\{ f \in \boldsymbol{\mathcal{A}} : \text{Re}\left(\frac{zf'(z)}{g(z)}\right) > \alpha,\ (z \in \mathbb{D}) \right\}. \tag{1.5}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belongs to star - like functions and so on.
Let $\mathcal{P}$ be an analytic and univalent function with positive real part in $\mathbb{D}, p(0) = 0, p'(0) = 1, \text{Re}(p(z)) > 0$ and $\mathcal{P}$ maps the unit disk $\mathbb{D}$ onto a region of star - like function with respect to symmetric points of the real axis. The Taylor series expansion of such that function.

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,\ |p_n| \leq 2. \tag{1.6}$$

where all the coefficients are real and $p_1 > 0$. Throughout this paper we assume that the function p satisfies the above conditions unless otherwise stated.
By $\mathcal{S}^*(p)$ and $\mathcal{K}(p)$ we denote the following classes of function

$$\mathcal{S}^*(p) = \left\{ f \in \boldsymbol{\mathcal{A}} : \text{Re}\left(\frac{zf'(z)}{f(z)}\right) \prec p(z),\ (z \in \mathbb{D}) \right\}. \tag{1.7}$$

$$\mathcal{K}(p) = \left\{ f \in \boldsymbol{\mathcal{A}} : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec p(z),\ (z \in \mathbb{D}) \right\}. \tag{1.8}$$

The classes $\mathcal{S}^*(p), \mathcal{K}(p)$ are the extension of classical set of star - like and convex functions (e.g., see Ma and Minda [31]). These functions serve as the common source from which these subclasses inherit their properties and all took their sources from the class of Carateòdory function $\mathcal{P}$. The work of Sokól and Stankiewicz [19], introduced a class denoted as $\mathcal{SL}^*$, which comprises normalized analytic functions f in $\mathbb{D}$ satisfying the condition

$$\left|\left[\frac{zf'(z)}{f(z)}\right]^2 - 1\right| < 1$$

This class is referred to as Sokól - Stankiewicz star-like functions. Additionally, Raza and Malik [17], have determined the upper bound of the third Hankel determinant $H_3(1)$ for the class $\mathcal{SL}^*$. Furthermore, Sahoo and Patel [18] obtained some upper bound to the second Hankel determinant for the class

$$\tilde{\mathcal{R}} = \{f \in \boldsymbol{\mathcal{A}}: |f'(z)^2 - 1| < 1,\ (z \in \mathbb{D})\}. \tag{1.9}$$

Motivated by the above-mentioned works obtained by earlier researchers, Trailokya Panigrahi and JanuszSokól [12], introduce the following subclass of analytical function.

Definition 1.1. A function $f \in \boldsymbol{\mathcal{A}}$ is said to be in the class $\boldsymbol{\mathcal{AR}_\lambda^*}$, $0 \leq \lambda \leq 1$, if it satisfies the condition

$$\left|\left[\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z}\right]^2 - 1\right| < 1,\ (z \in \mathbb{D}). \tag{1.10}$$

The family $\boldsymbol{\mathcal{A}(\lambda)}$ of new subclasses in analytical functions of type $\lambda; 0 \leq \lambda \leq 1$ provides a transition from the class of star - like functions to the class of functions of bounded boundary rotation. To see this, we note that for the choice of $\lambda = 0$, we have $\boldsymbol{\mathcal{A}(\lambda)} \equiv \mathcal{S}^*(0) \equiv \mathcal{S}^*$ the class of star - like functions $f \in \boldsymbol{\mathcal{A}}$, so that $\Re\left(\frac{zf'}{f}\right) > 0$ in $\mathbb{D}$ for the choice of $\lambda = 1$, we get the family of functions $\tilde{\mathcal{R}}$ of functions $f \in \boldsymbol{\mathcal{A}}$, of bounded boundary rotation so that $\Re(f') > 0$ in $\mathbb{D}$. (For further details see [3].) Note that for $\lambda = 0$, the class $\boldsymbol{\mathcal{AR}_0^*}$, reduces to the class $\mathcal{SL}^*$, studied by Raza and Malik [17] and while $\lambda = 1$, the class $\boldsymbol{\mathcal{AR}_1^*}$**,** reduces to $\tilde{\mathcal{R}}$ studied by Sahoo and Patel [18]. In terms of subordination, relation (1.10), can be written

$$\boldsymbol{\mathcal{A}(\lambda)} = \frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} \prec p(z),\ (z \in \mathbb{D}). \tag{1.11}$$

In this research paper, we setout on an investigation into the determinants of symmetric Toeplitz matrices, where their entries represent the coefficients a_n of star - like and close - to - convex functions. Toeplitz matrices are extensively studied structured matrices with applications in various fields such as mathematics, statistics, image processing, quantum mechanics and more (e.g., Ye and Lim [4]). We recall the definition of the Hankel determinant

$$H_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix}. \tag{1.12}$$

for example,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix},\ H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix},\ H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{1.13}$$

and define the symmetric Toeplitz determinant

$$T_k(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_n & \cdots & a_{n+k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_n \end{vmatrix}. \tag{1.14}$$

for example,

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix},\ T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},\ T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix},\ T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix} \tag{1.15}$$

For $f \in \boldsymbol{\mathcal{A}}$, the problem of finding the best possible bounds for $||a_{n+1}| - |a_n||$ has a long history [3]. It is well - known [3], that $||a_{n+1}| - |a_n|| \leq C$; however, finding exact values of the constant C for $\boldsymbol{\mathcal{A}}$ and its subclasses has proved difficult. It is clear from the definition that finding estimates for $T_k(n)$ is related to finding bounds for $|a_{n+1} - a_n|$.

The pivotal moment in the exploration of univalent functions occurred in 1985, when Louis de Branges successfully proved the renowned Bieberbach conjecture, $|a_n| = n$ for $n = 2$ [22]. While this marked the conclusion of an era, numerous unresolved issues persist, including the notable Zalcman conjecture, which pertains to the coefficients a_n. One such is the Zalcman conjecture is

$$|a_n - a_{2n-1}| \leq (n-1)^2, (n \in \mathbb{N}, n \geq 2). \tag{1.16}$$

Formulated in the early 1970s, Krushkal [23]., made significant strides in this direction, employing the complex geometry of the universal Teichm ̈ üller space. In 1999, a broader perspective on the Generalized Zalcman conjecture was introduced by Ma [24]. The Generalized Zalcman conjecture is

$$|a_m a_n - a_{m+n-1}| \leq (m-1)(n-1), (m, n \in \mathbb{N}, m \geq 2, n \geq 2). \tag{1.17}$$

Ma [23] successfully resolved the open problem within the realm of star-like functions and univalent functions with real coefficients. Ravichandran and Verma, as documented in [27], also tackled and closed the issue for star - like and convex functions of specified order, as well as for functions characterized by bounded turning. Ozaki and Nunokawa, as outlined in [25], established the univalence of functions within this class, deviating from the conventional characteristics observed in other univalent functions. Unlike the broad category of star - like functions, these exhibit unique patterns, adding intrigue to their study.

The class $\mathbb{D}$, being distinct, has garnered substantial interest over the previous decades. Chapter 12 of [28], provides a comprehensive summary of the noteworthy findings in this field. We have

$$\left|a_n^p - a_2^{p(n-1)}\right| \leq 2^{p(n-1)} - 2^p, (m, n \in \mathbb{N}, m \geq 2, n \geq 2). \tag{1.18}$$

over the class $\mathbb{D}$ for the cases $n=4, p=1$ and $n=5, p=1$. This inequality was introduced by Krushkal and proven for the whole class of univalent functions [23].

2. Definitions and Preliminaries

Lemma 2.1. [17] Let $p \in \mathcal{P}$, be given by (1.6), then

$$|p_n| \leq 2, \forall n \in \mathbb{N}. \tag{2.1}$$

and

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leq 2 - \frac{1}{2}|p_1|^2. \tag{2.2}$$

Lemma 2.2. [30], [16]
Let $p \in \mathcal{P}$, be given by (1.6), then for some complex valued x with $|x| \leq 1$, some complex valued ϱ with $|\varrho| \leq 1$ and some complex valued ψ with $|\psi| \leq 1$.

We have

$$2p_2 = p_1^2 + x(4 - p_1^2) \quad (2.3)$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\varrho \quad (2.4)$$
$$8p_4 = p_1^4 + (4 - p_1^2)x[p_1^2(x^2 - 3x + 3) + 4x] \quad (2.4)$$
$$-4(4 - p_1^2)(1 - |x|^2)[p(x - 1)\varrho + \bar{x}\varrho^2 - 1 - |\varrho|^2\psi]. \quad (2.5)$$

3. Coefficient estimates for Toeplitz determinant $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$

In our first theorem we determinant a sharp bound for the coefficient body $T_2(2)$.
Theorem 3.1. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$; $(0 \leq \lambda \leq 1)$. Then we have sharp bound

$$|T_2(2)| = |a_3^2 - a_2^2| \leq \frac{4}{(\lambda+2)^2}\max\left\{1, \left|\frac{-40\lambda^3 - 60\lambda^2 + 20}{(\lambda+1)^4}\right|\right\}.$$

Proof. First note that by equating the corresponding coefficients in the equation

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z} = p(z) \tag{3.1}$$

we get

$$a_2 = \frac{p_1}{\lambda+1}, \tag{3.2}$$

$$a_3 = \frac{p_1^2(1-\lambda)}{(\lambda+1)(\lambda+2)} + \frac{p_2}{\lambda+2}, \tag{3.3}$$

$$a_4 = \frac{p_1^3(1-\lambda)^2}{(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{p_1p_2(1-\lambda)(3+2\lambda)}{(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{p_3}{\lambda+3}, \tag{3.4}$$

$$a_5 = \frac{p_1^4(1-\lambda)^3}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} + \frac{p_1^2p_2(1-\lambda)^2(3+2\lambda)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} + \frac{p_1p_3(1-\lambda)}{(\lambda+3)(\lambda+4)} + \frac{p_1^2p_2(1-\lambda)^2}{(\lambda+1)(\lambda+2)(\lambda+4)} + \frac{p_2^2(1-\lambda)}{(\lambda+2)(\lambda+4)} + \frac{p_1p_3(1-\lambda)}{(\lambda+1)(\lambda+4)} + \frac{p_4}{\lambda+4}. \tag{3.5}$$

In the view of (3.2) and (3.3), a simple computation leads to

$$a_3^2 - a_2^2 = \frac{p_2^2}{(\lambda+2)^2} + \frac{p_1^4(1-\lambda)^2}{(\lambda+2)^2(\lambda+1)^2} + \frac{2p_2p_1^2(1-\lambda)}{(\lambda+2)^2(\lambda+1)} - \frac{p_1^2}{(\lambda+1)^2}. \tag{3.6}$$

Note that, by Lemma (2.2), we may write $2p_2 = p_1^2 + x(4-p_1^2)$, where without loss of generality, we let $0 \le p_1 = p \le 2$. Substitute this into the above equation we obtain the following quadratic equation in terms of x.

$$a_3^2 - a_2^2 = \frac{(4-p^2)^2}{4(\lambda+2)^2}x^2 + \frac{p^2(4-p^2)(\lambda-3)}{2(\lambda+2)^2(\lambda+1)}x + \frac{p^2[p^2(\lambda^4-4\lambda^3-2\lambda^2+12\lambda+9)-4(\lambda+2)^2(\lambda+1)^2]}{4(\lambda+2)^2(\lambda+1)^4}. \tag{3.7}$$

Using the triangular inequality, we gain

$$|a_3^2 - a_2^2| \le \frac{(4-p^2)^2}{4(\lambda+2)^2} + \frac{p^2(4-p^2)(\lambda-3)}{2(\lambda+2)^2(\lambda+1)} + \frac{p^2[p^2(\lambda^4-4\lambda^3-2\lambda^2+12\lambda+9)+4(\lambda+2)^2(\lambda+1)^2]}{4(\lambda+2)^2(\lambda+1)^4}$$
$$= \Upsilon(p,\lambda). \tag{3.8}$$

Differentiating $\Upsilon(p,\lambda)$ with respect to p, we obtain

$$\frac{\partial(\Upsilon(p,\lambda))}{\partial p} = p\left[\frac{16p^2 + (2\lambda^2-8\lambda-8)}{(\lambda+2)^2(\lambda+1)^2}\right] \tag{3.9}$$

Setting $\frac{\partial(\Upsilon(p,\lambda))}{\partial p} = 0$ yields either $p = 0$ or

$$p^2 = -\frac{2\lambda^2-8\lambda-8}{16} \tag{3.10}$$

but $-[2\lambda^2-8\lambda-8] < 0$ for $0 \le \lambda \le 1$.

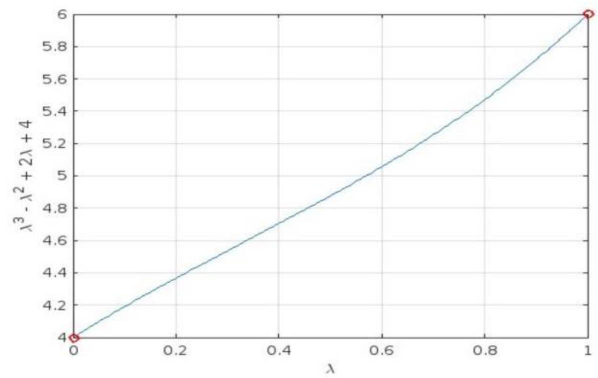

Figure 1. Graph of the bound $-2\lambda^2 + 8\lambda + 8$ in the range $\lambda \in [0,1]$.

Therefore, the maximum value of $|a_3^2 - a_2^2|$ is attained at the end points $p_1 = p \in [0,2]$.
For $p_1 = 0, p_2 = 2x$. Then, we have (3.6).

$$|a_3^2 - a_2^2| = \frac{4|x|^2}{(\lambda+2)^2} \le \frac{4}{(\lambda+2)^2} \tag{3.11}$$

For $p_1 = p_2 = 2$, we get

$$a_2 = \frac{2}{\lambda+2} \tag{3.12}$$

$$a_3 = \frac{4(1-\lambda)}{(\lambda+2)(\lambda+1)} + \frac{2}{\lambda+2} \tag{3.13}$$

which yields,

$$|a_3^2 - a_2^2| \le \left|\frac{-40\lambda^3 - 60\lambda^2 + 20}{(\lambda+1)^4(\lambda+2)^2}\right| \tag{3.14}$$

The result is sharp for the functions given by

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z} = \frac{1+z}{1-z} \tag{3.15}$$

Remark 3.2. Theorem (3.1), for $\lambda = 0$ yields the bound $|a_3^2 - a_2^2| \le 5$ for the class of star - like function $\mathcal{S}^*$ conforming the bound obtained by Thomous and Halim [1] and for $\lambda = 1$ yields the bound $|a_3^2 - a_2^2| \le \frac{5}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathcal{R}}$ conforming the bound obtained by Radhika et al. [2].

In our next theorem, we determine an upper bound for the coefficient body $T_2(3)$.

Theorem 3.3. Let f given by (1.1) be in the class $\boldsymbol{\mathcal{A}(\lambda)}$; $(0 \le \lambda \le 1)$. Then we have sharp bound

$$|T_2(3)| = |a_4^2 - a_3^2| \le \max\left\{|64R_1(\lambda) - 16R_2(\lambda)|, \frac{4}{(\lambda+2)^2}\right\}$$

$$R_1(\lambda) = \frac{\lambda^4 - 14\lambda^3 + 73\lambda^2 - 168\lambda + 144}{16(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}$$

$$R_2(\lambda) = \frac{\lambda^2 - 6\lambda + 9}{4(\lambda+2)^2(\lambda+1)^2}$$

Proof. First note that by equating the corresponding coefficient in the equation (3.1). In the view of (3.3), (3.4) and applying Lemma (2.2), denoting $X = 4 - p^2$ and $Y = (1 - |x|^2)\varrho$,
where $0 \le p \le 2$ and $|\varrho| < 1$, we get

$$
\begin{aligned}
a_4^2 - a_3^2 = &\left[\frac{\lambda^4 - 14\lambda^3 + 73\lambda^2 - 168\lambda + 144}{16(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}\right] p_1^6 + \left[\frac{-\lambda^2 + 6\lambda - 9}{4(\lambda+2)^2(\lambda+1)^2}\right] p_1^4 + \frac{X^2Y^2}{4(\lambda+3)^2} + \frac{p_1 x^2 X^2 Y}{4(\lambda+3)^2} \\
&+ \left[\frac{-\lambda^2 + 2\lambda + 5}{2(\lambda+3)^2(\lambda+2)(\lambda+1)}\right] p_1 x X^2 Y \\
&+ \left[\frac{\lambda^2 - 7\lambda + 12}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}\right] p_1^3 XY + \frac{p_1^2 x^4 X^2}{16(\lambda+3)^2} \\
&+ \left[\frac{-\lambda^2 + 2\lambda + 5}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}\right] p_1^2 x^3 X^2 \\
&+ \left[\frac{\lambda^4 - 4\lambda^3 - 6\lambda^2 + 20\lambda + 25}{4(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}\right] p_1^2 x^2 X^2 \\
&+ \frac{x^2X^2}{4(\lambda+2)^2} + \left[\frac{\lambda^2 - 7\lambda + 12}{8(\lambda+3)^2(\lambda+2)(\lambda+1)}\right] p_1^4 x^2 X \\
&+ \left[\frac{-\lambda^4 + 9\lambda^3 - 21\lambda^2 - 11\lambda + 60}{4(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}\right] p_1^4 x X \\
&+ \left[\frac{3 - \lambda}{2(\lambda+1)(\lambda+2)^2}\right] p_1^2 x X.
\end{aligned}
$$

As in the proof of theorem (3.1). Note that, by Lemma (2.2), where without loss of generality we let $0 \le p_1 = p \le 2$. Substitute this into the above equation, we get the following quadratic equation in terms of x.

$$
\begin{aligned}
\left|a_4^2 - a_3^2\right| \le &\frac{(2-p)^2(4-p^2)^2}{16(\lambda+3)^2}|x|^4 + \frac{(-\lambda^2 + 2\lambda + 5)(4-p^2)(p^2 - 2p)}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}|x|^3 \\
&+ \left[\frac{(\lambda^2 - 7\lambda + 12)(4-p^2)(p-2)p^3}{8(\lambda+3)^2(\lambda+2)(\lambda+1)} + \left[\frac{\lambda^4 - 4\lambda^3 - 6\lambda^2 + 20\lambda + 25}{4(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}\right] p^2(4-p^2)\right]|x|^2 \\
&+ \left[\frac{p(4-p^2)^2}{4(\lambda+3)^2} - \left[\frac{\lambda^2 + 2\lambda - 1}{4(\lambda+3)^2(\lambda+2)^2}\right](4-p^2)^2\right]|x|^2 \\
&+ \left[\frac{(3-\lambda)(4-p^2)p^2}{2(1+\lambda)(2+\lambda)^2} + \frac{(-\lambda^2 + 2\lambda + 5)(4-p^2)p}{2(1+\lambda)(2+\lambda)(3+\lambda)^2} + \frac{-\lambda^4 + 9\lambda^3 - 21\lambda^2 - 11\lambda + 60}{4(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2} p^4(4-p^2)\right]|x| \\
&+ |R_1(\lambda)p^6 - R_2(\lambda)p^4| + \left[\frac{\lambda^2 - 7\lambda + 12}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}\right] p^3(4-p^2) + \frac{(4-p^2)^2}{4(\lambda+3)^2}. \\
= &\, \Theta(p, |x|)
\end{aligned}
$$

Where,

$$
R_1(\lambda) = \frac{\lambda^4 - 14\lambda^3 + 73\lambda^2 - 168\lambda + 144}{16(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}
$$

$$
R_2(\lambda) = \frac{\lambda^2 - 6\lambda + 9}{4(\lambda+2)^2(\lambda+1)^2}
$$

It is necessary to prove that the maximum value of $\Theta(p,|x|)$ on $[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\Theta(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\Theta(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p=p_0=2$, which is contradiction. Thus, for the maximum of $\Theta(p,|x|)$, we need only to consider the end points of $[0,2]\times[0,1]$. For $p=0$, we obtain

$$\Theta(0,|x|)=\frac{4|x|^4}{(\lambda+3)^2}-\frac{4(\lambda^2+2\lambda-1)|x|}{(\lambda+3)(\lambda+2)^2}+\frac{4}{(\lambda+3)^2}\leq\frac{4}{(\lambda+2)^2} \tag{3.16}$$

For $p=2$, we obtain

$$\Theta(2,|x|)=|64R_1(\lambda)-16R_2(\lambda)| \tag{3.17}$$

For $|x|=0$, we get

$$\Theta(p,0)=|R_1(\lambda)p^6-R_2(\lambda)p^4|+\left[\frac{\lambda^2-7\lambda+12}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}\right]p_1^3(4-p^2)+\frac{(4-p^2)^2}{4(\lambda+3)^2} \tag{3.18}$$

which has the maximum value $|R_1(\lambda)p^6-R_2(\lambda)p^4|$ on $[0,2]$.

For $|x|=1$, we gain

$$\begin{aligned}\Theta(p,1)=&\frac{(2-p)^2(4-p^2)^2}{16(\lambda+3)^2}+\frac{(-\lambda^2+2\lambda+5)(4-p^2)(p^2-2p)}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}\\&+\left[\frac{(\lambda^2-7\lambda+12)(4-p^2)(p-2)p^3}{8(\lambda+3)^2(\lambda+2)(\lambda+1)}+\left[\frac{\lambda^4-4\lambda^3-6\lambda^2+20\lambda+25}{4(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}\right]p^2(4-p^2)\right]\\&+\left[\frac{p(4-p^2)^2}{4(\lambda+3)^2}-\left[\frac{\lambda^2+2\lambda-1}{4(\lambda+3)^2(\lambda+2)^2}\right](4-p^2)^2\right]\\&+\left[\frac{(3-\lambda)(4-p^2)p^2}{2(1+\lambda)(2+\lambda)^2}+\frac{(-\lambda^2+2\lambda+5)(4-p^2)p}{2(1+\lambda)(2+\lambda)(3+\lambda)^2}+\frac{-\lambda^4+9\lambda^3-21\lambda^2-11\lambda+60}{4(\lambda+3)^2(\lambda+2)^2(\lambda+1)^2}p^4(4-p^2)\right]\\&+|R_1(\lambda)p^6-R_2(\lambda)p^4|+\left[\frac{\lambda^2-7\lambda+12}{4(\lambda+3)^2(\lambda+2)(\lambda+1)}\right]p^3(4-p^2)+\frac{(4-p^2)^2}{4(\lambda+3)^2}.\end{aligned}$$

which has the maximum values $|64R_1(\lambda)-16R_2(\lambda)|$ for $p=2$ and $\frac{4}{(\lambda+2)^2}$ for $p=0$

Remark 3.4. Theorem (3.3), for $\lambda=0$ yields the bound $|a_4^2-a_3^2|\leq 7$ for the class of star - like function $\mathcal{S}^*$ conforming the bound obtained by Thomous and Halim [1]. and for $\lambda=1$ yields the bound $|a_4^2-a_3^2|\leq\frac{4}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathcal{R}}$ conforming the bound obtained by Radhika et al. [2].

Theorem 3.5. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}(\lambda)}$; $(0\leq\lambda\leq 1;\lambda\neq\lambda_0)$. Then we have sharp bound

$$|T_3(2)|=\left|\left(\begin{vmatrix}a_2&a_3&a_4\\a_3&a_2&a_3\\a_4&a_3&a_2\end{vmatrix}\right)\right|\leq\begin{cases}\max\left\{|R(\lambda)U(\lambda)|,\frac{8|R(\lambda)|}{(\lambda+2)^2}\right\};\ \text{if }\lambda\neq\lambda_0\\\max\left\{|U(\lambda)B(\lambda)|,\frac{8|B(\lambda)|}{(\lambda+2)^2}\right\};\ \text{if }\lambda=\lambda_0\end{cases}$$

Where $\lambda_0 \approx 0.5$ is the positive root of the polynomial $24x - 12 = 0$,

$$R(\lambda) = \frac{24x - 12}{(\lambda+3)(\lambda+2)(\lambda+1)} \tag{3.19}$$

$$U(\lambda) = \frac{4(8\lambda^2 + 32\lambda - 18)}{(\lambda+3)(\lambda+2)^2(\lambda+1)^2} \tag{3.20}$$

$$B(\lambda) = \frac{4\lambda^2 - 4\lambda + 36}{(\lambda+3)(\lambda+2)(\lambda+1)} \tag{3.21}$$

Proof. Write

$$|T_3(2)| = |a_2^3 - 2a_2a_3^2 + 2a_3^2a_4 - a_2a_4| \tag{3.22}$$

$$= |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)| \tag{3.23}$$

Using the same techniques as the theorem (3.1), one can obtain with simple computations that

$$|a_2 - a_4| \leq |R(\lambda)| \text{ for } \lambda \neq \lambda_0. \tag{3.24}$$

We need to show that

$$|a_2^3 - 2a_3^2 + a_2a_4| \leq |U(| \lambda)|. \tag{3.25}$$

In the view of (3.2),(3.3), (3.5) and Lemma (2.2), where we denote $X = 4 - p^2$ and $Y = (1 - |x|^2)\varrho$, where $0 \leq p \leq 2$ and $|\varrho| < 1$, one may easily get

$$|a_2^2 - 2a_3^2 + a_2a_4| = \left|\left[\frac{-\lambda^3 + \lambda^2 + 16\lambda - 30}{4(\lambda+3)(\lambda+2)^2(\lambda+1)^2}\right]p_1^4 + \frac{p_1^2}{(\lambda+1)^2} - \frac{p_1^2Xx^2}{4(\lambda+3)(\lambda+1)}\right| + \left|\frac{XYp_1}{2(\lambda+3)(\lambda+1)} - \frac{X^2x^2}{2(\lambda+2)^2} + \left[\frac{\lambda^3 + 2\lambda^2 - 9\lambda - 8}{2(\lambda+3)(\lambda+2)^2(\lambda+1)^2}\right]p_1^2Xx\right|$$

Applying the triangle inequality and assuming that $p_1 = p$, where $0 \leq p \leq 2$, we obtain

$$\begin{aligned}|a_2^2 - 2a_3^2 + a_2a_4| &\leq \left[\frac{p^2(4-p^2)}{4(\lambda+3)(\lambda+1)} + \frac{p^2(4-p^2)}{2(\lambda+3)(\lambda+1)} + \frac{(4-p^2)^2}{2(\lambda+2)^2}\right]|x|^2 \\ &+ \left[\left[\frac{\lambda^3 + 2\lambda^2 - 9\lambda - 8}{2(\lambda+3)(\lambda+2)(\lambda+1)^2}\right]p(4-p^2)\right]|x| + \frac{p(4-p^2)}{2(\lambda+3)(\lambda+1)} + \frac{p^2}{(\lambda+1)^2} \\ &+ \left[\frac{-\lambda^3 + \lambda^2 + 16\lambda - 30}{4(\lambda+3)(\lambda+2)^2(\lambda+1)^2}\right]p^4 \\ &= \Psi(p, |x|)\end{aligned}$$

We have to prove that the maximum value of $\Psi(p, |x|)$ on $[0,2] \times [0,1]$. First, assume that there is a maximum at an interior point $\Psi(p_0, |x_0|)$ of $[0,2] \times [0,1]$. Differentiating $\Psi(p, |x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$ which is contradiction. Thus, for the maximum of $\Psi(p, |x|)$, we need only to consider the end points of $[0,2] \times [0,1]$. For $p = 0$, we obtain

$$\Psi(0, |x|) = \frac{8|x|^2}{(\lambda+2)^2} \leq \frac{8}{(\lambda+2)^2} \tag{3.26}$$

For $p = 2$, we have

$$\Psi(2, |x|) = \frac{4(8\lambda^2 + 32\lambda - 18)}{(\lambda + 1)(\lambda + 2)^2(3 + \lambda)} = U(\lambda) \tag{3.27}$$

For $x = 0$, we brought

$$\Psi(p, 0) = \left|\frac{p^2}{(\lambda + 1)^2} + \frac{p(4 - p^2)}{2(\lambda + 3)(\lambda + 1)} + \frac{-\lambda^3 + \lambda^2 + 16\lambda - 30}{4(\lambda + 3)(\lambda + 2)^2(\lambda + 1)^2} p^4\right| \tag{3.28}$$

which has the maximum value $\Psi(p, 0) = U(\lambda)$ attained at the end point $p = 2$. Hence, for $|x| = 1$, we obtain

$$\Psi(p, 1) = \left[\frac{p^2(4 - p^2)}{4(\lambda + 3)(\lambda + 1)} + \frac{p^2(4 - p^2)}{2(\lambda + 3)(\lambda + 1)} + \frac{(4 - p^2)^2}{2(\lambda + 2)^2}\right] + \left[\frac{\lambda^3 + 2\lambda^2 - 9\lambda - 8}{2(\lambda + 3)(\lambda + 2)^2(\lambda + 1)^2}\right] p(4 - p^2) + \frac{p(4 - p^2)}{2(\lambda + 3)(\lambda + 1)} + \frac{p^2}{(\lambda + 1)^2} + \left[\frac{-\lambda^3 + \lambda^2 + 16\lambda - 30}{4(\lambda + 3)(\lambda + 2)^2(\lambda + 1)^2}\right] p^4.$$

which has maximum $\Psi(p, 1) = \frac{8}{(\lambda+2)^2}$ at $p = 0$ and $\Psi(p, 1) = U(\lambda)$ at $p = 2$.

$$|a_2^2 - 2a_3^2 + a_2 a_4| \le \max\left\{|U(\lambda)|, \frac{8}{(\lambda + 2)^2}\right\}. \tag{3.29}$$

Thus

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)| \le \max\left\{|R(\lambda)U(\lambda)|, \frac{8|R(\lambda)|}{(\lambda + 2)^2}\right\}. \tag{3.30}$$

For the case $\lambda = \lambda_0$, we compute $|a_2 - a_4|$ as follows

$$|a_2 - a_4| = \left|\frac{p_1}{\lambda + 1} - \left[\frac{p_1^3(1 - \lambda)^2}{(\lambda + 3)(\lambda + 2)(\lambda + 1)} + \frac{p_1 p_2(1 - \lambda)(3 + 2\lambda)}{(\lambda + 3)(\lambda + 2)(\lambda + 1)} + \frac{p_3}{\lambda + 3}\right]\right| \tag{3.31}$$

Since each $|p_n| \le 2$, an application of triangle inequality shows that

$$|a_2 - a_4| \le |B(\lambda)| = \frac{4\lambda^2 - 4\lambda + 36}{(\lambda + 3)(\lambda + 2)(\lambda + 1)}. \tag{3.32}$$

Therefore,

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)| \le \max\left\{|U(\lambda)B(\lambda)|, \frac{8|B(\lambda)|}{(\lambda + 2)^2}\right\}. \tag{3.33}$$

This completes the proof the Theorem (3.5).

Remark 3.6. Theorem (3.5), for $\lambda = 0$ yields the bound $|T_3(2)| \le 12$ for the class of star - like function $\mathcal{S}^*$ conforming the bound obtained by Thomous and Halim [1]. and for $\lambda = 1$ yields the bound

$|T_3(2)| \leq \frac{8}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathcal{R}}$ conforming the bound obtained by Radhika et al. [2].

Theorem 3.7. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}(\lambda)}$; $0 \leq \lambda \leq 1$. Then we have sharp bound

$$|T_3(1)| = \left|\left(\begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}\right)\right| \leq \max\left\{1 + \frac{1}{4(\lambda+2)^2}, |N(\lambda)|\right\}$$

Where,

$$N(\lambda) = \frac{\lambda^5 + 7\lambda^4 + 7\lambda^3 - 11\lambda^2 - 44\lambda + 32}{(\lambda+2)^2(\lambda+1)^3}$$

Proof. Expanding the determinant by using equation (3.1), we get (3.2) and (3.3), we have

$$\begin{aligned} T_3(1) &= 1 + 2a_2^2(a_3 - 1) - a_3^2 \\ &= 1 + \frac{2p_1^2}{(\lambda+1)^2}\left(\frac{p_1^2(1-\lambda)}{(\lambda+2)(\lambda+1)} + \frac{p_2}{\lambda+2} - 1\right) \\ &- \left[\frac{p_1^4(1-\lambda)}{(\lambda+2)^2(\lambda+1)^2} + \frac{p_2^2}{(\lambda+2)^2} + \frac{2(1-\lambda)p_1^2 p_2}{(\lambda+2)^2(\lambda+1)}\right]. \\ &= 1 + \left[\frac{-\lambda^3 + \lambda^2 + \lambda + 15}{4(\lambda+2)^2(\lambda+1)^3}\right]p_1^4 + \left[\frac{\lambda^2+1}{2(\lambda+2)^2(\lambda+1)^2}\right]p_1^2 xX \\ &- \frac{2p_1^2}{(\lambda+1)^2} - \frac{x^2X^2}{4(\lambda+2)^2}. \end{aligned}$$

Note that, by Lemma (2.2), without loss of generality we let $0 \leq p_1 = p \leq 2$. Substitute this into the above equation and applying the triangle inequality, we obtain the following quadratic equation in terms of x.

$$\begin{aligned} T_3(1) &\leq \left[\frac{(4-p^2)^2}{4(\lambda+2)^2}\right]|x|^2 + \left[\frac{p^2(4-p^2)(\lambda^2+1)}{2(\lambda+2)^2(\lambda+1)^2}\right]|x| \\ &\quad + \left[1 + \frac{8(\lambda+2)^2(\lambda+1) + (-\lambda^3+\lambda^2+\lambda+15)p^2}{4(\lambda+2)^2(\lambda+1)^3}p^2\right]. \\ T_3(1) &\leq \left[\frac{(4-p^2)}{4(\lambda+2)^2}\right] + \left[\frac{p^2(4-p^2)(\lambda^2+1)}{2(\lambda+2)^2(\lambda+1)^2}\right] \\ &\quad + \left[1 + \frac{8(\lambda+2)^2(\lambda+1) + (-\lambda^3+\lambda^2+\lambda+15)p^2}{4(\lambda+2)^2(\lambda+1)^3}p^2\right]. \\ &\quad = \Xi(p,\lambda). \end{aligned}$$

Differentiating $\Xi(p,\lambda)$ with respect to p, we obtain

$$\frac{\partial(\Xi(p,\lambda))}{\partial p} = p\left[\frac{p^2(-2\lambda^3 + 2\lambda^2 + 2\lambda + 14) + (4\lambda^3 + 12\lambda^2 + 24\lambda + 16)}{(\lambda+2)^2(\lambda+1)^3}\right]. \tag{3.34}$$

Setting $\frac{\partial(\Xi(p,\lambda))}{\partial p} = 0$ yields either $p = 0$ or

$$p^2 = \frac{-4\lambda^3 - 12\lambda^2 - 24\lambda - 16}{-2\lambda^3 + 2\lambda^2 + 2\lambda + 14}. \tag{3.35}$$

but $-4\lambda^3 - 12\lambda^2 - 24\lambda - 16 < 0$ for $0 \le \lambda \le 1$ and
therefore the maximum value of $T_3(1)$ is attained at the end points $p_1 = p \in [0,2]$.

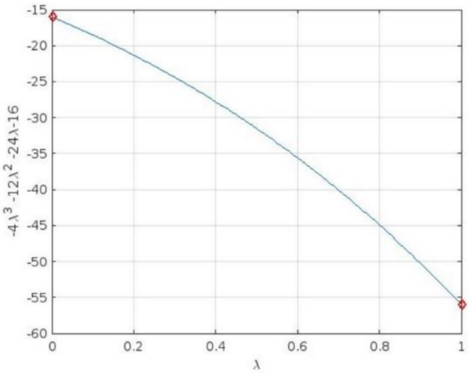

Figure 2. Graph of the bound $-4\lambda^3 - 12\lambda^2 - 24\lambda - 16$ in the range $\lambda \in [0,1]$.

For $p_1 = 0$ and $p_2 = 2x$. Then, we have

$$|1 + 2a_2^2(a_3 - 1) - a_3^2| = 1 - \frac{4|x|^2}{(\lambda + 2)^2} \le 1 + \frac{4}{(\lambda + 2)^2}. \tag{3.36}$$

In the view of (3.2), (3.3) and $p_1 = p_2 = 2$, we get

$$2a_2^2a_3 = \frac{32(1-\lambda)}{(\lambda+2)(\lambda+1)^3} + \frac{16}{(\lambda+2)(\lambda+1)^2}. \tag{3.37}$$

$$-2a_2^2 = \frac{-8}{(\lambda+1)^2}. \tag{3.38}$$

$$-a_3^2 = -\frac{16(1-\lambda)^2}{(\lambda+2)(\lambda+1)^2} - \frac{4}{(2+\lambda)^2} - \frac{16(1-\lambda)}{(\lambda+2)^2(\lambda+1)}. \tag{3.39}$$

Substitute the values of (3.37), (3.38) and (3.39) in (3.22), we may get

$$|1 + 2a_2^2(a_3 - 1) - a_3^2| \le \left|\frac{\lambda^5 + 7\lambda^4 + 7\lambda^3 - 11\lambda^2 - 44\lambda + 32}{(\lambda+2)^2(\lambda+1)^3}\right| \le N(\lambda). \tag{3.40}$$

where,

$$N(\lambda) = \left|\frac{\lambda^5 + 7\lambda^4 + 7\lambda^3 - 11\lambda^2 - 44\lambda + 32}{(\lambda+2)^2(\lambda+1)^3}\right|. \tag{3.41}$$

This completes the proof of the theorem (3.7).

Remark 3.8. Theorem (3.7), for $\lambda = 0$ yields the bound $|1 + 2a_2^2(a_3 - 1) - a_3^2| \le 8$ for the class of star - like function $\mathcal{S}^*$ conforming the bound obtained by Thomous and Halim [1]. and for $\lambda = 1$ yields the bound $|1 + 2a_2^2(a_3 - 1) - a_3^2| \le \frac{13}{9}$ for the class of functions with bounded boundary rotation $\tilde{\mathcal{R}}$ conforming the bound obtained by Radhika et al. [2].

4. Zalcman Conjecture For the class $\mathcal{A}(\lambda)$

Theorem 4.1. Let f given by (1.1), be in the class $\mathcal{A}(\lambda)$; $(0 \leq \lambda \leq 1)$. Then we have sharp bound

$$|a_2^2 - a_3| \leq \max\left\{\frac{2}{(\lambda+2)}, \frac{\mathcal{T}(\lambda)}{(\lambda+1)^2(\lambda+2)}\right\} \tag{4.1}$$

where

$$\mathcal{T}(\lambda) = 2(\lambda^2+1) \tag{4.2}$$

Proof. First note that by equating the corresponding coefficients in the equation (3.1). We get, in the view of (3.2) and (3.3), a simple computation leads to

$$a_2^2 - a_3 = \left[\frac{p_1}{\lambda+1}\right]^2 - \left[\frac{p_1^2(1-\lambda)}{(\lambda+1)(\lambda+2)} + \frac{p_2}{\lambda+2}\right]$$

$$= \frac{p^2{}_1}{(\lambda+1)^2} - \frac{p_1^2(1-\lambda)}{(\lambda+1)(\lambda+2)} - \frac{p_2}{\lambda+2} \quad . \tag{4.3}$$

Note that, by Lemma (2.2), we may write $2p_2 = p_1^2 + x(4 - p_1^2)$, we can easily get

$$= \left[\frac{\lambda^2+1}{2(\lambda+1)(\lambda+2)}\right] p_1^2 - \frac{Xx}{2(\lambda+2)}. \tag{4.4}$$

Without loss of generality, we let $0 \leq p_1 = p \leq 2$. Substitute this into the above equation, we obtain the following quadratic equation in terms of x.

$$|a_2^2 - a_3| = \frac{4-p^2}{2(\lambda+2)}|x| + \left[\frac{\lambda^2+1}{2(\lambda+1)^2(\lambda+2)}\right] p_1^2. \tag{4.5}$$

$$= ¥(p, |x|). \tag{4.6}$$

We required to prove that the maximum value of $¥(p, |x|)$ on $[0,2] \times [0,1]$. First, assume that there is a maximum at an interior point $¥(p_0, |x_0|)$ of $[0,2] \times [0,1]$. Differentiating $¥(p, |x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$ which is contradiction.

Thus, for the maximum $¥(p, |x|)$, we must consider the end points of $[0,2] \times [0,1]$. For $p = 0$, we obtain

$$¥(0, |x|) = \frac{4}{2(\lambda+2)}|x|^2 \leq \frac{2}{\lambda+2}. \tag{4.7}$$

For $p = 2$, we owe

$$¥(2, |x|) = \left[\frac{2(\lambda^2+1)}{(\lambda+1)^2(\lambda+2)}\right]. \tag{4.8}$$

For $|x| = 0$, we receive

$$¥(p, 0) = \left[\frac{\lambda^2+1}{2(\lambda+1)^2(\lambda+2)}\right] p^2. \tag{4.9}$$

which has maximum value $\frac{|\mathcal{T}(\lambda)|}{(\lambda+1)^2(\lambda+2)}$ attained at the end point $p = 2$.
For $|x| = 1$, we obtained

$$¥(p,1) = \left[\frac{\lambda^2+1}{2(\lambda+1)^2(\lambda+2)}\right]p^2 + \frac{4-p^2}{2(\lambda+1)}. \tag{4.10}$$

which is maximum value of $¥(p,1) = \frac{2}{\lambda+1}$ at $p = 0$ and $\frac{|\mathcal{T}(\lambda)|}{(\lambda+1)^2(\lambda+2)}$ at $p = 2$. Hence

$$|a_2^2 - a_3| \leq \max\left\{\frac{2}{4(\lambda+2)}, \frac{|\mathcal{T}(\lambda)|}{(\lambda+1)^2(\lambda+2)}\right\} \tag{4.11}$$

where

$$\mathcal{T}(\lambda) = 2(\lambda^2+1) \tag{4.12}$$

Theorem 4.2. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda})$; $(0 \leq \lambda \leq 1)$. Then we have sharp bound

$$|a_3^2 - a_5| \leq \max\left\{\frac{(3+2\lambda)4}{(\lambda+2)(\lambda+4)} + \frac{6}{\lambda+4}, \frac{|\mathcal{G}(\lambda)|}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right\}$$

where,

$$\mathcal{G}(\lambda) = 2(\lambda^5 - 7\lambda^4 + 15\lambda^3 + 12\lambda^2 - 104\lambda + 96) \tag{4.13}$$

Proof. First note that by equating the corresponding coefficients in the equation (3.1). We get, in the view of (3.3), (3.5) and Lemma (2.2), we may write $2p_2 = p_1^2 + x(4-p_1^2)$, $Y = (1-|x|^2)\varrho$, a simple computation leads to

$$\begin{aligned} a_3^2 - a_5 = &\left[\frac{\lambda^5 - 7\lambda^4 + 15\lambda^3 + 12\lambda^2 - 104\lambda + 96}{8(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right]p_1^4 \\ &+ \left[\frac{3+2\lambda}{4(2+\lambda)(4+\lambda)}\right]x^2X^2 + \left[\frac{-\lambda^2+8\lambda+17}{8(1+\lambda)(3+\lambda)(4+\lambda)}\right]p_1^2x^2X \\ &+ \left[\frac{-7\lambda^3+2\lambda^2+35\lambda+58}{8(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)}\right]p_1^2xX + \left[\frac{\lambda^2-2\lambda-7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p_1XY - \frac{p_1^2x^3X}{8(\lambda+4)} \\ &- \frac{x^2X}{2(\lambda+4)} + \frac{xXYp_1}{2(\lambda+4)} + \frac{XY\bar{x}}{2(\lambda+4)}. \end{aligned}$$

Without loss of generality, we let $0 \leq p_1 = p \leq 2$. Substitute this into the above equation, we obtain the following quadratic equation in terms of x.

$$
\begin{aligned}
|a_3^2 - a_5| \leq & \left[\frac{p^2(4-p^2)}{8(\lambda+4)} - \frac{p(4-p^2)}{2(\lambda+4)}\right]|x|^3 \\
& + \left[\frac{(3+2\lambda)(4-p^2)^2}{4(2+\lambda)(4+\lambda)} + \frac{(-\lambda^2+8\lambda+17)(4-p^2)p^2}{8(1+\lambda)(\lambda+3)(\lambda+4)} - \frac{(4-p^2)\bar{x}}{2(\lambda+4)}\right]|x|^2 \\
& + \left[\frac{(4-p^2)}{2(\lambda+4)} - \frac{(\lambda^2-2\lambda-7)p(4-p^2)}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]|x|^2 \\
& + \left[\left[\frac{-7\lambda^3+2\lambda^2+35\lambda+58}{8(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}\right]p^2(4-p^2) + \frac{p(4-p^2)}{2(\lambda+4)}\right]|x| \\
& + \left[\frac{\lambda^5-7\lambda^4+15\lambda^3+12\lambda^2-104\lambda+96}{8(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right]p^4 \\
& + \left[\frac{\lambda^2-2\lambda-7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2) + \frac{(4-p^2)\bar{x}}{2(\lambda+4)}. \\
& = C(p,|x|).
\end{aligned}
$$

We want to prove that the maximum value of $C(p,|x|)$ on $[0,2] \times [0,1]$. First, assume that there is a maximum at an interior point $C(p_0,|x_0|)$ of $[0,2] \times [0,1]$. Differentiating $C(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$ which is contradiction. Thus for the maximum of $C(p,|x|)$, we need to consider the end points of $[0,2] \times [0,1]$.

For $p = 0$, we obtain

$$
\begin{aligned}
C(0,|x|) &= \left[\frac{(3+2\lambda)4}{(\lambda+2)(\lambda+4)} - \frac{2}{(\lambda+4)}\bar{x}\right]\left|x\right|^2 + \left[\frac{4}{2(\lambda+4)}\right]\left|x\right|^2 + \frac{4}{2(4+x)}\bar{x} \\
&\leq \frac{(3+2\lambda)4}{(\lambda+2)(\lambda+4)} + \frac{6}{(\lambda+4)} \qquad (4.14)
\end{aligned}
$$

For $p = 2$, we acquire

$$C(2,|x|) = \frac{|\mathcal{G}(\lambda)|}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)} \qquad (4.15)$$

where,

$$\mathcal{G}(\lambda) = 2(\lambda^5 - 7\lambda^4 + 15\lambda^3 + 12\lambda^2 - 104\lambda + 96) \qquad (4.16)$$

For $|x| = 0$, we attained

$$C(p,0) = \left[\frac{\lambda^5-7\lambda^4+15\lambda^3+12\lambda^2-104\lambda+96}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right]p^4 + \left[\frac{-\lambda^2-2\lambda-7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2). \qquad (4.17)$$

For $|x| = 1$, we have

$$C(p,1)=\left[\frac{p^2(4-p^2)}{8(\lambda+4)}-\frac{p(4-p^2)}{2(\lambda+4)}\right]$$
$$+\left[\frac{(3+2\lambda)(4-p^2)^2}{4(2+\lambda)(4+\lambda)}+\frac{(-\lambda^2+8\lambda+17)(4-p^2)p^2}{8(1+\lambda)(\lambda+3)(\lambda+4)}-\frac{(4-p^2)\bar{x}}{2(\lambda+4)}\right]$$
$$+\left[\frac{(4-p^2)}{2(\lambda+4)}-\frac{(\lambda^2-2\lambda-7)p(4-p^2)}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]$$
$$+\left[\frac{-7\lambda^3+2\lambda^2+35\lambda+58}{8(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}p^2(4-p^2)+\frac{p(4-p^2)}{2(\lambda+2)}\right]$$
$$+\left[\frac{\lambda^5-7\lambda^4+15\lambda^3+12\lambda^2-104\lambda+96}{8(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right]p^4+\left[\frac{-\lambda^2-2\lambda-7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2)+\frac{4-p^2}{2(\lambda+4)}.$$

(4.18)

which has maximum value $\frac{|\mathcal{G}(\lambda)|}{(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}$ attained at the end point $p=2$ and $\frac{(3+2\lambda)4}{(\lambda+2)(\lambda+4)}+\frac{6}{(\lambda+4)}$ at $p=0$.

5. Generalized Zalcman Conjecture for the class $\mathcal{A}(\lambda)$

Theorem 5.1. Let f given by (1.1), be in the class $\mathcal{A}(\lambda)$; $(0\le\lambda\le1)$. Then we have sharp bound

$$|a_2a_3-a_4|\le\max\left\{\frac{4}{(\lambda+3)},\frac{|\mathcal{H}(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)}\right\}$$

where

$$\mathcal{H}(\lambda)=2(-\lambda^3+4\lambda^2-5\lambda+6)\tag{5.1}$$

Proof. First note that by equating the corresponding coefficients in the equation (3.1), we bring, in the view of (3.2), (3.3) and (3.4), a simple computation leads to

$$a_2a_3-a_4=\left[\frac{p_1}{\lambda+1}\right]\left[\frac{p_1^2(1-\lambda)}{(\lambda+1)(\lambda+2)}+\frac{p_2}{\lambda+2}\right].$$
$$-\left[\frac{p_1^3(1-\lambda)^2}{(\lambda+1)(\lambda+2)(\lambda+3)}+\frac{p_1p_2(1-\lambda)(3+2\lambda)}{(\lambda+1)(\lambda+2)(\lambda+3)}+\frac{p_3}{\lambda+3}\right].\tag{5.2}$$

Note that, by Lemma (2.2), we may write

$$a_2a_3-a_4=\frac{p_1Xx^2}{4(\lambda+3)}+\frac{XY}{2(\lambda+3)}+\left[\frac{\lambda^2-\lambda-2}{2(\lambda+1)(\lambda+2)(\lambda+3)}\right]p_1xX+\left[\frac{-\lambda^3+4\lambda^2-5\lambda+6}{4(\lambda+1)^2(\lambda+2)(\lambda+3)}\right]p_1^3.$$

Without loss of generality, we let $0\le p_1=p\le2$. Substitute this into the above equation, we obtain the following quadratic equation in terms of x.

$$|a_2a_3-a_4|\le\left[\frac{p(4-p^2)}{4(\lambda+3)}-\frac{(4-p^2)}{2(\lambda+3)}\right]|x|^2+\left[\frac{\lambda^2-\lambda-2}{2(\lambda+1)(\lambda+2)(\lambda+3)}\right](4-p^2)xp$$
$$+\left[\frac{-\lambda^3+4\lambda^2-5\lambda+6}{4(\lambda+1)^2(\lambda+2)(\lambda+3)}\right]p^3+\frac{4-p^2}{2(\lambda+3)}=£(p,|x|)\tag{5.3}$$

We prove that the maximum value of $£(p,|x|)$ on $[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $£(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $£(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p=p_0=2$ which is contradiction. Thus, for the maximum of $£(p,|x|)$, we need to consider the end points of $[0,2]\times[0,1]$. For $p=0$, we obtain

$$£(0,|x|)=\frac{-4}{2(\lambda+3)}|x|^2+\frac{4}{2(\lambda+3)}\leq\frac{4}{\lambda+3} \tag{5.4}$$

For $p=2$, we obtain

$$£(2,|x|)=\frac{2(-\lambda^3+4\lambda^2-5\lambda+6)}{(\lambda+1)^2(\lambda+2)(\lambda+3)} \tag{5.5}$$

For $|x|=0$, we get

$$£(p,0)=\left[\frac{-\lambda^3+4\lambda^2-5\lambda+6}{4(\lambda+1)^2(\lambda+2)(\lambda+3)}\right]p^3+\frac{4-p^2}{2(\lambda+3)} \tag{5.6}$$

which has maximum value $\frac{|\mathcal{H}(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)}$ attained at the end point $p=2$. For $|x|=1$, we obtain

$$£(p,1)=\left[\frac{p(4-p^2)}{4(\lambda+3)}-\frac{(4-p^2)}{2(\lambda+3)}\right]+\left[\frac{\lambda^2-\lambda-2}{2(\lambda+3)(\lambda+1)}\right](4-p^2)p$$
$$+\left[\frac{-\lambda^3+4\lambda^2-5\lambda+6}{4(\lambda+1)^2(\lambda+\lambda)(\lambda+3)}\right]p^3+\frac{4-p^2}{2(\lambda+3)}\quad. \tag{5.7}$$

which is maximum value of $£(p,1)=\frac{4}{\lambda+1}$ at $p=0$ and $£(p,1)=\frac{|\mathcal{H}(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)}$ at $p=2$.
Hence

$$|a_2a_3-a_4|\leq\max\left\{\frac{4}{(\lambda+3)},\frac{|\mathcal{H}(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)}\right\}. \tag{5.8}$$
$$\mathcal{H}(\lambda)=2(-\lambda^3+4\lambda^2-5\lambda+6). \tag{5.9}$$

Theorem 5.2. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}(\lambda)}$; $(0\leq\lambda\leq1)$. Then we have sharp bound

$$|a_2a_4-a_5|\leq\max\left\{\frac{4}{(\lambda+2)},\frac{|\partial(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}\right\}.$$

where,

$$\mathcal{J}(\lambda)=2(\lambda^4-9\lambda^3+29\lambda^2-45\lambda+36). \tag{5.10}$$

Proof.

First note that by equating the corresponding coefficients in the equation (3.1). In view of (3.2), (3.4) and (3.5), we may write $2p_2=p_1^2+x(4-p_1^2)$, $Y=(1-|x|^2)\varrho$ and applying Lemma (2.2), a simple computation leads to

$$a_2a_4 - a_5 = \left[\frac{\lambda^4-9\lambda^3+29\lambda^2-45\lambda+36}{8(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}\right]p_1^4$$
$$+\left[\frac{-3\lambda^4+11\lambda^3+9\lambda^2-35\lambda-6}{8(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}\right]p_1^2xX + \left[\frac{-\lambda^2+6\lambda+9}{8(\lambda+1)(\lambda+3)(\lambda+4)}\right]p_1^2x^2X$$
$$+\left[\frac{3\lambda^2+7\lambda+3}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p_1XY + \left[\frac{1-\lambda}{4(\lambda+2)(\lambda+4)}\right]x^2X^2 - \frac{p_1^2x^3X}{8(\lambda+4)}$$
$$-\frac{x^2X}{2(\lambda+4)} - \frac{xXp_1}{2(\lambda+4)} + \frac{XY\bar{x}}{2(\lambda+4)} \tag{5.11}$$

Without loss of generality, we let $0 \leq p_1 = p \leq 2$. Substitute this into the above equation, we obtain the following quadratic equation in terms of x.

$$|a_2a_4 - a_5| \leq \left[\frac{p^2(4-p^2)}{8(\lambda+4)} - \frac{p(4-p^2)}{2(\lambda+4)}\right]|x|^3$$
$$+\left[\frac{(-\lambda^2+6\lambda+9)(4-p^2)p^2}{8(\lambda+1)(\lambda+3)(\lambda+4)} + \frac{4-p^2}{2(\lambda+4)}\right]|x|^2$$
$$+\left[\frac{(4-p^2)^2}{4(2+\lambda)(\lambda+4)} - \frac{(3\lambda^2+7\lambda+3)(4-p^2)p}{2(\lambda+1)(\lambda+3)(\lambda+4)} - \frac{(4-p^2)\bar{x}}{2(\lambda+4)}\right]|x|^2$$
$$+\left[\frac{-3\lambda^4+11\lambda^3+9\lambda^2-35\lambda-6}{8(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}p^2(4-p^2) + \frac{p(4-p^2)}{2(\lambda+2)}\right]|x|$$
$$+\left[\frac{3\lambda^2+7\lambda+3}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2)$$
$$+\frac{(4-p^2)\bar{x}}{2(\lambda+4)} + \left[\frac{\lambda^4-9\lambda^3+29\lambda^2-45\lambda+36}{8(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}\right]p^4.$$
$$= \Delta(p,|x|). \tag{5.12}$$

We need to prove that the maximum value of $\Delta(p,|x|)$ on $[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\Delta(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\Delta(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$ which is contradiction. Thus, for the maximum of $\Delta(p,|x|)$, we should consider the end points of $[0,2]\times[0,1]$.

For $p = 0$, we obtain

$$\Delta(0,|x|) = \left[\frac{2}{(4+\lambda)} + \frac{4}{(\lambda+2)(4+\lambda)} - \frac{2\bar{x}}{(4+\lambda)}\right]|x|^2 + \frac{2\bar{x}}{(4+\lambda)} \quad . \tag{5.13}$$

For $p = 2$, we have

$$\Delta(2,|x|) = \frac{|\mathcal{J}(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}. \tag{5.14}$$

For $|x| = 0$, we gain

$$\Delta(p,0) = \left[\frac{3\lambda^3+7\lambda+3}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2)$$
$$+\left[\frac{\lambda^4-9\lambda^3+29\lambda^2-45\lambda+36}{8(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right]p^4 \,. \tag{5.15}$$

For $|x| = 1$, we obtain

$$\Delta(p,1) = \left[\frac{p^2(4-p^2)}{8(\lambda+4)} - \frac{p(4-p^2)}{2(\lambda+4)}\right] + \left[\frac{(-\lambda^2+6\lambda+9)(4-p^2)p^2}{8(\lambda+1)(\lambda+3)(\lambda+4)} + \frac{4-p^2}{2(\lambda+4)}\right] + \left[\frac{(4-p^2)^2}{4(2+\lambda)(\lambda+4)} - \frac{(3\lambda^2+7\lambda+3)(4-p^2)p}{2(\lambda+1)(\lambda+3)(\lambda+4)} - \frac{(4-p^2)}{2(\lambda+4)}\right] + \left[\frac{-3\lambda^4+11\lambda^3+9\lambda^2-35\lambda-6}{8(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}p^2(4-p^2) + \frac{p(4-p^2)}{2(\lambda+2)}\right] + \left[\frac{3\lambda^2+7\lambda+3}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2) + \frac{4-p^2}{2(\lambda+4)} + \left[\frac{\lambda^4-9\lambda^3+29\lambda^2-45\lambda+36}{8(\lambda+1)^2(\lambda+2)^2(\lambda+3)(\lambda+4)}\right]p^4 \quad . \tag{5.16}$$

which has maximum value $\frac{|\partial(\lambda)|}{(\lambda+1)^2(\lambda+2)(\lambda+3)(\lambda+4)}$ attained at the end point $p = 2$ and $\frac{4}{\lambda+2}$ at $p = 0$. Where,

$$\mathcal{J}(\lambda) = 2(\lambda^4 - 9\lambda^3 + 29\lambda^2 - 45\lambda + 36). \tag{5.17}$$

6. Krushkal Inequality for the class $\boldsymbol{\mathcal{A}(\lambda)}$

Theorem 6.1. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}(\lambda)}$; $(0 \le \lambda \le 1)$. Then we have sharp bound

$$|a_4 - a_2^3| = \max\left\{\frac{4}{\lambda+3}, \frac{|\mathcal{L}(\lambda)|}{(\lambda+1)^3(\lambda+2)(\lambda+3)}\right\}.$$

where,

$$\mathcal{L}(\lambda) = \lambda^4 - 5\lambda^3 - 5\lambda^2 - 3\lambda - 12. \tag{6.1}$$

Proof. First note that by equating the corresponding coefficients in the equation (3.1). We get, in the view of (3.2) and (3.4), we may write $2p_2 = p_1^2 + x(4-p_1^2), Y = (1-|x|^2)\varrho$ and applying Lemma (2.2), a simple computation leads to

$$a_4 - a_2^3 = \left[\frac{p_1^3(1-\lambda)^2}{(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{p_1p_2(1-\lambda)(3+2\lambda)}{(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{p_3}{\lambda+3}\right] - \left[\frac{p_1}{\lambda+1}\right]^3. \tag{6.2}$$

Note that, by Lemma (2.2), we have

$$a_4 - a_2^3 = \left[\frac{(1-\lambda)^2}{(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{(1-\lambda)(3+2\lambda)}{2(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{1}{4(\lambda+3)} - \frac{1}{(\lambda+1)^3}\right]p_1^3 + \left[\frac{(1-\lambda)(3+2\lambda)}{2(\lambda+1)(\lambda+2)(\lambda+3)} + \frac{1}{2(\lambda+3)}\right]p_1Xx - \frac{p_1Xx}{4(\lambda+3)} + \frac{XY}{2(\lambda+3)} \quad . \tag{6.3}$$

Without loss of generality, we let $0 \le p_1 = p \le 2$. Substitute this into the above equation, we obtain the following quadratic equation in terms of x.

$$|a_4 - a_2^3| \leq \left[\frac{p(4-p^2)}{4(\lambda+3)} - \frac{4-p^2}{2(\lambda+3)}\right]|x|^2 + \left[\left[\frac{-\lambda^2+2\lambda+5}{2(1+\lambda)(2+\lambda)(3+\lambda)}\right]p(4-p^2)\right]|x| + \left[\frac{\lambda^4-5\lambda^3-5\lambda^2-3\lambda-12}{4(\lambda+1)^3(\lambda+2)(\lambda+3)}\right]p^3 + \frac{4-p^2}{2(\lambda+3)}. = \Gamma(p,|x|) \quad . \tag{6.4}$$

So, the maximum value of $\Gamma(p,|x|)$ on $[0,2]\times[0,1]$. First, assume that there is a maximum at an interior point $\Gamma(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\Gamma(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$ which is contradiction. Thus, for the maximum value of $\Gamma(p,|x|)$, we need to consider the end points of $[0,2]\times[0,1]$.
For $p = 0$, we obtain

$$\Gamma(0,|x|) = \left[\frac{-4}{2(\lambda+3)}\right]|x|^2 + \frac{4}{2(\lambda+3)} \leq \frac{4}{\lambda+3} \quad . \tag{6.5}$$

For $p = 2$, we get

$$\Gamma(2,|x|) = \frac{|\mathcal{L}(\lambda)|}{(\lambda+1)^3(\lambda+2)(\lambda+3)}. \tag{6.6}$$

For $|x| = 0$, we gain

$$\Gamma(p,0) = \left[\frac{\lambda^4-5\lambda^3-5\lambda^2-3\lambda-12}{4(\lambda+1)^3(\lambda+2)(\lambda+3)}\right]p^3 + \frac{4-p^2}{2(\lambda+3)}. \tag{6.7}$$

For $|x| = 1$, we have

$$\Gamma(p,1) = \left[\frac{p(4-p^2)}{4(\lambda+3)} - \frac{4-p^2}{2(\lambda+3)}\right] + \left[\frac{-\lambda^2+2\lambda+5}{2(1+\lambda)(2+\lambda)(3+\lambda)}\right]p(4-p^2) + \left[\frac{\lambda^4-5\lambda^3-5\lambda^2-3\lambda-12}{4(\lambda+1)^3(\lambda+2)(\lambda+3)}\right]p^3 + \frac{4-p^2}{2(\lambda+3)} \quad . \tag{6.8}$$

which has maximum value $\frac{|\mathcal{L}(\lambda)|}{(\lambda+1)^3(\lambda+2)(\lambda+3)}$ attained at the end point $p = 2$ and $\frac{4}{\lambda+3}$ at $p = 0$.
Where,

$$\mathcal{L}(\lambda) = \lambda^4 - 5\lambda^3 - 5\lambda^2 - 3\lambda - 12. \tag{6.9}$$

Theorem 6.2. Let f given by (1.1), be in the class $\boldsymbol{\mathcal{A}(\lambda)}$; $(0 \leq \lambda \leq 1)$. Then we have sharp bound

$$|a_5 - a_2^4| \leq \max\left\{\frac{4(1+\lambda)}{(\lambda+2)(\lambda+4)} + \frac{6}{\lambda+4}, \frac{|Q(\lambda)|}{(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}\right\}. \tag{6.10}$$

where

$$\mathcal{Q}(\lambda) = 2(\lambda^6 + 21\lambda^5 + 6\lambda^4 - 54\lambda^3 - 43\lambda^2 - 87\lambda - 132). \tag{6.11}$$

Proof. First note that by equating the corresponding coefficients in the equation (3.1) we get, in the view of (3.2) and (3.5), we may write

$$a_5 - a_2^4 = \frac{p_1^4(1-\lambda)^3}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} + \frac{p_1^2 p_2(1-\lambda)^2(3+2\lambda)}{(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)} \tag{6.12}$$

$$+\frac{p_1p_3(1-\lambda)}{(\lambda+3)(\lambda+4)} + \frac{p_1^2p_2(1-\lambda)^2}{(\lambda+1)(\lambda+2)(\lambda+4)} + \frac{p_2^2(1-\lambda)}{(\lambda+2)(\lambda+4)} + \frac{p_1p_3(1-\lambda)}{(\lambda+1)(\lambda+4)}$$

$$+\frac{p_4}{\lambda+4} - \left(\frac{p_1}{\lambda+1}\right)^4. \tag{6.13}$$

Note that, by Lemma (2.2) and we have $2p_2 = p_1^2 + x(4-p_1^2), Y = (1-|x|^2)\varrho$, a simple computation leads to

$$a_5 - a_2^4 = \left[\frac{\lambda^6+21\lambda^5+6\lambda^4-54\lambda^3-43\lambda^2-87\lambda-132}{8(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}\right]p_1^4 + \left[\frac{3\lambda^3-14\lambda^2-7\lambda+90}{8(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)}\right]p^2xX$$

$$+\left[\frac{-\lambda^2+2\lambda+7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p_1XY - \left[\frac{-\lambda^2+8\lambda+17}{8(\lambda+1)(\lambda+3)(\lambda+4}\right]p_1^2Xx^2 \tag{6.14}$$

$$+\left[\frac{(1-\lambda)}{4(\lambda+2)(\lambda+4)}\right]X^2x^2 + \frac{{p_1}^2Xx^3}{8(\lambda+4)} + \frac{Xx^2}{2(\lambda+4)} - \frac{XYp_1x}{2(\lambda+4)} - \frac{XY\bar{x}}{2(\lambda+4)}$$

without loss of generality, we let $0 \le p_1 = p \le 2$. Substitute this into the above equation, we obtain the following quadratic equation in terms of x.

$$\begin{aligned}
|a_5 - a_2^4| \le &\left[\frac{p^2(4-p^2)}{8(\lambda+4)} - \frac{p(4-p^2)}{2(\lambda+4)}\right]|x|^3 \\
&+\left[\frac{(-\lambda^2+2\lambda+7)p(4-p^2)}{2(\lambda+1)(\lambda+3)(\lambda+4)} + \frac{(-\lambda^2+8\lambda+17)p^2(4-p^2)}{8(\lambda+1)(\lambda+3)(\lambda+4)}\right]|x|^2 \\
&+\left[\frac{(4-p^2)^2(1-\lambda)}{4(\lambda+2)(\lambda+4)} + \frac{(4-p^2)}{2(\lambda+4)} - \frac{(4-p^2)\bar{x}}{2(\lambda+4)}\right]|x|^2 \\
&+\left[\left[\frac{(3\lambda^3-14\lambda^2-7\lambda+90)p^2}{8(\lambda+1)(\lambda+3)(\lambda+4)(\lambda+2)} + \frac{p}{2+\lambda}\right](4-p^2)\right]|x| \\
&+\left[\left[\frac{-\lambda^2+2\lambda+7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p + \frac{\bar{x}}{2(\lambda+4)}\right](4-p^2) \\
&+\left[\frac{\lambda^6+21\lambda^5+6\lambda^4-54\lambda^3-43\lambda^2-87\lambda-132}{8(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}\right]p^4 \\
&= \Lambda(p,|x|)
\end{aligned}$$

We need to prove that the maximum value of $\Lambda(p,|x|)$ on $[0,2]\times[0,1]$. First assume, that there is a maximum at an interior point $\Lambda(p_0,|x_0|)$ of $[0,2]\times[0,1]$. Differentiating $\Lambda(p,|x|)$ with respect to $|x|$ and equating it to 0 implies that $p = p_0 = 2$ which is contradiction. Thus, for the maximum of $\Lambda(p,|x|)$, we have to consider the end points of $[0,2]\times[0,1]$. For $p = 0$ we obtain

$$\Lambda(0,|x|) = \left[\frac{4(1-\lambda)}{(\lambda+2)(\lambda+4)} + \frac{2}{(\lambda+4)} - \frac{2\bar{x}}{(\lambda+4)}\right]|x|^2 + \frac{2\bar{x}}{(\lambda+4)}. \tag{6.15}$$

$$\leq \frac{4(1-\lambda)}{(\lambda+2)(\lambda+4)} + \frac{6}{\lambda+4}. \tag{6.16}$$

For $p = 2$, we obtain

$$\Lambda(2,|x|) = \frac{|Q(\lambda)|}{(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}. \tag{6.17}$$

For $|x| = 0$, we have

$$\Lambda(p,0) = \left[\frac{\lambda^6 + 21\lambda^5 + 6\lambda^4 - 54\lambda^3 - 43\lambda^2 - 87\lambda - 132}{8(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}\right]p^4$$
$$+ \left[\frac{-\lambda^2 + 2\lambda + 7}{2(\lambda+1)(\lambda+3)(\lambda+4)}\right]p(4-p^2)\ .$$

(6.18)

For $|x| = 1$, we get

$$\begin{aligned}\Lambda(p,1) = &\left[\frac{p^2(4-p^2)}{8(\lambda+4)} - \frac{p(4-p^2)}{2(\lambda+4)}\right]\\ &+ \left[\frac{(-\lambda^2+2\lambda+7)p(4-p^2)}{2(\lambda+1)(\lambda+3)(\lambda+4)} + \frac{(-\lambda^2+8\lambda+17)p^2(4-p^2)}{8(\lambda+1)(\lambda+3)(\lambda+4)}\right]\\ &+ \left[\frac{(4-p^2)^2(1-\lambda)}{4(\lambda+2)(\lambda+4)} + \frac{(4-p^2)}{2(\lambda+4)} - \frac{(4-p^2)}{2(\lambda+4)}\right]\\ &+ \left[\left[\frac{(3\lambda^3 - 14\lambda^2 - 7\lambda + 90)p^2}{8(\lambda+1)(\lambda+3)(\lambda+4)(\lambda+2)} + \frac{p}{2+\lambda}\right](4-p^2)\right]\\ &+ \left[\left[\frac{-\lambda^2+2\lambda+7}{2(\lambda+1)(\lambda+3)(\lambda+4}\right]p + \frac{1}{2(\lambda+4)}\right](4-p^2)\\ &+ \left[\frac{\lambda^6 + 21\lambda^5 + 6\lambda^4 - 54\lambda^3 - 43\lambda^2 - 87\lambda - 132}{8(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}\right]p^4.\end{aligned}$$

which has maximum value $\frac{|Q(\lambda)|}{(\lambda+1)^4(\lambda+2)(\lambda+3)(\lambda+4)}$ attained at the end point $p = 2$ and $\frac{4(1-\lambda)}{(\lambda+2)(\lambda+4)} + \frac{6}{\lambda+4}$ at $p = 0$.
Where,

$$Q(\lambda) = 2(\lambda^6 + 21\lambda^5 + 6\lambda^4 - 54\lambda^3 - 43\lambda^2 - 87\lambda - 132). \tag{6.19}$$

Conclusion

In this Present paper Toeplitz matrices characterized by coefficients from novel subclasses. The investigation establishes upper limits for the initial four determinants of these matrices, presenting innovative and unique findings. Notably, our results parallel recent works by Thomas and Halim [1], specifically in the context of star-like and close - to - convex functions, as well as by Radhika et al. [2], which concentrate on functions with bounded boundary rotation. Additionally, we have determined the Zalcman and Generalized Zalcman conjecture, along with Krushkal inequalities for certain parameters. This contributes to the existing body of knowledge in the field and demonstrates the novelty of our results in comparison to recent literature. The result obtained perhaps give an opportunity for researchers to further investigate inequalities problems for functions of the class $\mathcal{A}$ as well as other subclasses $\mathcal{S}$.

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